ON THE MECHANICAL AND ELECTRODYNAMICAL PROPERTIES OF THE ELECTRON.

BY MEGH NAD SAHA.

THE object of the present paper is to extend Minkowski's method[1] of four-dimensional analysis to the investigation of the mechanical and electrodynamical problems connected with the electron. As is well-known, Minkowski's four-dimensional analysis is based on the principle of relativity, and we have thereby to abandon two time-honored concepts of physics, *i. e.*, absolute independence of time and space, and the constancy of mass. The correctness of these two principles is no longer a matter of hypothesis, but is founded on experiments. It is therefore to be hoped that the results of these investigations will be helpful to us for the elucidation of the mechanical and electrical problems connected with the electron, though sometimes difficulty may be encountered in putting proper interpretation on these results.

The notation is the same as that adopted by Minkowski, and for the convenience of the reader, it is explained at the very outset.

I.

$(x, y, z, l \pm ict)$ denotes the space and time coördinates of any point in the four-dimensional world

$$(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\left[\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, i\right]$$

denotes the velocity four-vector of the point.

We put

$$ds^2 = -(dx^2 + dy^2 + dz^2 + dl^2)$$

therefore we have

$$(\omega_1, \omega_2, \omega_3, \omega_4) = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dl}{ds}\right), \text{ and } \sqrt{-}\,(\omega_1, \omega_2, \omega_3, \omega_4)$$

[1] Minkowski's method of four-dimensional analysis is expounded in two papers: (1) Raum und Zeit, published in the Phys. Zeits., and (2) Die Grundgleichungen für die Electro-magnetischen Vorgänge in bewegten Körpern-Gött. Nach, 1908. These two papers have been translated by me, and are being published by the Calcutta University.

denote the direction cosines of the four-dimensional tangent to the path of the particle. c = velocity of light in space.

We put

$$(f_{23}, f_{31}, f_{12}) = (Hx, Hy, Hz),$$

the components of the magnetic field, and

$$(f_{14}, f_{24}, f_{34}) = -i[Ex, Ey, Ez],$$

the components of the electric field. Minkowski has shown that f constitutes a six-vector.

$$(a_1, a_2, a_3, a_4) = [F, G, H, l\phi],$$

are the components of the potential four-vector; (F, G, H) are the vector potentials Φ is the scalar potential.

ρ = electrical space-density;

$$\rho\left[\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, i\right] = \rho_0(\omega_1, \omega_2, \omega_3, \omega_4)$$

are the components of the stream four-vector s;

$$\rho_0 = \rho\sqrt{1 - \frac{u^2}{c^2}}$$

is known as the rest-density of electricity.

The vector operator

$$\Box = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + \frac{\partial}{\partial l}\right),$$

is known as the lor and the scalar operator

$$\Box^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial l^2}\right)$$

is known as the generalized D'alembertian.

The equations of electrodynamics can be written in the forms

$$\text{lor}\, f = s, \qquad \text{lor}\, f^* = 0,$$

$$f = \text{Curl}\, a; \qquad \Box^2 a = -s, \qquad \Box a = 0.$$

2. The Scalar and Vector Potentials of a Moving Electron.

Lienard,[1] and almost simultaneously Wiechert[2] showed that the scalar and vector potentials are given by the expressions

$$\phi = -\frac{e}{r\left(1 - \frac{ur}{c}\right)}, \qquad (F, G, H) = -\frac{e(u_1, u_2, u_3)}{cr\left(1 - \frac{ur}{c}\right)}, \tag{1}$$

[1] Lienard, L'eclairage electrique, 16 (1898), pp. 5, 53 and 106.

[2] Wiechert, Arch. Neérl., (2), 5 (1900).

If P be the point at which the potentials are calculated at the time t and M be the position of the electron at the time t_0, where $MP = c(t-t_0)$, the distance MP is denoted by r and $[u]$ denotes the velocity in the position M, and (ur) its component in the direction of r.

The formulæ are deduced from the theory of retarded potential and do not involve the principle of relativity.

Several investigators[1] have shown that the formulæ can also be deduced from the theory of relativity and can be thrown into the compact form

$$a = \frac{e[\omega]}{[R\omega]},$$

R being the four-vector joining the two points, $[R\cdot\omega]$ denoting the scalar product of R and ω.

It is quite clear that the forms (1) and (2) are quite equivalent.

