

## On some classes of 3-dimensional generalized $(\kappa, \mu)$ -contact metric manifolds

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**Abstract:** The object of the present paper is to obtain a necessary and sufficient condition for a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold to be locally  $\phi$ -symmetric in the sense of Takahashi and the condition is verified by an example. Next we characterize a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold satisfying certain curvature conditions on the concircular curvature tensor. Finally, we construct an example of a generalized  $(\kappa, \mu)$ -contact metric manifold to verify Theorem 1 of our paper.

**Key words:** Generalized  $(\kappa, \mu)$ -contact metric manifolds, concircular curvature tensor,  $\xi$ -concircularly flat, locally  $\phi$ -concircularly symmetric

### 1. Introduction

In a 3-dimensional Riemannian manifold the curvature tensor has a special form. These 3-dimensional Riemannian manifolds have certain characteristic properties that are different from that of a manifold of dimension greater than 3. Due to the interesting properties of 3-dimensional Riemannian manifolds, several authors have studied such manifolds. It is well known that a Sasakian manifold is a normal contact metric manifold  $(M, \phi, \xi, \eta, g)$  whose curvature tensor  $R$  satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for any  $X, Y \in \chi(M)$ , where  $\chi(M)$  denotes the Lie algebra of all smooth vector fields on  $M$ . As a generalization of the Sasakian manifold, Blair et al. [7] introduced the notion of a contact metric manifold called a  $(\kappa, \mu)$ -contact metric manifold satisfying the condition

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \quad (1.1)$$

for any  $X, Y \in \chi(M)$ , where  $\kappa$  and  $\mu$  are constants on  $M$  and  $h = \frac{1}{2}L_{\xi}\phi$  (here  $L_{\xi}$  is the Lie derivative in the direction of  $\xi$ ).  $(\kappa, \mu)$ -contact metric manifolds have been studied by several authors [1, 7, 9, 11, 12, 16, 20, 15].

In 2000, Koufogiorgos and Tsihlias [13] generalized the notion of a  $(\kappa, \mu)$ -contact metric manifold by taking the constants  $\kappa$  and  $\mu$  in (1.1) to be smooth functions on  $M$ , called a generalized  $(\kappa, \mu)$ -contact metric manifold. Furthermore, the same authors [14] studied 3-dimensional generalized  $(\kappa, \mu)$ -contact metric

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manifolds with  $\xi\mu = 0$ . Three-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds were also studied by De and Ghosh [10] and De and Sarkar [17]. In a recent paper [18], Shaikh et al. proved that a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold is locally  $\phi$ -symmetric provided  $\kappa$  and  $\mu$  are constants. In a recent paper [6], Blair et al. studied concircular curvature tensor in  $N(k)$ -contact metric manifolds. In 2005, Tripathi and Kim [20] studied the concircular curvature tensor of a  $(\kappa, \mu)$ -contact metric manifold.

The present paper is organized as follows: after preliminaries in Section 3, we improve the result of paper [18] and prove that a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold is locally  $\phi$ -symmetric if and only if  $\kappa$  and  $\mu$  are constants. In Section 4, we prove that a  $\xi$ -concircularly flat 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold is either flat or the manifold reduces to a  $(\kappa, \mu)$ -contact metric manifold. In the next section, it is shown that a locally  $\phi$ -concircularly symmetric 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold is a  $(\kappa, \mu)$ -contact metric manifold. In Section 6, we study concircularly semisymmetric 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds. In the next section, we consider the curvature condition  $\tilde{C} \cdot S = 0$  in a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold, where  $\tilde{C}$  is a concircular curvature tensor and  $S$  is the Ricci tensor. This result generalizes the results of Tripathi and Kim [20]. Finally, we give an example of a generalized  $(\kappa, \mu)$ -contact metric manifold to verify the Theorem of Section 3.

## 2. Preliminaries

In this section, we present some basic facts about contact metric manifolds. We refer the reader to [4] for a more detailed treatment. A differentiable manifold  $M$  of dimension  $2n + 1$  is called a contact manifold if it carries a global 1-form  $\eta$  such that  $\eta \wedge (d\eta)^{2n+1} \neq 0$  everywhere on  $M$ . The form  $\eta$  is usually called the contact form of  $M$ . It is well known that a contact metric manifold admits an almost contact metric structure  $(\phi, \xi, \eta, g)$ , i.e. a global vector field  $\xi$ , which is called the characteristic vector field, a  $(1, 1)$ -tensor field  $\phi$ , and a Riemannian metric  $g$  such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields  $X, Y \in \chi(M)$ . Moreover,  $(\phi, \xi, \eta, g)$  can be chosen such that

$$d\eta(X, Y) = g(X, \phi Y), \quad X, Y \in \chi(M),$$

and we then call the structure a contact metric structure. A manifold  $M$  carrying such a structure is said to be a contact metric manifold and it is denoted by  $(M, \phi, \xi, \eta, g)$ . As a consequence of the above relations, we have  $\eta(\xi) = 1$ ,  $\phi\xi = 0$ ,  $\eta \circ \phi = 0$ , and  $d\eta(\xi, X) = 0$ . If  $\nabla$  denotes the Riemannian connection of  $(M, \phi, \xi, \eta, g)$ , then following [2], we define the  $(1, 1)$ -tensor fields  $h$  and  $l$  by  $h = \frac{1}{2}(L_\xi\phi)$  and  $l = R(\cdot, \xi)\xi$ , where  $L_\xi$  is the Lie differentiation in the direction of  $\xi$  and  $R$  is the curvature tensor, which is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z,$$

for all vector fields  $X, Y, Z \in \chi(M)$ . The tensor fields  $h, l$  are self-adjoint and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad trh = tr\phi h = 0, \quad \phi h + h\phi = 0.$$

