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CONFORMALLY RECURRENT SPACE-TIMES ADMITTING A PROPER CONFORMAL VECTOR FIELD

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Abstract. In this paper we study the properties of conformally recurrent pseudo Riemannian manifolds admitting a proper conformal vector field with respect to the scalar field σ, focusing particularly on the 4-dimensional Lorentzian case. Some general properties already proven by one of the present authors for pseudo conformally symmetric manifolds endowed with a conformal vector field are proven also in the case, and some new others are stated. Moreover interesting results are pointed out; for example, it is proven that the Ricci tensor under certain conditions is *Weyl compatible*: this notion was recently introduced and investigated by one of the present authors. Further we study conformally recurrent 4-dimensional Lorentzian manifolds (space-times) admitting a conformal vector field: it is proven that the covector σ_j is null and unique up to scaling; moreover it is shown that the same vector is an eigenvector of the Ricci tensor. Finally, it is stated that such space-time is of Petrov type N with respect to σ_j.

1. Introduction

Recurrent manifolds have been of great interest and were investigated by many geometers (see for example [1], [14], [15] and [33]). In particular, Walker studied manifolds on which the Riemannian curvature tensor is recurrent [33] while conformally recurrent manifolds were investigated by Adati and Miyazawa [1]. Mc Lenaghan and Leroy [22] and then Mc Lenaghan and Thompson [23] investigated deeply space-times with complex-recurrent conformal curvature tensor. They showed that such spaces belong to types D and N of the Petrov classification and found the metric forms of these spaces in the case in which the recurrence vector is real. Conformally recurrent semi-Riemannian manifolds were studied in some detail also by Suh and Kwon [32]. An n-dimensional pseudo-Riemannian manifold is said to be *conformally recurrent* $(CR)_n$ [1] if

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it is not conformally flat and satisfies the following condition:

$$\nabla_i C_{jklm} = \lambda_i C_{jklm}, \tag{1}$$

where λ_i is a non null covector called associated vector and C_{jkl}^m is the conformal curvature tensor whose local components are given by [26]:

$$\begin{aligned} C_{jkl}^m &= R_{jkl}^m + \frac{1}{n-2}(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j^m g_{kl} - R_k^m g_{jl}) \\ &- \frac{R}{(n-1)(n-2)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}). \end{aligned} \tag{2}$$

In the previous expression the Ricci tensor is defined as $R_{kl} = -R_{mkl}^m$ and the scalar curvature as $R = g^{ij} R_{ij}$. An n-dimensional pseudo Riemannian manifold is said to admit an infinitesimal conformal vector field (or a proper conformal motion) ξ if the Lie derivative of the metric g_{ij} along ξ satisfies the following condition (see [31] page 564 and [12]):

$$\pounds_\xi g_{ij} \equiv \nabla_i \xi_j + \nabla_j \xi_i = 2\sigma g_{ij}, \tag{3}$$

where σ is a scalar function. If $\sigma_j \equiv \nabla_j \sigma \neq 0$ the motion is called proper, if σ is constant the vector is called homothetic [31]. Several authors investigated different space-times structures admitting conformal vector fields. For example Sharma [28] studied conformally symmetric space-times, i.e., 4-dimensional Lorentzian manifolds with $\nabla_i C_{jklm} = 0$, equipped with a conformal vector field: he found that such spaces are necessarily of Petrov types N or O. Moreover, the same author extended these results to space-times with divergence free Weyl tensor, i.e., to 4-dimensional Lorentzian manifolds with $\nabla_m C_{jkl}^m = 0$ and equipped with a proper conformal vector field (see [29]). Also Barua and De [4] extended the result of Sharma.