In a paper published elsewhere, it has been shown that from Mimkowski's four-dimensional analysis we obtain

$$a = \frac{e[\omega]}{P}. \tag{3}$$

In this formula, (x, y, z, l) denote the time-space coördinates of the electron (A), $(\omega_1, \omega_2, \omega_3, \omega_4)$ its velocity-components, $(x'\ y'\ z'\ l')$ denote the space-time coördinates of the point B at which the potentials are estimated,

P denotes the four-dimensional perpendicular distance of B from the axis of motion of (A); since the direction-cosines of this axis are $-i\ (\omega_1, \omega_2, \omega_3, \omega_4)$, we have

$$\begin{aligned} P^2 &= (x-x')^2 + (y-y')^2 + (z-z')^2 + (l-l')^2 \\ &\quad + [(x-x')\omega_1 + (y-y')\omega_2 + (z-z')\omega_3 + (l-l')\omega_4]^2 \\ &= R^2 + [R\omega]^2. \end{aligned}$$

Now if we make the assumption that the time coördinates are so chosen that

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 + (l-l')^2 = 0$$

i. e.,

$$c^2(t-t')^2 = r^2, \tag{4}$$

i. e.,

$$c(t-t') = r,$$

the formula (3) becomes the same as (2) and therefore (1). Also the assumption which we make here about the interval between the time-coördinates is identical with the premises of Lienard and Wiechert.

[1] Sommerfeld, Über die Relativitäts-theorie, Ann. d. Physik, Vols. 32 and 33.

I am not quite certain whether this assumption (4) which is made here is at all essential. I am inclined to think that it is not essential, but necessary only for the interpretation of the result to those three-dimensional beings whose senses are not sharpened enough to enable them to grasp a result expressed in four-dimensional figures.

3. The Electric and Magnetic Fields Due to a Moving Electron.

If a denote the potential four-vector, the components of the six-vector f giving the electric and magnetic fields are given by

$$f = \text{Curl } a = \begin{vmatrix} \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'} & \frac{\partial}{\partial'} \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix}.$$

Thus

$$f_{12} = \frac{\partial a_2}{\partial x'} - \frac{\partial a_1}{\partial y'} = -\frac{e}{P^3}[\omega_3\alpha_2 - \omega_2\alpha_3], \text{ etc.} \tag{5}$$

where

$$\alpha_1 = P\frac{\partial P}{\partial x'}, \qquad \alpha_2 = P\frac{\partial P}{\partial y'} \ldots \text{ etc.}$$

we can easily verify that if we put $c(t - t') = r$, we have

$$Hx = \frac{e\beta^2}{r^3\lambda^3}\left[\frac{u_2}{c}(z - z') - \frac{u_3}{c}(y - y')\right],$$

where

$$\lambda = \left(1 - \frac{Ur}{c}\right),$$

$$= \frac{e\beta^2}{r^3\lambda^3}[\mathbf{u} \times \mathbf{r}]. \tag{6}$$

The electric forces are given by

$$f_{14} = -iEx = \frac{\partial a_4}{\partial x'} - \frac{\partial a_1}{\partial l'},$$

$$= -\frac{e}{P^3}[\omega_1(l - l') - \omega_4(x - x')],$$

$$= -i\frac{e\beta^2}{r^3\lambda^3}\left[r\frac{u_1}{c} - (x - x')\right], \tag{7}$$

and generally

$$Ex = \frac{e\beta^2}{r^3\lambda^3}\left[(x - x') - \frac{ru_1}{c}\right], \qquad E = \frac{e\beta^2}{r^3\lambda^3}\left[\mathbf{r} - \frac{r\mathbf{u}}{c}\right].$$

These values are widely different and simpler than the values obtained from the older theories, for example, compare the values given by Crehore.[1]

The discrepancy is due to the fact that in these older theories, we always assume that the equation

$$(x - x')^2 + (y - y')^2 + (z - z')^2 + (l - l')^2 = 0,$$

is an essential condition. But in performing differentiations with regard to (x', y', z', l') we here assume that they are quite independent of (x, y, z, l). I am not quite definite as to which of these two standpoints is correct but I am inclined to think that my standpoint is more in accordance with Minkowski's ideas of time and space. However it is preferable to keep an open mind on this point.

