Research Article

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Some curvature properties of paracontact metric manifolds

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Abstract: The purpose of this paper is to study Ricci semisymmetric paracontact metric manifolds satisfying $\nabla_\xi h = 0$ and such that the sectional curvature of the plane section containing ξ equals a non-zero constant c. Also, we study paracontact metric manifolds satisfying the curvature condition $Q \cdot R = 0$, where Q and R are the Ricci operator and the Riemannian curvature tensor, respectively, and second order symmetric parallel tensors in paracontact metric manifolds under the same conditions. Several consequences of these results are discussed.

Keywords: Paracontact metric manifolds, Ricci semisymmetric, second order parallel tensor, Einstein manifold

MSC 2010: Primary 53B30, 53C15, 53C25; secondary 53D10, 53D15

1 Introduction

Among the geometric properties of manifolds, symmetry is an important one. From the local point of view, it was introduced by Shirokov [15] who studied a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where ∇ is the Levi-Civita connection. An extensive study of symmetric Riemannian manifolds was carried out by Cartan in 1927. A Riemannian manifold or pseudo-Riemannian manifold is called semisymmetric if the curvature tensor R satisfies $R(X, Y) \cdot R = 0$, where R is the Riemannian curvature tensor and $R(X, Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Semisymmetric manifolds were locally classified by Szabó in [16].

A Riemannian manifold or pseudo-Riemannian manifold is said to be *Ricci semisymmetric* if $R(X,Y) \cdot S = 0$, where S denotes the Ricci tensor of type $(0, 2)$. A general classification of these manifolds has been worked out by Mirzoyan [13].

An example of a curvature condition of semisymmetry type is $Q \cdot R = 0$, where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$. The curvature condition $Q \cdot R = 0$ was studied in [17].

One of the main aims of this paper is to study the Eisenhart problem. In 1923, Eisenhart [9] proved that if a positive definite Riemannian manifold admits a second order parallel symmetric covariant tensor, other than a constant multiple of the associated metric tensor, then it is reducible. In 1925, Levy [11] proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the associated metric tensor. Sharma [14] considered the Eisenhart problem on contact geometry. De [6], and Wang and Liu [18] studied second order parallel tensors in P-Sasakian manifolds and almost Kenmotsu manifolds with nullity distributions, respectively. More recently, De and Mandal [8] studied second order parallel tensors on generalized (k, μ)-contact metric manifolds. Motivated by the above studies, we study paracontact metric manifolds admitting a second order symmetric parallel tensor assuming certain conditions.

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The present paper is structured as follows. In Section 2 we give some basic results of paracontact metric manifolds. Section 3 is devoted to studying Ricci semisymmetric paracontact metric manifolds satisfying $\nabla_\xi h = 0$ and such that the sectional curvature of the plane section containing ξ equals a non-zero constant c. Also, we prove that under these conditions a paracontact metric manifold is Ricci semisymmetric if and only if the manifold is Einstein. In the next sections we consider paracontact metric manifolds satisfying the curvature condition $Q \cdot R = 0$, where Q and R are the Ricci operator and Riemannian curvature tensor, respectively, and second order symmetric parallel tensor in paracontact metric manifolds under the same conditions.

2 Preliminaries

A smooth manifold M^{2n+1} has an almost paracontact structure (ϕ, ξ, η) if it admits a $(1, 1)$-type tensor field ϕ, a vector field ξ (called the Reeb vector field) and a 1-form η satisfying the following conditions (see [10]):
(i) $\phi^2 X = X - \eta(X)\xi$.
(ii) $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\eta(\xi) = 1$.
(iii) The tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have the same dimension n.

An almost paracontact manifold, equipped with a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.1}$$

for all $X, Y \in \chi(M)$, is called *almost paracontact metric manifold*, and (ϕ, ξ, η, g) is said to be an *almost paracontact metric structure*.

