PHASE SPACE ANALYSIS OF AN INTERACTING FERMI GAS

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The concept of momentum anisotropy in the phase space of a system of interacting particles interconnects the momentum deformation with the nuclear interaction energy and deduces an expression for the single particle level density of the system. Using our RGM interactions, our predictions compare favourably with the experimental level density parameters.

In the free Fermi gas model [1], the nucleus is assumed to be a noninteracting system of A particles confined to move in a spherical nuclear volume $V_0 = \frac{4}{3}\pi r_0^3 A$ with an isotropic momentum distribution. The phase space occupied by the system in the ground state, assuming a continuous single particle level density $g_0(\epsilon)$ to exist, may be expressed as

$$A = \int_0^{\epsilon_0} g_0(\epsilon)\mathrm{d}\epsilon = 4\int_0^{\epsilon_0} \frac{V_0}{h^3}\cdot 4\pi p^2 \frac{\mathrm{d}p}{\mathrm{d}\epsilon}\mathrm{d}\epsilon\ , \quad (1)$$

where the factor 4 outside the integral accounts for the spin and isospin weights of the nucleons. From eq. (1), the Fermi energy ϵ_0, and the single particle level density g_0 at ϵ_0 is found to be

$$\epsilon_0 = (\tfrac{9}{8}\pi)^{2/3}\hbar^2/2mr_0^2, \qquad g_0 = 3A/2\epsilon_0, \quad (1a)$$

where ϵ_0 is independent of the number of noninteracting particles; the excitation energy of the system is measured from ϵ_0.

When nuclear interactions between A particles are to be included, the situation gets complicated since the Fermi surface ϵ_F will no longer be a universal constant; it will depend on the exact nature of interactions and will vary from nucleus to nucleus. One obtains, in principle, the Fermi energy ϵ_F from the phase space equation of this interacting system by defining a single particle density of states function $g(\epsilon)$ such that

$$A = \int_0^{\epsilon_F} g(\epsilon)\mathrm{d}\epsilon = \frac{4V\Omega}{h^3} = \frac{4V}{h^3}\int_0^{\epsilon_F} f(p)\frac{\mathrm{d}p}{\mathrm{d}\epsilon}\mathrm{d}\epsilon \quad (2)$$

where V and Ω are the coordinate and momentum space volumes respectively, and $f(p)$ is an elementary momentum volume.

In this note, we introduce the concept of momentum anisotropy as a result of nuclear interactions on a free Fermi gas system. Using a radial type deformation expansion [2,3] for a deformed momentum volume, the Fermi momentum $p_F = (2m\epsilon_F)^{1/2}$ has been analytically related with the momentum deformation. An expression for the single particle level density $g(\epsilon_F)$ at the Fermi surface of the interacting system ϵ_F has been deduced. The effects of nuclear shape deformation have been included when necessary. Using the nuclear interaction energies of our renormalised gas model (RGM) [4], the final results of our analysis have been computed in the form of the statistical level density parameter $a = \pi^2 g(\epsilon_F)/6$ and have been compared with the experimental data.

We start with the constraints that (a) the total number of particles A is the same in the free and interacting systems (i.e., the total phase space is conserved) and that (b) the nuclear volume is conserved ($V = V_0$ even when the nuclear shape is nonspherical); these imply that the total momentum volume is conserved:

$$\tfrac{4}{3}\pi p_0^3 = \Omega_0 = \Omega = \int_0^{\epsilon_F} f(p)\frac{\mathrm{d}p}{\mathrm{d}\epsilon}\mathrm{d}\epsilon\ . \quad (3)$$

On switching off the nuclear interactions, $\epsilon = \epsilon_0$, $f(p) = 4\pi p^2$ and eq. (3) is a trivial identity. The function $f(p)$ for an interacting system thus must contain a momentum "deformation", i.e., the momentum distribution of A interacting particles must be anisotropic in contrast with the isotropic momentum distribution of the corresponding free Fermi gas.

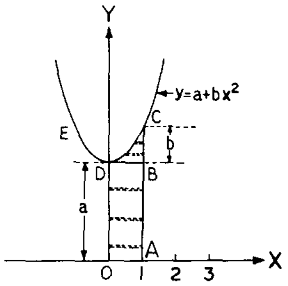

Fig. 1. Limits of the variables x and y used in the momentum space integration in eq. (8). Parameters a and b have been defined in the text

The volume-conserving space deformation problems have been extensively studied from the Swiatecki-type radial expansion [3]

$$R = R_0\left[1+\sum_{n=1}^{\infty}\beta_n P_n(\cos\theta')\right]\lambda_\beta^{-1} \tag{4}$$

around the spherical shape R_0 with λ_β as the volume-conserving parameter (the other symbols carry their ususal meanings). A momentum deformation expansion similar to eq.(4) leads to ambiguities of interpretation connected with the momentum coordinate variables. We therefore introduce a coordinate y which is a function of (and is of the dimension of) the local momentum p and take the deformation to be

$$Y = p_0\left[1+\sum_{n=1}^{\infty}\alpha_n P_n(\cos\theta)\right]\lambda^{-1} . \tag{5}$$

Our momentum volume now takes the form

$$\Omega = \Omega_0 = \int_0^{\epsilon_F} f(p)\,\frac{dp}{d\epsilon}\,d\epsilon = \int_0^{\epsilon_F} F(y)\,\frac{dy}{d\epsilon}\,d\epsilon . \tag{6}$$

Our aim is to relate p_F (or ϵ_F) with Y.