3. Maxwell's Stresses, Poynting-Vector, etc.

Minkowski has shown that if we multiply f by its own matrix, we obtain a matrix

$$ff = \begin{vmatrix} S_{11} - L^2 & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} - L^2 & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} - L^2 & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} - L^2 \end{vmatrix}$$

where

$$S_{11} = \tfrac{1}{2}[f_{23}^2 + f_{34}^2 + f_{42}^2 - f_{12}^2 - f_{13}^2 - f_{14}^2],$$

$$S_{12} = [f_{13}f_{32} + f_{14}f_{42}],$$

$$L = \tfrac{1}{2}[f_{23}^2 + f_{32}^2 + f_{12}^2 + f_{14}^2 + f_{24}^2 + f_{34}^2],$$

and the matrix

$$\frac{1}{4\pi}\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} = \begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix}$$

denote the Maxwellian stresses, $i(S_{14}, S_{24}, S_{34})$ denote the components of the Poynting vector, and S_{44} is the energy function. We have generally

$$\left.\begin{aligned} X_x &= \frac{1}{8\pi}[f_{23}^2 + f_{34}^2 + f_{42}^2 - f_{12}^2 - f_{13}^2 - f_{14}^2], \\ X_y &= \frac{1}{4\pi}[f_{13}f_{32} + f_{14}f_{42}], \end{aligned}\right\} \quad (8)$$

etc.

Now on the standpoint taken up by me, it is quite easy to calculate these qualities. It can be shown that

$$X_x = \frac{e^2}{8\pi P^2}[- P^2(1 + 2\omega_1^2) + \alpha_1^2], \quad X_y = \frac{e^2}{8\pi P^6}[- \omega_1\omega_2 P^2 + \alpha_1\alpha_2], \quad (9)$$

[1] Phys. Rev., July, 1917, p. 448.

The Poynting-vector

$$(X_\iota,\ Y_\iota,\ Z_\iota) = \frac{e^2}{8\pi P^6}[(-\ \omega_1\omega_4P^2 + \alpha_1\alpha_4),\quad (-\ \omega_2\omega_4P^2 + \alpha_2\alpha_4),\quad (-\ \omega_3\omega_4P^2 + \alpha_3\alpha_4)]$$

and the energy function

$$S_{44} = L = \frac{e^2}{8\pi P^6}[-\ P^2(1 + 2\omega_4^2) + \alpha_4^2]$$

where

$$\alpha_1 = P\frac{\partial P}{\partial x'},\quad \alpha_2 = P\frac{\partial P}{\partial y'},\quad \text{etc.}$$

and

$$\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = P^2.$$

4. The Law of Attraction between Two Moving Electrons.

We can now proceed to find out the attraction which one moving electron exerts upon another moving electron.

According to Lorentz's theorem the components of the force acting on an electron (A) moving in any electromagnetic field are

$$\begin{aligned} X &= e[\omega_2 f_{12} + \omega_3 f_{13} + \omega_4 f_{14}],\\ Y &= e[\omega_1 f_{21} + \omega_3 f_{23} + \omega_4 f_{24}],\\ Z &= e[\omega_1 f_{31} + \omega_2 f_{32} + \omega_4 f_{34}], \end{aligned} \tag{10}$$

and we can also add the fourth or the time component

$$L = -\frac{ie}{c}[Xu_1 + Yu_2 + Zu_2],\quad \beta = \sqrt{1 - \frac{v^2}{c^2}}$$

which is proportional to the rate at which work is done by the moving charge,—we have

$$L = e[\omega_1 f_{41} + \omega_2 f_{42} + \omega_3 f_{43}].$$

In this case, the field is due to the second electron (charge e', position $x'\ y'\ z'\ l'$, velocity components $\omega_1'\ \omega_2'\ \omega_3'\ \omega_4'$).

According to the last section, the potential four-vector

$$a = \frac{e'[\omega']}{P'},\quad \text{where}\quad P'^2 = R^2 + [R\omega']^2.$$

We have now, since $f = \text{Curl } a$,

$$\begin{aligned} X = ee'\Bigg[& \omega_2\left\{\frac{\partial}{\partial x}\left(\frac{\omega_2'}{P'}\right) - \frac{\partial}{\partial y}\left(\frac{\omega_1'}{P}\right)\right\} + \omega_3\left\{\frac{\partial}{\partial x}\left(\frac{\omega_3'}{P'}\right) - \frac{\partial}{\partial z}\left(\frac{\omega_1'}{P'}\right)\right\} \\ & + \omega_4\left\{\frac{\partial}{\partial x}\left(\frac{\omega_4'}{P'}\right)\frac{\partial}{\partial l}\left(\frac{\omega_1'}{P'}\right)\right\} \\ = ee'\Bigg[& \frac{\partial}{\partial x}\left(\frac{\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4'}{P'}\right) \\ & - \left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z} + \omega_4\frac{\partial}{\partial l}\right)\left(\frac{\omega_1'}{P'}\right)\Bigg], \end{aligned}$$