Since  $h$  anticommutes with  $\phi$ , if  $X \neq 0$  is an eigenvector of  $h$  corresponding to the eigenvalue  $\lambda$ , then  $\phi X$  is also an eigenvector of  $h$  corresponding to the eigenvalue  $-\lambda$ . Therefore on any contact metric manifold

$(M, \phi, \xi, \eta, g)$ , the following formulas are valid:

$$\begin{aligned} \nabla \xi &= -\phi - \phi h \quad (\text{and so } \nabla_{\xi} \xi = 0), \\ \nabla_{\xi} h &= \phi - \phi l - \phi h^2, \\ \phi l \phi - l &= 2(\phi^2 + h^2). \end{aligned}$$

A contact metric structure  $(\phi, \xi, \eta, g)$  on  $M$  gives rise to an almost complex structure on the product  $M \times \mathfrak{R}$ . If this structure is integrable, then the contact metric manifold  $(M, \phi, \xi, \eta, g)$  is said to be Sasakian. Equivalently, a contact metric manifold  $(M, \phi, \xi, \eta, g)$  is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all  $X, Y \in \chi(M)$ .

By a generalized  $(\kappa, \mu)$ -manifold, we mean a 3-dimensional contact metric manifold such that

$$R(X, Y)\xi = \{\kappa I + \mu h\}[\eta(Y)X - \eta(X)Y], \tag{2.1}$$

for all  $X, Y \in \chi(M)$ , where  $\kappa, \mu$  are smooth nonconstant real functions on  $M$ . In the special case where  $\kappa, \mu$  are constant, then  $(M, \phi, \xi, \eta, g)$  is called a  $(\kappa, \mu)$ -manifold. We note that in any Sasakian manifold  $h = 0$  and  $\kappa = 1$ .

**Lemma 1** [13, 14] *In any generalized  $(\kappa, \mu)$ -manifold  $(M, \phi, \xi, \eta, g)$  the following formulas are valid :*

$$\begin{aligned} h^2 &= (\kappa - 1)\phi^2, \quad \kappa = \frac{tr l}{2} \leq 1, \\ \xi \kappa &= 0, \quad \xi r = 0, \\ hgrad \mu &= grad \kappa. \end{aligned}$$

**Lemma 2** [3] *A  $(2n + 1)$ -dimensional contact metric manifold satisfying  $R(X, Y)\xi = 0$  is locally isometric to  $E^{n+1}(0) \times S^n(4)$  for  $n > 1$  and flat if  $n = 1$ .*

**Lemma 3** [13] *Let  $M$  be a non-Sasakian, generalized  $(\kappa, \mu)$ -contact metric manifold. If  $\kappa, \mu$  satisfy the condition  $a\kappa + b\mu = c$  ( $a, b, c$  are constants), then  $\kappa, \mu$  are constants.*

**Lemma 4** [5] *A 3-dimensional contact metric manifold  $(M^3, \phi, \xi, \eta, g)$  with  $Q\phi = \phi Q$  is either Sasakian, flat, or of constant  $\xi$ -sectional curvature  $\kappa < 1$  and constant  $\phi$ -sectional curvature  $-\kappa$ .*

Furthermore, in a generalized  $(\kappa, \mu)$ -contact metric manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$ , we have the following [2]:

$$\begin{aligned} (\nabla_X h)Y &= \{(1 - \kappa)g(X, \phi Y) - g(X, \phi h Y)\}\xi - \eta(Y)\{(1 - \kappa)\phi X + \phi h X\} \\ &\quad - \mu \eta(X)\phi h Y, \end{aligned} \tag{2.2}$$

$$(\nabla_X \phi)Y = \{g(X, Y) + g(X, h Y)\}\xi - \eta(Y)(X + h X). \tag{2.3}$$

For a 3-dimensional Riemannian manifold, the conformal curvature tensor  $C = 0$ . Thus, the curvature tensor  $R$  and Ricci tensor  $S$  for a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold are given by [18]:

$$\begin{aligned}
 R(X, Y)Z &= -(\kappa + \mu)\{g(Y, Z)X - g(X, Z)Y\} \\
 &+ (2\kappa + \mu)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\
 &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\
 &+ \mu\{g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y\},
 \end{aligned}
 \tag{2.4}$$

$$S(X, Y) = -\mu g(X, Y) + \mu g(hX, Y) + (2\kappa + \mu)\eta(X)\eta(Y),
 \tag{2.5}$$

respectively. We see that on a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold the scalar curvature  $r$  is equal to

$$r = 2(\kappa - \mu).
 \tag{2.6}$$

In an  $n$ -dimensional Riemannian manifold, the concircular curvature tensor  $\tilde{C}$  is defined by [21, 22]

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}.
 \tag{2.7}$$

We observe from (2.7) that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature is a measure of the failure of a Riemannian manifold to be of constant curvature.