In this work, we assume for simplicity that the momentum deformation is symmetric and retain only the lowest even amplitude α_2:

$$Y = p_0[1+\alpha_2 P_2(\cos\theta)]\lambda^{-1} = a + bx^2 , \tag{7}$$

where (7a)

$$a = p_0(1-\alpha_2/2)/\lambda, \quad b = 3p_0\alpha_2/2\lambda, \quad x = \cos\theta ,$$

and

$$\lambda^3 = \tfrac{1}{2}\int_{x=-1}^{1}[1+\alpha_2 P_2(x)]^3\,dx = 1 + \tfrac{3}{5}\alpha_2^2 + \tfrac{2}{35}\alpha_2^3 . \tag{7b}$$

The interacting momentum volume can now be written as

$$\Omega = \int_{\theta=0}^{\pi}\int_{y=0}^{a+b\cos^2\theta}\int_{\phi=0}^{2\pi} y^2 \sin\theta\, d\theta\, dy\, d\phi =$$

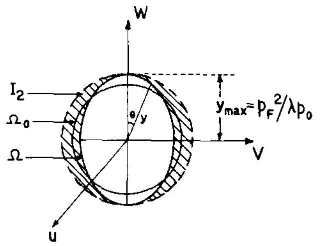

Fig. 2. Interconnection between the y-space and the p-space drawn in the orthogonal coordinate system u, v and w in the y-space. The deformed momentum volume $\Omega = \Omega_0$, the free gas volume (full lines). The relationship between y and the local momentum p is $y = p^2/\lambda p_0$. A sphere of radius $Y_{max} = p_F^2/\lambda p_0$ does not conserve the momentum volume and a reduction by an amount I_2 (shaded area) is necessary. The maximum elongation Y_{max} corresponds to the Fermi momentum $p_F = (\lambda p_0 Y_{max})^{1/2} = p_0(1+\alpha_2)^{1/2}$.

$$= \int_{x=0}^{1}\int_{y=0}^{a+bx^2} 4\pi y^2\, dx\, dy . \tag{8}$$

In eq.(8), since the upper limit of y is a function of x, we have to perform the y-integration first. This, however, gives the triviality that $\Omega = \frac{4}{3}\pi p_0^3$. Our aim is to determine $F(y)$ in eq. (6). This forces us to change the order of integration in eq.(8) and involves finding new limits of x as function of y and of limits of y independent of x, as illustrated in fig. 1. The lower and upper limits of y-integration are now the x-axis and the parabola EDC ($y = a + bx^2$), respectively. The limits of the x-integration are the y-axis and the straight line AC ($x = 1$). The area of integration is OACD. Change in the order of integration divides the area into narrow strips parallel to the x-axis instead of that of y. The integral breaks up into two parts: (i) one over the area OABD (with new limits of x as $x = 0$ and $x = 1$, and those of y as $y = 0$ and $y = a$), and (ii) the other part over BCD (with limits of x as $x = [(y-a)/b]^{1/2}$ and $x = 1$, and $y = a$ and $y = a + b$). Completion of the x-integration in (8) now gives

$$\Omega = \int_{y=0}^{a+b} 4\pi y^2 dy - \int_{y=a}^{a+b} 4\pi y^2\left(\frac{y-a}{b}\right)^{1/2} dy. \tag{9}$$

We try to combine the two integrals into a common functional under common limits of integration to make use of eq. (6). Writing the second integral in eq. (9) in the form

$$I_2 = \int_a^{a+b} 4\pi y^2 \left(\frac{y-a}{b}\right)^{1/2} dy = \int_a^{a+b} \mathcal{F}(y) dy =$$

$$= \int_0^{a+b} \mathcal{F}(my+c) m dy \ , \qquad (10)$$

we make use of theorem no. 406 of Landau [5]. We obtain $m = b/(a+b)$ and $c = a$ from the condition that when a is the lower limit of the integral in $\mathcal{F}(y)$, that of $\mathcal{F}(my+c)$ is zero, and note that both the upper limits remain unaltered. Rewriting eq. (9) in terms of these functions and limits, the total phase space equation may now be explicitly written from eqs. (2), (3), (6) and (7) as

$$\int_0^{\epsilon_F} \frac{dA}{dy} \cdot \frac{dy}{d\epsilon} d\epsilon = A = \int_0^{\epsilon_F} \frac{16\pi V}{h^3} \times$$

$$\left\{ y^2 - \frac{b}{a+b} \left(\frac{b \cdot y}{a+b} + a\right)^2 \left(\frac{y}{a+b}\right)^{1/2} \right\} dy \ . \qquad (11)$$