Now putting

$$\Phi = ee'(\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4')/P',$$

we find that

$$X = \frac{\partial\Phi}{\partial x} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dx}{ds}}\right),$$

where $\partial/\partial x$ denotes differentiation in which x is explicitly involved, similarly with

$$\frac{\partial}{\partial\omega_1} = \frac{\partial}{\partial\frac{dx}{ds}}.$$

$$\frac{d}{ds} = \left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z} + \omega_4\frac{\partial}{\partial l}\right),$$

as is easily seen. We have similarly

We have similarly

$$Y = \frac{\partial\Phi}{\partial y} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dy}{ds}}\right), \qquad Z = \frac{\partial\Phi}{dz} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dz}{ds}}\right),$$

$$L = \frac{\partial\Phi}{\partial l} - \frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dl}{ds}}\right). \tag{11}$$

We can say that Φ is the kinetic-potential of the electron (A) in the field of the electron (B). Similarly if Φ' denote the Kinetic-Potential of the electron (B) in the field of (A),

$$\Phi' = ee'(\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4')/P,$$

$$P^2 = R^2 + (R\omega)^2.$$

and we have similarly

$$X' = \frac{\partial\Phi'}{\partial x'} - \frac{d}{ds'}\left(\frac{\partial\Phi'}{\partial\frac{dx'}{ds'}}\right). \tag{12}$$

Let us now interpret the results in three dimensions. We have

$$X = \frac{ee'\beta'^2}{r^3\lambda^3\beta'}(x - x')\left(1 - \frac{uv\cos\theta}{c^2}\right) - \frac{ee'\beta'^2}{r^2\lambda^3\beta c}\left(1 - \frac{Ur}{c}\right)u_1', \tag{13}$$

where

$$\beta = \sqrt{1 - \frac{v^2}{c^2}}, \qquad \beta' = \sqrt{1 - \frac{v'^2}{c^2}}, \qquad \lambda = \left(1 - \frac{Ur'}{c}\right).$$

In three dimensions, the forces are equivalent to a force of repulsion

$$\frac{ee'\beta'^2}{r^3\lambda^3\beta}\left(1 - \frac{uv\cos\theta}{c^2}\right)\mathbf{r},$$

in the direction of the line joining the two points, and a force

$$\frac{ee'\beta'^2}{r^2\lambda^3\beta c}\left(1 - \frac{Ur}{c}\right)\mathbf{u}', \tag{14}$$

in the direction of the velocity of the second or the attracting point.

We thus perceive that the force which comes out in a very simple form in four dimensions takes a very complicated form in three dimensions.

The kinetic potential

$$\phi = \frac{ee'\left(1 - \frac{vv'\cos\theta}{c^2}\right)}{r\left(1 - \frac{U'r}{c}\right)\beta}. \tag{15}$$

This kinetic potential is practically coincident with the kinetic potential assumed by Clausius in order to find out the law of attraction between two moving charges of electricity; Clausius has shown that this kinetic potential leads us to the celebrated electrodynamic laws of Ampere. A short résumé of the work done in this connection is given below for the purpose of comparison. The problem was first enunciated by Gauss in the year 1835, and was called by him the fundamental keystone of electrodynamics.[1]

1. Gauss (1835): The forces are the derivatives with regard to (x, y, z) of the potential function,

$$\phi = \frac{ee'}{r}\left(1 - \frac{3}{2c}\frac{d^2r}{dt^2}\right).$$

2. Weber (1843) takes the potential function

$$\phi = \frac{ee'}{r}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt}\right).$$

Both of these forms have been long discredited. Later writers have pointed out that the force cannot be simply the derivations with regard to (x, y, z) of some potential function, but are the Lagrangian derivatives of a certain kinetic-potential. We give the form of this kinetic potential according to different investigators.

[1] For the literature on the subject, see Maxwell, Electricity and Magnetism, Vol. 2, Chap. XXIII., and J. J. Thomson, Application of Dynamics to Problems of Physics and Chemistry, pp. 35.