Hence, using (2.4) and (2.6) in (2.7), we get

$$\begin{aligned}
 \tilde{C}(X, Y)Z &= -\left(\frac{4\kappa + 2\mu}{3}\right)\{g(Y, Z)X - g(X, Z)Y\} + (2\kappa + \mu)\{g(Y, Z)\eta(X)\xi \\
 &- g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\
 &+ \mu\{g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y\}.
 \end{aligned}
 \tag{2.8}$$

Using (2.8), we can give the following formulas:

$$\tilde{C}(X, Y)\xi = \frac{2\kappa + \mu}{3}\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},
 \tag{2.9}$$

$$\tilde{C}(X, \xi)Z = \frac{2\kappa + \mu}{3}\{\eta(Z)X - g(X, Z)\xi\} + \mu\{\eta(Z)hX - g(hX, Z)\xi\},
 \tag{2.10}$$

$$\tilde{C}(X, \xi)\xi = \frac{2\kappa + \mu}{3}\{X - \eta(X)\xi\} + \mu hX,
 \tag{2.11}$$

$$S(X, \xi) = 2\kappa\eta(X).
 \tag{2.12}$$

### 3. Locally $\phi$ -symmetric 3-dimensional generalized $(\kappa, \mu)$ -contact metric manifolds

**Definition 1** A contact metric manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  is said to be locally  $\phi$ -symmetric in the sense of Takahashi [19] if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ .

In a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold, if we take  $X, Y, Z, W$  to be horizontal vector fields, that is, if  $X, Y, Z, W$  are orthogonal to  $\xi$ , then we obtain [18]

$$\begin{aligned} \phi^2(\nabla_W R)(X, Y)Z &= (W\kappa + W\mu)\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad - (W\mu)\{g(Y, Z)hX - g(X, Z)hY \\ &\quad + g(hY, Z)X - g(hX, Z)Y\}. \end{aligned} \tag{3.1}$$

Suppose that  $\kappa$  and  $\mu$  are constants; then a 3-dimensional generalied  $(\kappa, \mu)$ -contact metric manifold is locally  $\phi$ -symmetric in the sense of Takahashi. Conversely, let the manifold under consideration be locally  $\phi$ -symmetric. Then, from (3.1), we obtain

$$\begin{aligned} &(W\kappa + W\mu)\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\} \\ &- (W\mu)\{g(Y, Z)g(hX, U) - g(X, Z)g(hY, U) + g(hY, Z)g(X, U) \\ &- g(hX, Z)g(Y, U)\} = 0. \end{aligned} \tag{3.2}$$

Contracting  $X$  and  $Z$  in (3.2), we obtain

$$2n(W\kappa + W\mu)g(Y, U) + (2n - 1)(W\mu)g(Y, hU) = 0, \tag{3.3}$$

which implies

$$2n(W\kappa + W\mu)Y + (2n - 1)(W\mu)hY = 0. \tag{3.4}$$

Applying  $h$  in (3.4) and taking the trace of  $h$  we get

$$(W\mu) = 0, \tag{3.5}$$

since trace  $h^2 \neq 0$ .

That is,  $\mu = \text{constant}$ . Using  $\mu = \text{constant}$  in (3.4), we get  $\kappa = \text{constant}$ . Thus, we can state the following:

**Theorem 1** *A 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold is locally  $\phi$ -symmetric if and only if  $\kappa$  and  $\mu$  are constants.*

Now we obtain the following:

**Corollary 1** *A 3-dimensional  $(\kappa, \mu)$ -contact metric manifold is always locally  $\phi$ -symmetric.*

#### 4. $\xi$ -conircularly flat 3-dimensional $(\kappa, \mu)$ -contact metric manifolds

Let  $M$  be an almost contact metric manifold equipped with an almost contact metric structure  $(\phi, \xi, \eta, g)$ . At each point  $p \in M$ , decompose the tangent space  $T_pM$  into direct sum  $T_pM = \phi(T_pM) \oplus \{\xi_p\}$ , where  $\{\xi_p\}$  is the 1-dimensional linear subspace of  $T_pM$  generated by  $\{\xi_p\}$ . Thus, the conformal curvature tensor  $C$  is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M.$$

It may be natural to consider the following particular cases:

1.  $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow L(\xi_p)$ , i.e. the projection of the image of  $C$  in  $\phi(T_pM)$  is zero.
2.  $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \phi(T_p(M))$ , i.e. the projection of the image of  $C$  in  $L\xi_p$  is zero. This condition is equivalent to

$$C(X, Y)\xi = 0. \tag{4.1}$$

3.  $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \longrightarrow L(\xi_p)$ , i.e. when  $C$  is restricted to  $\phi(T_pM) \times \phi(T_pM) \times \phi(T_pM)$ , the projection of the image of  $C$  in  $\phi(T_p(M))$  is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{4.2}$$

A Riemannian manifold satisfying (4.1) is called  $\xi$ -conformally flat. The Riemannian manifold satisfying cases (4.1) and (4.2) was considered in [23, 24, 8], respectively.