An energy difference $\Delta\epsilon = \epsilon_F - \epsilon_0$, ($\Delta\epsilon/\epsilon_0 \ll 1$), may also be used as a measure of deformation. We relate this to α_2 as

$$\alpha_2 = \Delta\epsilon/\epsilon_0 = (\epsilon_F/\epsilon_0) - 1 \ . \qquad (12)$$

Remembering that $\epsilon = 0$ when $y = 0$, and $\epsilon = \epsilon_F$ when $y = a+b$, the upper limit in eqs.(8) to (12) is $Y_{max} = a+b = p_0\epsilon_F/\lambda\epsilon_0 = p_F^2/\lambda p_0$. The choice of the functions

$$y = (p_0/\lambda) \cdot (\epsilon/\epsilon_0) \ , \qquad dy/d\epsilon = p_0/\lambda\epsilon_0 \qquad (13)$$

is consistent with our entire procedure and also conserves the momentum volume. This is illustrated in fig. 2.

From eqs. (2) and (12) the single particle level density for the interacting system may now be expressed as

$$g(\epsilon) = \frac{16\pi V}{h^3} \cdot \frac{p_0}{\lambda\epsilon_0} \times$$

$$\left[y^2 - \left(\frac{by}{a+b} + a\right)^2 \frac{b}{a+b} \left(\frac{y}{a+b}\right)^{1/2} \right] , \qquad [14]$$

from which the single particle level density of the system at the Fermi surface ϵ_F is given by

$$g(\epsilon_F) = 2g_0(\epsilon_F/\epsilon_0) \cdot (1 - \alpha_2/2)/\lambda \ . \qquad (15)$$

Since $\Delta\epsilon$ in eq. (12) is a measure of the change in kinetic energy of the system, it is clear that any attactive interaction will reduce the momentum deformation α_2, while a repulsive one will increase it. A shape deformed nucleus with a ground state deformation β_2 is more stable than a spherical one by the magnitude of the deformation energy E_β. In this case, the momentum deformation may be taken as

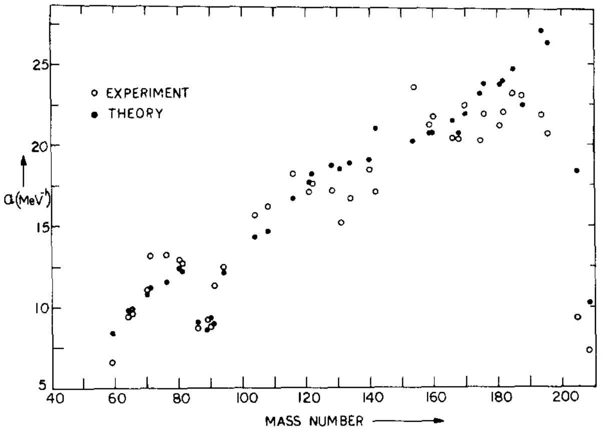

Fig. 3. The single particle statistical level density parameter $a = \frac{1}{6}\pi^2 g$ plotted as a function of the nuclear mass A. The RGM-predicted values from refs. [4] and [6] are compared with the observed slow neutron resonance data [7].

$$\alpha_2' = (\Delta\epsilon - E_\beta)/\epsilon_0 \ , \qquad (16)$$

which modifies y as a function of ϵ. The corresponding single particle level density at the Fermi surface is

$$g'(\epsilon_F) = 2g_0 \frac{(\epsilon_F - E_\beta)^2}{\epsilon_0 \cdot \epsilon_F} \cdot \frac{(1-\alpha_2/2)}{\lambda^3}$$

$$\approx 2g_0 \left(1 - \frac{2E_\beta}{\epsilon_F}\right) \frac{1 - \frac{1}{2}\alpha_2'}{\lambda^3} \qquad (17)$$

which reduces to eq. (15) when $E_\beta = 0$.

As a test of this analysis, we have calculated the statistical level density parameter $a = \frac{1}{6}\pi^2 g(\epsilon_F)$ from eqs. (15) and (17) in the mass region $60 < A < 208$ using the values of ϵ_F and E_β from the renormalized gas model (RGM) [4,6] (by setting $\Delta\epsilon = \partial\epsilon_{RGM}$). We compare these with the experimental a-parameters of Facchini and Saetta-Menichella [7] in fig. 3. We note that the overall fit is impressive. In the deformed (rare earth) region, use of (17) reduces the single particle densities to about 60% of (15); satisfactory agreement is found with this prediction.

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