1. Clausius (1881);

$$\phi = \frac{ee'}{r}\left(1 - \frac{uu'\cos\theta}{c^2}\right),$$

where v and u' are the velocities of the two electrons, and θ is the angle between their lines of motion. In two papers communicated to the Crelle's journal,[1] Clausius deduces Ampere's laws of electrodynamical action between two currents from this law.

2. J. J. Thomson

$$\phi = \frac{ee'}{r}\left(1 - \frac{\mu}{3c^2}uu'\cos\theta\right) \qquad (\mu = \text{magnetic permeability} = 1).$$

Crehore[2] has calculated the forces components according to J. J. Thomson's theory.[3] He finds that the forces are equivalent to

$F_1 = \frac{ee'}{r^2}$, a repulsion along the line joining the centers.

$F_2 = \frac{ee'}{c^2r^2}uu'\cos\theta$, an attraction along the same line.

$F_3 = \frac{ee'}{r}\dot{u}'$, a force in the direction opposite to the acceleration of the second charge.

$F_4 = ee'u'\frac{d}{dt}\left(\frac{1}{r}\right)$, a force in a direction opposite to the velocity of the second charge.

3. Sommerfeld[4] has also calculated the ponderomotive forces, assuming that the value of the potential four-vector

$$a = \frac{e[\omega]}{(R\omega)},$$

and using the condition

$$(x - x')^2 + (y - y')^2 + (z - z')^2 + (l - l')^2 = 0$$

in course of differentiation. Their forms are a bit too complicated.

5. Equations of Motion of the Electron.

Minkowski[5] deduces the equations of motions of a ponderable particle by means of a variational process in which the function

$$\int m_0c^2ds, \text{ where } ds^2 = -(dx^2 + dy^2 + dz^2 + dl^2) = c^2dt^2\left(1 - \frac{u^2}{c^2}\right)$$

is used instead of the three-dimensional $\int T dt$.

[1] Vols. 82 and 83.

[2] It will be seen that forces F_1, F_2, F_4 are, but for some minor details, represented in our formula. Force F_3 does not occur at all.

[3] Phil. Mag., 1913.

[4] Ann. d. Phys., vols. 32 and 33, Über die Relativitäts theories.

[5] Minkowski, *loc. cit.* Anhaup, Mechanics.

He obtains

$$m_0c^2\frac{d^2x}{ds^2} = X, \qquad m_0c^2\frac{d^2y}{ds^2} = Y, \qquad m_0c^2\frac{d^2z}{ds^2} = Z, \qquad m_0c^2\frac{d^2l}{ds^2} = L. \tag{16}$$

Now we have

$$X = e[\omega_2 f_{12} + \omega_3 f_{13} + \omega_4 f_{14}],$$

according to Lorentz's theorem. We have also

$$\begin{aligned}\frac{d^2x}{ds^2} &= \left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z} + \omega_4\frac{\partial}{\partial l}\right)\omega_1 \\ &= \omega_1\frac{\partial}{\partial x}[\tfrac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2)] + \omega_2\Bigg[\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right) \\ &\qquad + \omega_3\left(\frac{\partial\omega_1}{\partial z} - \frac{\partial\omega_3}{\partial x}\right) + \omega_4\left(\frac{\partial\omega_1}{\partial l} - \frac{\partial\omega_4}{\partial x}\right)\Bigg] \\ &= -(\omega_2\Omega_{12} + \omega_3\Omega_{13} + \omega_4\Omega_{14}), \text{ putting } \Omega_{hk} = \frac{\partial\omega k}{\partial xh} - \frac{\partial\omega h}{\partial xk}.\end{aligned}$$

Hence we have the four equations, putting $\mu = c^2m_0/e$,

$$\left.\begin{array}{l} \omega_2(f_{12} + \mu\Omega_{12}) + \omega_3(f_{13} + \mu\Omega_{13}) + \omega_4(f_{14} + \mu\Omega_{14}) = 0 \\ \omega_1(f_{21} + \mu\Omega_{21}) + \omega_3(f_{23} + \mu\Omega_{23}) + \omega_4(f_{24} + \mu\Omega_{24}) = 0 \\ \omega_1(f_{31} + \mu\Omega_{31}) + \omega_2(f_{32} + \mu\Omega_{32}) + \omega_4(f_{34} + \mu\Omega_{34}) = 0 \\ \omega_1(f_{41} + \mu\Omega_{41}) + \omega_2(f_{42} + \mu\Omega_{42}) + \omega_3(f_{43} + \mu\Omega_{43}) = 0 \end{array}\right\}.$$

Of these, only three are independent; the fourth can be deduced from the first three.