Analogous to the consideration of a conformal curvature tensor, here we can define the following:

**Definition 2** A  $(2n + 1)$ -dimensional contact metric manifold is said to be  $\xi$ -concircularly flat if

$$\tilde{C}(X, Y)\xi = 0. \tag{4.3}$$

In this section we prove the following:

**Theorem 2** A  $\xi$ -concircularly flat 3-dimensional non-Sasakian generalized  $(\kappa, \mu)$ -contact metric manifold is either flat or reduces to a  $(\kappa, \mu)$  contact metric manifold.

**Proof** For a  $\xi$ -concircularly flat manifold from (2.9), we get

$$\frac{2\kappa + \mu}{3} \{ \eta(Y)X - \eta(X)Y \} + \mu \{ \eta(Y)hX - \eta(X)hY \} = 0. \tag{4.4}$$

If we take  $X = \xi$  in (4.4), we obtain

$$\frac{2\kappa + \mu}{3} \{ \eta(Y)\xi - Y \} - \mu hY = 0. \tag{4.5}$$

Taking  $Y = hY$  in (4.5), we have

$$-\frac{2\kappa + \mu}{3} hY - \mu h^2 Y = 0. \tag{4.6}$$

Taking the trace of  $h$  we get  $\mu = 0$ . Using  $\mu = 0$  in (4.4) we get  $\kappa = 0$ . Thus,  $R(X, Y)\xi = 0$ , and hence, from Lemma 2, we get that the manifold is flat. Taking the inner product with  $U$  in the equation (4.4) and contracting  $X$  and  $U$ , we obtain  $2\kappa + \mu = 0$ . Applying Lemma 3, we get  $k, \mu = \text{constant}$ . Thus, the manifold reduces to a  $(k, \mu)$ -contact metric manifold.  $\square$

**5. Locally  $\phi$ -concircularly symmetric 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds**

In this section, first we give the following:

**Definition 3** A contact metric manifold  $M$  is said to be locally  $\phi$ -concircularly symmetric in the sense of Takahashi [19] if it satisfies

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = 0. \tag{5.1}$$

Now we investigate locally  $\phi$ -concircularly symmetric 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds. First we can write the following:

$$\begin{aligned} (\nabla_W \tilde{C})(X, Y)Z &= \nabla_W \tilde{C}(X, Y)Z - \tilde{C}(\nabla_W X, Y)Z \\ &\quad - \tilde{C}(X, \nabla_W Y)Z - \tilde{C}(X, Y)\nabla_W Z. \end{aligned} \tag{5.2}$$

Using (2.8) in (5.2), we obtain

$$\begin{aligned} (\nabla_W \tilde{C})(X, Y)Z &= -\frac{1}{3}((W\kappa) + (W\mu))\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + (2(W\kappa) + (W\mu))\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad + (W\mu)\{g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y\} \\ &\quad + (2\kappa + \mu)\{g(Y, Z)g(X, \nabla_W \xi)\xi + g(Y, Z)\eta(X)\nabla_W \xi \\ &\quad - g(X, Z)g(Y, \nabla_W \xi)\xi - g(X, Z)\eta(X)\xi + g(Y, \nabla_W \xi)\eta(Z)X \\ &\quad + \eta(Y)g(Z, \nabla_W \xi)X - g(X, \nabla_W \xi)\eta(Z)Y - \eta(X)g(Z, \nabla_W \xi)Y\} \\ &\quad + \mu\{g(Y, Z)(\nabla_W h)X + g((\nabla_W h)Y, Z)X \\ &\quad - g((\nabla_W h)X, Z)Y - g(X, Z)(\nabla_W h)Y\}. \end{aligned} \tag{5.3}$$

Applying  $\phi^2$  to both sides of (5.3) and taking  $X, Y, Z,$  and  $W$  as horizontal vector fields, we get

$$\begin{aligned} &\frac{1}{3}(4(W\kappa) + 2(W\mu))\{g(Y, Z)X - g(X, Z)Y\} \\ &+ (W\mu)\{g(X, Z)hY - g(Y, Z)hX - g(hY, Z)X + g(hX, Z)Y\} = 0. \end{aligned} \tag{5.4}$$

Applying  $h$  to (5.4) and taking the trace of  $h$ , we have

$$(W\mu)\{g(X, Z)traceh^2Y - g(Y, Z)traceh^2X\} = 0. \tag{5.5}$$

Contracting  $X$  and  $Z$  in (5.5), we obtain

$$2n(W\mu)traceh^2Y = 0. \tag{5.6}$$

Since  $traceh^2 \neq 0$ , from (5.6) it follows that  $\mu = \text{constant}$ . Using  $\mu = \text{constant}$  in (5.4) yields  $\kappa = \text{constant}$ . Hence, the manifold reduces to a  $(\kappa, \mu)$ -contact metric manifold. Thus, we can give the following:

**Theorem 3** A locally  $\phi$ -concircularly symmetric 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold reduces to a  $(\kappa, \mu)$ -contact metric manifold.