An almost paracontact structure is said to be *normal* (see [19]) if and only if for the $(1, 2)$-type torsion tensor N_ϕ, we have

$$N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0, \quad \text{where } [\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

An almost paracontact structure is said to be a *paracontact* structure if $g(X, \phi Y) = d\eta(X, Y)$ (see [19]). Any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ-basis (see [19]), that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, \dots, E_n, \phi E_1, \phi E_2, \dots, \phi E_n\}$, where $\xi, E_1, E_2, \dots, E_n$ are space-like vector fields, and then, by (2.1), the vector fields $\phi E_1, \phi E_2, \dots, \phi E_n$ are time-like. In a paracontact metric manifold one can easily define a symmetric trace-free $(1, 1)$-tensor $h = \frac{1}{2}\pounds_\xi \phi$

$$\begin{aligned} &\phi h + h\phi = 0, \quad h\xi = 0, \\ &\nabla_X \xi = -\phi X + \phi h X \quad \text{for all } X \in \chi(M). \end{aligned} \tag{2.2}$$

Clearly h vanishes identically if and only if ξ is a Killing vector field, and then (ϕ, ξ, η, g) is said to be a *K-paracontact structure*. An almost paracontact metric manifold is said to be a *paraSasakian* manifold if and only if (see [19])

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is paraSasakian and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.3}$$

for any $X, Y \in \chi(M)$ but, unlike contact metric geometry, the relation (2.3) does not imply that the paracontact manifold is paraSasakian. It is well known that every paraSasakian manifold is K-paracontact. The converse is not always true, but it holds in the three-dimensional case (see [1]). Paracontact metric manifolds have been studied extensively, see, e.g., [1–5, 7, 12].

Moreover, the following properties hold in paracontact metric manifolds (see [1]):

$$\nabla_\xi \xi = 0, \quad \nabla_\xi \phi = 0,$$
$$(\nabla_\xi h)X = -\phi X - \phi l X + h^2 \phi X, \tag{2.4}$$
$$\phi l \phi X + lX = 2(h^2 - \phi^2)X, \tag{2.5}$$
$$\operatorname{Tr} l = g(Q\xi, \xi) = \operatorname{Tr} h^2 - 2n.$$

Assuming $\nabla_\xi h = 0$ and that the sectional curvature $K(\xi, X)$ is equal to a non-zero constant c, from (2.4) and (2.5), we have

$$h^2 X = (1 - c)\phi^2 X. \tag{2.6}$$

Making use of (2.4) and (2.6), we obtain

$$R(X, \xi)\xi = -c\{X - \eta(X)\xi\}. \tag{2.7}$$

Contracting X in (2.7), one gets

$$S(\xi, \xi) = -2nc. \tag{2.8}$$

3 Ricci semisymmetric paracontact metric manifolds

In this section we discuss about Ricci semisymmetric paracontact metric manifolds such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. Hence, we have