We have now identically

$$\begin{vmatrix} & f_{12} + \mu\Omega_{12} & f_{13} + \mu\Omega_{13} & f_{14} + \mu\Omega_{14} \\ f_{21} + \mu\Omega_{21} & & f_{23} + \mu\Omega_{23} & f_{24} + \mu\Omega_{24} \\ f_{31} + \mu\Omega_{31} & f_{32} + \mu\Omega_{32} & & f_{34} + \mu\Omega_{34} \\ f_{41} + \mu\Omega_{41} & f_{42} + \mu\Omega_{42} & f_{43} + \mu\Omega_{43} & \end{vmatrix} = 0,$$

i. e.,

$$\begin{aligned}(f_{12} + \mu\Omega_{12})(f_{34} + \mu\Omega_{34}) + (f_{23} + \mu\Omega_{23})(f_{14} + \mu\Omega_{14}) \\ + (f_{31} + \mu\Omega_{31})(f_{24} + \mu\Omega_{24}) = 0.\end{aligned} \tag{17}$$

The condition is evidently satisfied if we put

$$-\mu = \frac{f_{12}}{\Omega_{12}} = \frac{f_{23}}{\Omega_{23}} = \frac{f_{31}}{\Omega_{31}} = \frac{f_{14}}{\Omega_{14}} = \frac{f_{24}}{\Omega_{24}} = \frac{f_{34}}{\Omega_{34}}. \tag{18}$$

If of these equations, any three are satisfied the remaining three come out automatically from the equations of motion. But we cannot possibly be sure of the authenticity of these relations unless it can be deduced from an independent source. For this purpose let us take the original variational equations.

Let (X, Y, Z, L) represent the components of the force four-vector at any point, which is subjected to a virtual displacement δx, δy, δz, δl.

Then

$$\delta W = X\delta x + Y\delta y + Z\delta z + L\delta l,$$

i. e., if we call

$$W = \frac{\partial A}{\partial s}, \qquad A = \int W ds,$$

$$\delta A = \int \delta W ds = \int (X\delta x + Y\delta y + Z\delta z + L\delta l) ds$$
$$= e\int [f_{12}(dy\delta x - \delta y dz) + f_{23}(dz\delta y - \delta z dy) + f_{31}(dx\delta z - \delta x dz)$$
$$+ f_{14}(dl\delta x - \delta l dx) + f_{24}(dl\delta y - \delta l dy) + f_{34}(dl\delta z - \delta z dl)].$$

Now the function $\int m_0 c^2 ds$ can also be subjected to a variational process. Since

$$ds = \omega_1 dx + \omega_2 dy + \omega_3 dz + \omega_4 dl,$$

we find

$$\delta \int m_0 c^2 ds = - m_0 c^2 \int [\Omega_{12}\delta S_{12} + \Omega_{23}\delta S_{23} + \Omega_{31}\delta S_{31}$$
$$+ \Omega_{14}\delta S_{14} + \Omega_{24}\delta S_{24} + \Omega_{34}\delta S_{34}],$$

where

$$\delta S_{hk} = dx_k \delta x_h - \delta x_k dx_h.$$

Thus from the variational equation

$$\delta \int m_0 c^2 ds + \int \delta W . ds = 0;$$

i. e., from the principle of least action, keeping the initial and final points fixed, we obtain the original equation

$$\int [(f_{12} + \mu\Omega_{12})\delta S_{12} + \dots] = 0.$$

The relations (18) thus seem to be borne out by independent evidence.

Difficulty is encountered here about the interpretation of the terms $(\Omega_{12}, \Omega_{23}, \dots)$ in three dimensions Ω is evidently a six-vector being the four-dimensional curl of the velocity four-vector. The components $[\Omega_{23}, \Omega_{31}, \Omega_{12}]$ are evidently connected with rotations

$$\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \quad \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \quad \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right]$$

and $[\Omega_{14}, \Omega_{24}, \Omega_{34}]$ are connected with the accelerations

$$\left[\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2}\right]$$

but the exact interpretation in three dimensions has not yet been obtained. We can style Ω as the acceleration six-vector.

On a future occasion, I hope to communicate the result of my investigations on the orbits of the electron under different conditions.

In conclusion, I wish to express my best thanks to my friend, Mr. S. N. Basu, for much help, and useful criticism.

UNIVERSITY COLLEGE OF SCIENCE,
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