**6. Three-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds satisfying the condition  $R \cdot \tilde{C} = 0$**

Let  $M$  be a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold satisfying the condition  $R \cdot \tilde{C} = 0$ . Then we can write

$$\begin{aligned} (R(X, Y) \cdot \tilde{C})(U, V)W &= R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W \\ &\quad - \tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W = 0. \end{aligned} \tag{6.1}$$

If we take  $X = V = \xi$  in (6.1), we get

$$\begin{aligned} (R(\xi, Y) \cdot \tilde{C})(U, \xi)W &= R(\xi, Y)\tilde{C}(U, \xi)W - \tilde{C}(R(\xi, Y)U, \xi)W \\ &\quad - \tilde{C}(U, R(\xi, \xi)V)W - \tilde{C}(U, \xi)R(\xi, Y)W = 0. \end{aligned} \tag{6.2}$$

Using (2.4) in (6.2), we have

$$\begin{aligned} &\kappa g(Y, \tilde{C}(U, \xi)W)\xi - \kappa \eta(\tilde{C}(U, \xi)W)Y + \mu g(hY, \tilde{C}(U, \xi)W)\xi \\ &\quad - \mu \eta(\tilde{C}(U, \xi)W)hY - \kappa g(Y, U)\tilde{C}(\xi, \xi)W + \kappa \eta(U)\tilde{C}(Y, \xi)W \\ &\quad - \mu g(hY, U)\tilde{C}(\xi, \xi)W + \mu \eta(U)\tilde{C}(hY, \xi)W - \kappa \eta(Y)\tilde{C}(U, \xi)W \\ &\quad + \kappa \tilde{C}(U, Y)W + \mu \tilde{C}(U, hY)W = 0. \end{aligned} \tag{6.3}$$

Using (2.10) in (6.3), we write

$$\begin{aligned} &\kappa \left(\frac{2\kappa + \mu}{3}\right) \{ \eta(W)g(Y, U)\xi - g(U, W)\eta(Y)\xi - g(Y, W)\eta(U)\xi \\ &\quad - \eta(Y)\eta(W)U - \eta(Y)g(U, W)\xi \} \\ &\quad + \kappa \mu \{ \eta(W)g(hY, U)\xi - \eta(U)g(hY, W)\xi + \eta(U)\eta(W)hY \\ &\quad - \eta(U)g(hY, W)\xi - \eta(Y)\eta(W)hU + g(Y, W)hU - g(U, W)hY + g(hY, W)U \} \\ &\quad - \mu \left(\frac{4\kappa + 2\mu}{3}\right) \{ g(hY, W)U - g(U, W)hY \} + \mu^2 \{ \eta(W)g(hY, hU)\xi \\ &\quad + \eta(U)\eta(W)h^2Y - \eta(U)g(h^2Y, W)\xi + g(hY, W)hU \\ &\quad - g(U, W)h^2Y - g(h^2Y, W)U \} \\ &\quad - \kappa \left(\frac{4\kappa + 2\mu}{3}\right) \{ g(Y, W)U - g(U, W)Y \} + \kappa(2\kappa + \mu) \{ g(Y, W)\eta(U)\xi \\ &\quad - g(U, W)\eta(Y)\xi + \eta(Y)\eta(W)U - \eta(U)\eta(W)hY \} \\ &\quad + \mu \left(\frac{2\kappa + \mu}{3}\right) \{ \eta(W)g(hY, U)\xi - g(hY, W)\xi \} \\ &\quad + \mu(2\kappa + \mu) \{ g(hY, W)\eta(U)\xi - \eta(U)\eta(W)hY \} = 0 \end{aligned} \tag{6.4}$$

Taking the inner product with  $T$  and contracting  $Y$  and  $T$  in (6.4), we obtain

$$\begin{aligned} &\left\{ 5\kappa \frac{2\kappa + \mu}{3} + \mu^2 g(h^2e_i, e_i) \right\} (\eta(U)\eta(W) - g(W, U)) \\ &\quad + \left\{ \kappa \mu - \mu \left(\frac{4\kappa + 2\mu}{3}\right) \right\} g(hW, U) = 0. \end{aligned} \tag{6.5}$$



Instead of (6.5), we can write

$$\begin{aligned} & \left\{5\kappa\frac{2\kappa+\mu}{3} + \mu^2g(h^2e_i, e_i)\right\}(\eta(U)\xi - U) \\ & + \left\{\kappa\mu - \mu\left(\frac{4\kappa+2\mu}{3}\right)\right\}hU = 0. \end{aligned} \tag{6.6}$$

Operating  $h$  and taking the trace of  $h$ , we obtain  $\kappa\mu - \mu\left(\frac{4\kappa+2\mu}{3}\right) = 0$ . Therefore, either  $\mu = 0$  or  $\kappa + 2\mu = 0$ . By Lemma 3, we get from  $\kappa + 2\mu = 0$  that the manifold is a  $(\kappa, \mu)$ -contact manifold. We also know  $Q\phi - \phi Q = 2\mu h\phi$ . Therefore,  $\mu = 0$  implies  $Q\phi = \phi Q$ . Then by Lemma 4, we get that the manifold is either Sasakian, flat, or of constant  $\xi$ -sectional curvature  $\kappa < 1$  and constant  $\phi$ -sectional curvature  $-1$ .