$$(R(X, Y) \cdot S)(U, V) = 0$$

for any $X, Y, U, V \in \chi(M)$. This implies

$$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \tag{3.1}$$

Putting $X = U = \xi$ in (3.1), we obtain

$$S(R(\xi, Y)\xi, V) + S(\xi, R(\xi, Y)V) = 0. \tag{3.2}$$

Making use of (2.7) and (3.2), we have

$$c\{S(Y, V) - \eta(Y)S(\xi, V)\} + S(\xi, R(\xi, Y)V) = 0. \tag{3.3}$$

Again for $V = \xi$, the above equation implies

$$2c\{S(Y, \xi) - \eta(Y)S(\xi, \xi)\} = 0.$$

Since c is non-zero, we infer that

$$S(Y, \xi) = \eta(Y)S(\xi, \xi)$$

for any vector field Y. Using (2.8), the above equation is equivalent to

$$S(Y, \xi) = -2nc\eta(Y) \tag{3.4}$$

for any vector filed Y. Substituting (3.4) in (3.3), we get

$$c\{S(Y, V) + 2nc\eta(Y)\eta(V)\} - 2nc\eta(R(\xi, Y)V) = 0.$$

Since c is non-zero, the above equation yields

$$S(Y, V) + 2nc\eta(Y)\eta(V) - 2n\eta(R(\xi, Y)V) = 0. \tag{3.5}$$

From (2.7), it follows that

$$\eta(R(\xi, Y)V) = -c\{g(Y, V) - \eta(Y)\eta(V)\}. \tag{3.6}$$

Using (3.6) in (3.5), we have

$$S(Y, V) = -2ncg(Y, V).$$

Thus, the manifold is Einstein. Conversely, if the manifold is Einstein, then obviously $R \cdot S = 0$.

By the above discussions we have the following theorem.

Theorem 3.1. *Let M be a paracontact metric manifold such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. Then the manifold M is Ricci semisymmetric if and only if the manifold is Einstein.*

Again Ricci symmetry ($\nabla S = 0$) implies Ricci semisymmetry ($R \cdot S = 0$), thus we have the following corollary.

Corollary 3.2. *Let M be a paracontact metric manifold such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. Then the manifold M is Ricci symmetric if and only if the manifold is Einstein.*

Moreover, semisymmetry ($R \cdot R = 0$) implies Ricci semisymmetry ($R \cdot S = 0$), therefore we can state the following corollary.

Corollary 3.3. *Let M be a paracontact metric manifold such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. If the manifold M is semisymmetric, then the manifold is Einstein.*

4 Paracontact metric manifolds satisfying the curvature condition $Q \cdot R = 0$

This section is devoted to studying paracontact metric manifolds such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c and satisfies the curvature condition $Q \cdot R = 0$, that is, $(Q \cdot R)(X, Y)Z = 0$ for all vector fields X, Y, and $Z \in \chi(M)$. This is equivalent to

$$Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ = 0. \tag{4.1}$$

Substituting $X = Z = \xi$ in (4.1), we have

$$Q(R(\xi, Y)\xi) - R(Q\xi, Y)\xi - R(\xi, QY)\xi - R(\xi, Y)Q\xi = 0.$$

Since, by hypothesis, $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c, applying (2.7) in the above equation yields

$$c\{QY - \eta(Y)Q\xi\} - R(Q\xi, Y)\xi - c\{QY - \eta(QY)\xi\} - R(\xi, Y)Q\xi = 0. \tag{4.2}$$

In (4.2), taking the inner product with ξ, we get

$$-c\eta(Y)S(\xi, \xi) - g(R(Q\xi, Y)\xi, \xi) + c\eta(QY) - g(R(\xi, Y)Q\xi, \xi) = 0. \tag{4.3}$$

Now from (2.7), we obtain

$$g(R(\xi, Y)Q\xi, \xi) = -c\{S(Y, \xi) + 2nc\eta(Y)\} \tag{4.4}$$

and

$$g(R(Q\xi, Y)\xi, \xi) = 0. \tag{4.5}$$

Making use of (2.8), (4.4) and (4.5) in (4.3) yields

$$S(Y, \xi) = -2nc\eta(Y), \tag{4.6}$$

which implies

$$Q\xi = -2nc\xi. \tag{4.7}$$

Using (4.6) and (4.7) in (4.2), we infer that

$$cR(\xi, Y)\xi = 0,$$

which implies either $c = 0$ or $R(\xi, Y)\xi = 0$.

Since, by hypothesis, $c \neq 0$, it follows that $R(\xi, Y)\xi = 0$. Again, if $R(\xi, Y)\xi = 0$, then from (2.7) we have that $c = 0$, which is a contradiction. Hence, we obtain the following theorem.

Theorem 4.1. *Let M be a paracontact metric manifold such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. Then M does not satisfy the curvature condition $Q \cdot R = 0$.*

5 Second order symmetric parallel tensor

Definition 5.1 ([11]). A tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the covariant differentiation with respect to the associated metric tensor.

Let α be a symmetric $(0, 2)$-tensor field on a paracontact metric manifold M such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(W, X)Y, Z) + \alpha(Y, R(W, X)Z) = 0 \tag{5.1}$$

for any vector fields $W, X, Y, Z \in \chi(M)$.

Substituting $W = Y = Z = \xi$ in (5.1) and noticing that α is symmetric implies

$$\alpha(R(\xi, X)\xi, \xi) = 0. \tag{5.2}$$

Let us assume that the manifold M is connected. Applying (2.7) in (5.2) yields

$$c\{\alpha(X, \xi) - \eta(X)\alpha(\xi, \xi)\} = 0.$$

Since $c \neq 0$, we have

$$\alpha(X, \xi) - \eta(X)\alpha(\xi, \xi) = 0. \tag{5.3}$$

Taking the covariant differentiation of (5.3) along Y, we obtain

$$\alpha(\nabla_Y X, \xi) + \alpha(X, \nabla_Y \xi) = g(\nabla_Y X, \xi)\alpha(\xi, \xi) + g(X, \nabla_Y \xi)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\nabla_Y \xi, \xi). \tag{5.4}$$

Replacing X by $\nabla_Y X$ in (5.3), we get

$$g(\nabla_Y X, \xi)\alpha(\xi, \xi) - \alpha(\nabla_Y X, \xi) = 0. \tag{5.5}$$

Making use of (5.4) and (5.5), we have

$$\alpha(X, \nabla_Y \xi) = g(X, \nabla_Y \xi)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\nabla_Y \xi, \xi). \tag{5.6}$$

Applying (2.2) and (5.3) in (5.6), it follows that

$$\alpha(X, \phi Y) - \alpha(X, \phi hY) - g(X, \phi Y)\alpha(\xi, \xi) + g(X, \phi hY)\alpha(\xi, \xi) - 2\eta(X)\alpha(\phi Y, \xi) + 2\eta(X)\alpha(\phi hY, \xi) = 0. \tag{5.7}$$

If we put $X = \phi X$ in (5.3), then we get

$$\alpha(\phi X, \xi) = 0.$$

Using the above equation, from (5.7), we have

$$\alpha(X, \phi Y) - \alpha(X, \phi hY) - g(X, \phi Y)\alpha(\xi, \xi) + g(X, \phi hY)\alpha(\xi, \xi) = 0. \tag{5.8}$$

Substituting $Y = \phi Y$ in (5.8) and using (5.3), we have

$$\alpha(X, Y) + \alpha(X, hY) - g(X, Y)\alpha(\xi, \xi) - g(X, hY)\alpha(\xi, \xi) = 0. \tag{5.9}$$

Replacing Y by hY in (5.9) and making use of (2.6), we obtain

$$\alpha(X, hY) - g(X, hY)\alpha(\xi, \xi) + (1 - c)\{\alpha(X, Y) - g(X, Y)\alpha(\xi, \xi)\} = 0. \tag{5.10}$$

Subtracting (5.9) from (5.10) and since $c \neq 0$, it follows that

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y).$$

Since α and g are both parallel tensor fields, $\alpha(\xi, \xi)$ must be constant.

Hence, we can state the following theorem.

Theorem 5.2. *Let M be a connected paracontact metric manifold such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. If the manifold M admits a second order symmetric parallel tensor, then the second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

As an application, let us consider a paracontact metric manifold which is Ricci symmetric, that is, $\nabla S = 0$. Since the Ricci tensor is a symmetric $(0, 2)$-tensor, applying Theorem 5.2, we have the following corollary.

Corollary 5.3. *Let M be a connected paracontact metric manifold such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals a non-zero constant c. If the manifold M is Ricci symmetric, then the manifold is Einstein.*

Note that the above corollary is a weaker version of Corollary 3.2.

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