Hence, we can give the following:

**Theorem 4** *A 3-dimensional non-Sasakian generalized  $(\kappa, \mu)$ -contact metric manifold satisfying the condition  $R \cdot \tilde{C} = 0$  is either a  $(\kappa, \mu)$ -contact manifold, flat, or of constant  $\xi$ -sectional curvature  $\kappa < 1$  and constant  $\phi$ -sectional curvature  $-1$ .*

**7. Three-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds satisfying the condition  $\tilde{C} \cdot S = 0$**

Let  $M$  be a 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold satisfying the condition  $\tilde{C}(\xi, X) \cdot S = 0$ . Then we can write

$$(\tilde{C}(\xi, X) \cdot S)(Y, W) = -S(\tilde{C}(\xi, X)Y, W) - S(Y, \tilde{C}(\xi, X)W) = 0,$$

or equivalently,

$$S(\tilde{C}(\xi, X)Y, W) + S(Y, \tilde{C}(\xi, X)W) = 0. \tag{7.1}$$

Using (2.10) in (7.1), we get

$$\begin{aligned} & -\mu\frac{2\kappa+\mu}{3}\{g(X, Y)\eta(W) - g(X, W)\eta(Y) - \eta(Y)g(hX, W)\} \\ & + \mu^2\{\eta(Y)g(hX, W) - g(hX, Y)\eta(W)\} \\ & + \frac{(2\kappa+\mu)^2}{3}\{g(X, Y)\eta(W) - \eta(X)\eta(W)\eta(Y)\} + \mu(2\kappa+\mu)g(hX, Y)\eta(W) \\ & + \mu^2(k-1)g(X, W)\eta(Y) - \mu_2(k-1)\eta(Y)\eta(W)\eta(X) - \mu\frac{2\kappa+\mu}{3}\{g(X, W)\eta(Y) \\ & - g(X, Y)\eta(W) + \eta(W)g(hY, X)\} + \mu^2\{\eta(W)g(hX, Y) - g(hX, W)\eta(Y)\} \\ & + \{g(X, W)\eta(Y) - \eta(W)\eta(Y)\eta(X)\} + \mu(2\kappa+\mu)g(hX, W)\eta(Y) \\ & + \mu^2(k-1)g(X, Y)\eta(W) - \mu_2(k-1)\eta(Y)\eta(W)\eta(X) = 0, \end{aligned} \tag{7.2}$$

or equivalently,

$$\begin{aligned} & \mu\frac{2\kappa+\mu}{3}\{\eta(Y)g(hX, W) + \eta(W)g(hY, X)\eta(W)\} \\ & - \mu(2\kappa+\mu)\{g(hX, Y)\eta(W) + g(hX, W)\eta(Y)\} + \frac{(2\kappa+\mu)^2}{3} \\ & + \mu^2(\kappa-1)\{g(X, Y)\eta(W) - 2\eta(X)\eta(Y)\eta(W) + g(X, W)\eta(Y)\} = 0. \end{aligned} \tag{7.3}$$

Taking  $W = \xi$  in (7.3), we obtain

$$2\mu \frac{2\kappa + \mu}{3} g(hY, X) + \left(\frac{(2\kappa + \mu)^2}{3} + \mu^2(\kappa - 1)\right) \{g(X, Y) - 2\eta(X)\eta(Y) + \eta(X)\eta(Y)\} = 0. \tag{7.4}$$

Equation (7.4) can be written as

$$2\mu \frac{2\kappa + \mu}{3} hY + \left(\frac{(2\kappa + \mu)^2}{3} + \mu^2(\kappa - 1)\right) \{Y - \eta(Y)\xi\} = 0. \tag{7.5}$$

Operating  $h$  in (7.5), we get

$$2\mu \frac{2\kappa + \mu}{3} h^2Y + \left(\frac{(2\kappa + \mu)^2}{3} + \mu^2(\kappa - 1)\right) hY = 0. \tag{7.6}$$

Taking the trace of  $h$  and using  $trh = 0$  in (7.6), we get  $\mu(2\kappa + \mu) = 0$ , since  $trh^2 \neq 0$ . Therefore, either  $\mu = 0$  or  $2\kappa + \mu = 0$ . By Lemma 3, we get from  $2\kappa + \mu = 0$  that the manifold becomes a  $(\kappa, \mu)$ -contact metric manifold. Also, if  $\mu = 0$ , then by Lemma 4, we get that the manifold is either Sasakian, flat, or of constant  $\xi$ -sectional curvature  $\kappa < 1$  and constant  $\phi$ -sectional curvature  $-1$ .

Thus, we are in a position to state the following:

**Theorem 5** *A 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifold satisfying the condition  $\tilde{C} \cdot S = 0$  is either Sasakian, flat, or of constant  $\xi$ -sectional curvature  $\kappa < 1$  and constant  $\phi$ -sectional curvature  $-1$ , or a  $(\kappa, \mu)$ -contact metric manifold.*

### 8. An example of generalized $(\kappa, \mu)$ -contact metric manifolds

Let  $k : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function defined on an open interval  $I$ , such that  $k(z) < 1$  for any  $z \in I$ . Then we construct a generalized  $(\kappa, \mu)$ -contact metric manifold  $(M, \phi, \xi, \eta, g)$  in the set  $M = \mathbb{R}^2 \times I \subset \mathbb{R}^3$  as follows:

We put  $\lambda(z) = \sqrt{1 - k(z)} > 0$ ,  $\lambda'(z) = \frac{\partial \lambda}{\partial z}$  and we consider three linearly independent vector fields  $e_1, e_2$ , and  $e_3$  on  $M$  as follows:

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y},$$

$$e_3 = [2y + f(z)] \frac{\partial}{\partial x} + [2\lambda(z)x - \frac{\lambda'(z)}{2\lambda(z)}y + h(z)] \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where  $f(z)$  and  $h(z)$  are arbitrary functions of  $z$ . We define the tensor fields  $\phi, \eta, g$  as follows:

$g$  is the Riemannian metric on  $M$  with respect to which the vector fields  $e_1, e_2, e_3$  are orthonormal;  $\eta$  is the 1-form on  $M$  defined by  $\eta(Z) = g(Z, e_1)$  for any  $Z \in \chi(M)$ ;  $\phi$  is the  $(1, 1)$ -tensor field defined by  $\phi e_1 = 0, \phi e_2 = e_3, \phi e_3 = -e_2$ .

Now we calculate the following:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 2\lambda(z)e_2, \quad [e_2, e_3] = -\frac{\lambda'(z)}{2\lambda(z)}e_2 + 2e_1.$$

Since  $(\eta \wedge d\eta)(e_1, e_2, e_3) \neq 0$  everywhere on  $M$ , we conclude that  $\eta$  is a contact form. From the definition of  $\phi$ ,  $g$  and the relations of (8.1) it is easy to verify that

$$\begin{aligned} \phi^2 Z &= -Z + \eta(Z)e_1, \\ g(\phi Y, \phi Z) &= g(Y, Z) - \eta(Y)\eta(Z), \\ d\eta(Y, Z) &= g(Y, \phi Z), \end{aligned}$$

for any  $Y, Z \in \chi(M)$ . Therefore,  $(M, \phi, \xi, \eta, g)$  is a contact metric manifold for  $\xi = e_1$ . Using Koszul's formula we calculate the following:

$$\begin{aligned} \nabla_{e_1} e_1 &= 0 \quad \nabla_{e_1} e_2 = -\{1 + \lambda(z)\}e_3, \quad \nabla_{e_1} e_3 = \{1 + \lambda(z)\}e_2 \\ \nabla_{e_2} e_1 &= -\{1 + \lambda(z)\}e_3, \quad \nabla_{e_2} e_3 = \frac{\lambda'(z)}{2\lambda(z)}e_3, \quad \nabla_{e_2} e_3 = -\frac{\lambda'(z)}{2\lambda(z)}e_2 + \{1 + \lambda(z)\}e_1 \\ \nabla_{e_3} e_1 &= \{1 - \lambda(z)\}e_2, \quad \nabla_{e_3} e_2 = -\{1 - \lambda(z)\}e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned} \tag{8.1}$$

Comparing with the relation  $\nabla_X e_1 = -\phi X - \phi hX$  and the relations obtained in (8.1), we see that

$$he_1 = 0, \quad he_2 = \lambda(z)e_2, \quad he_3 = -\lambda(z)e_3.$$

Using the formula  $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$ , we calculate the following:

$$\begin{aligned} R(e_2, e_1)e_1 &= \{1 + \lambda(z)\}^2 e_2 \\ &= \{1 + 2\lambda(z) + (\lambda(z))^2\}e_2 \\ &= [\{1 - (\lambda(z))^2\} + 2\{1 + \lambda(z)\}\lambda(z)]e_2 \\ &= \{1 - (\lambda(z))^2\}(\eta(e_1)e_2 - \eta(e_2)e_1) + 2\{1 + \lambda(z)\}(\eta(e_1)he_2 - \eta(e_2)he_1), \end{aligned}$$

$$\begin{aligned} R(e_3, e_1)e_1 &= \{1 - 2\lambda(z) - 3(\lambda(z))^2\}e_3 \\ &= [\{1 - (\lambda(z))^2\} + 2\{1 + \lambda(z)\}(-\lambda(z))]e_3 \\ &= \{1 - (\lambda(z))^2\}(\eta(e_1)e_3 - \eta(e_3)e_1) + 2\{1 + \lambda(z)\}(\eta(e_1)he_3 - \eta(e_3)he_1), \end{aligned}$$

$$\begin{aligned} R(e_2, e_3)e_1 &= 0 \\ &= \{1 - (\lambda(z))^2\}(\eta(e_3)e_2 - \eta(e_2)e_3) + 2\{1 + \lambda(z)\}(\eta(e_3)he_2 - \eta(e_2)he_3). \end{aligned}$$

In view of the above expressions of curvature tensors we can easily conclude that the manifold  $M$  is a generalized  $(\kappa, \mu)$ -contact metric manifold with  $k = \{1 - (\lambda(z))^2\}$  and  $\mu = 2\{1 + \lambda(z)\}$ .

Other curvature tensors are given below:

$$\begin{aligned}
 R(e_1, e_2)e_3 &= 0, \\
 R(e_1, e_2)e_2 &= \{1 + \lambda(z)\}e_1, \\
 R(e_1, e_3)e_3 &= \{1 - 2\lambda(z) - 3(\lambda(z))^2\}e_1, \\
 R(e_3, e_2)e_2 &= \left\{ \frac{\lambda''(z)}{2(\lambda(z))^2} - \frac{3(\lambda'(z))^2}{4(\lambda(z))^2} - 2\{1 + \lambda(z)\} - \{1 - (\lambda(z))^2\} \right\}e_3 \\
 R(e_2, e_3)e_3 &= \left\{ \frac{\lambda''(z)}{2(\lambda(z))^2} - \frac{3(\lambda'(z))^2}{4(\lambda(z))^2} - 2\{1 + \lambda(z)\} - \{1 - (\lambda(z))^2\} \right\}e_2.
 \end{aligned}$$

If we consider  $\lambda(z) = \text{constant} > 0$ , then  $\kappa$  and  $\mu$  become constants and hence the manifold becomes a  $(\kappa, \mu)$ -contact metric manifold.

It can be easily verified that such a  $(\kappa, \mu)$ -contact metric manifold is locally  $\phi$ -symmetric. Then Theorem 1 is verified.

### References

- [1] Arslan K, Murathan C, Özgür C. On  $\phi$ -conformally flat contact metric manifolds. *Balkan J Geo Appl* 2000; 5: 1–6.
- [2] Blair DE. *Contact manifolds in Riemannian geometry*. New York, NY, USA: Springer-Verlag, 1976.
- [3] Blair DE. [Two remarks on contact metric structures](#). *Tohoku Math J* 1977; 29: 319–324.
- [4] Blair DE. *Riemannian geometry of contact and symplectic manifolds*. Boston, MA, USA: Birkhauser, 2002.
- [5] Blair DE, Chen H. A classification of 3-dimensional contact metric manifolds with  $Q\phi = \phi Q$ . *Bull Math Acad Sinica* 1992; 20: 379–383.
- [6] Blair DE, Kim JS, Tripathi MM. On the concircular curvature tensor of a contact metric manifold. *Korean J Math Soc* 2005; 42: 883–892.
- [7] Blair DE, Koufogiorgos T, Papantoniou BJ. [Contact metric manifolds satisfying a nullity condition](#). *Israel J Math* 1995; 91: 189–214.
- [8] Cabrerizo JL, Fernandez LM, Fernandez M. [The structure of a class of  \$K\$ -contact manifolds](#). *Acta Math Hung* 1999; 82: 331–340.
- [9] Carriazo A, Martin-Molina V. Generalized  $(\kappa, \mu)$ -space forms and  $D_a$ -homothetic deformations. *Balkan J Geo Appl* 2011; 16: 59–69.
- [10] Ghosh S, De UC. On a class of generalized  $(\kappa, \mu)$ -contact metric manifolds. *P Jangjeon Math Soc* 2010; 13: 337–347.
- [11] Gouli-Andreou F, Xenos PJ. A class of contact metric 3-manifolds with  $\xi \in N(\kappa, \mu)$  and  $\kappa, \mu$  functions. *Algebras Groups and Geometries* 2000; 17: 401–407.
- [12] Jin JE, Park JH, Sekigawa K. Notes on some classes of 3-dimensional contact metric manifolds. *Balkan J Geo Appl* 2012; 17: 54–65.
- [13] Koufogiorgos T, Tsihlias C. [On the existence of a new class contact metric manifolds](#). *Can Math Bull* 2000; 43: 440–447.
- [14] Koufogiorgos T, Tsihlias C. [Generalized  \$\(\kappa, \mu\)\$ -contact metric manifolds with  \$\|grad\kappa\| = \text{constant}\$](#) . *J Geo* 2003; 78: 83–91.
- [15] Papantoniou BJ. Contact Riemannian manifolds satisfying  $R(\xi, X).R = 0$  and  $\xi \in (\kappa, \mu)$ -nullity distribution. *Yokohama Math J* 1993; 40: 149–161.
- [16] Perrone D. Weakly  $\phi$ -symmetric contact metric spaces, *Balkan J Geo Appl* 2002; 7: 67–77.

- [17] Sarkar A, De UC, Sen M. Some results on generalized  $(\kappa, \mu)$ -contact metric manifolds. *Acta Universitatis Apulensis* 2012; 32: 49–59.
- [18] Shaikh AA, Arslan K, Murathan C, Baishya KK. On 3-dimensional generalized  $(\kappa, \mu)$ -contact metric manifolds. *Balkan J Geo Appl* 2007; 2: 122–134.
- [19] Takahashi T. Sasakian  $\phi$ -symmetric spaces. *Tohoku Math J* 1977; 29: 91–113.
- [20] Tripathi MM, Kim JS. On the concircular curvature tensor of a  $(\kappa, \mu)$ -manifold. *Balkan J Geo Appl* 2004; 2: 104–114.
- [21] Yano K. Concircular geometry I. Concircular transformations. *P Imp Acad* 1940; 16: 195–200.
- [22] Yano K, Bochner S. Curvature and Betti Numbers. Princeton, NJ, USA: Princeton University Press, 1953.
- [23] Zhen G. On conformal symmetric  $K$ -contact manifolds. *Chinese Quart J Math* 1992; 7: 5–10.
- [24] Zhen G, Cabrerizo JL, Fernandez LM, Fernandez M. On  $\xi$ -conformally flat  $K$ -contact manifolds. *Indian J Pure Appl Math* 1997; 28: 725–734.