Investigating pseudo conformally symmetric manifolds (see [5]), i.e., pseudo Riemannian manifolds with the Weyl tensor subjected to the condition

$$\nabla_i C_{jklm} = 2A_i C_{jklm} + A_j C_{iklm} + A_k C_{jilm} + A_l C_{jkim} + A_m C_{jkli}, \tag{4}$$

being A_i a non null vector, De and Mazumder proved in [6] the following results:

1) if a pseudo conformally symmetric manifold admits a proper conformal motion with respect to the scalar field σ, then the manifold is either conformally flat or $\nabla_j \sigma$ is a null vector;

2) if a pseudo conformally symmetric space-time $(PCS)_4$ admits a proper conformal motion, then $(PCS)_4$ is either of Petrov type N or O. Pseudo Riemannian manifolds satisfying (4) were recently studied by one of the present authors in [21].

In this paper we investigate the properties of conformally recurrent manifolds admitting a proper conformal vector field and focusing particularly on the 4-dimensional Lorentzian case. In Section 2, we extend the results of De and Mazumdar to conformally recurrent pseudo-Riemannian manifolds; some general properties are proven: in particular, it is shown that the Ricci tensor of a conformally recurrent pseudo-Riemannian manifold is *Weyl compatible.* This notion was recently introduced and investigated by one of the present authors (see [18], [19] and [20]); other interesting results are then derived. In Section 3, we study conformally recurrent 4-dimensional Lorentzian manifolds (space-times) with conformal motion: it is proven that the covector σ_j is unique up to a scaling; moreover it is shown that the same vector is an eigenvector of the Ricci tensor. Finally, it is stated that such space-time is of Petrov type N with respect to the null vector σ_j. Throughout the paper all manifolds under considerations are assumed to be connected Hausdorff manifolds endowed with a non-degenerate metric of arbitrary signature, i.e., n-dimensional pseudo-Riemannian manifolds; in Section 3, we will specialize to a metric of signature $s = +2$, i.e., to 4-dimensional Lorentz manifold [9]. It is always assumed that $\nabla_j g_{kl} = 0$ (Levi Civita connection) and that the space-matter content is described by the stress-energy tensor and related to the Ricci tensor by Einstein's equations

$R_{kl} - \frac{R}{2} g_{kl} = \kappa T_{kl}$, being $\kappa = \frac{8\pi G}{c^4}$ the Einstein gravitational constant (see [9], [30], [31]).

2. Conformally recurrent pseudo Riemannian manifolds with conformal vector field: general properties

In this section, we extend the results obtained by De and Mazumdar [6] to n-dimensional conformally recurrent pseudo Riemannian manifolds equipped with a proper conformal vector field. A proper conformal vector field is known to satisfy the following properties (see [34]):

$$\begin{aligned} &\pounds_\xi \Gamma^a_{bc} = \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{ab}\sigma^a, \\ &\pounds_\xi C^a_{bcd} = 0, \end{aligned} \tag{5}$$

where Γ^a_{bc} are the Christoffel symbols of the Levi-Civita connection. Now we consider the commutation relation:

$$\begin{aligned} \pounds_\xi(\nabla_l C^h_{ijk}) - \nabla_l(\pounds_\xi C^h_{ijk}) = {}& (\pounds_\xi \Gamma^h_{al}) C^a_{ijk} - (\pounds_\xi \Gamma^a_{li}) C^h_{ajk} \\ & - (\pounds_\xi \Gamma^a_{lj}) C^h_{iak} - (\pounds_\xi \Gamma^a_{lk}) C^h_{ija}. \end{aligned} \tag{6}$$

Using equation (5) and $\nabla_l C^h_{ijk} = \lambda_l C^h_{ijk}$ that provides $\pounds_\xi(\nabla_l C^h_{ijk}) = \mu_l C^h_{ijk}$, being $\mu_l = \pounds_\xi \lambda_l$, we infer easily:

$$\begin{aligned} \mu_l C^h_{ijk} = {}& -2\sigma_l C^h_{ijk} + \delta^h_l \sigma_a C^a_{ijk} - \sigma^h C_{lijk} - \sigma_i C^h_{ljk} \\ & - \sigma_j C^h_{ilk} - \sigma_k C^h_{ijl} + g_{li}\sigma^a C^h_{ajk} + g_{lj}\sigma^a C^h_{iak} + g_{lk}\sigma^a C^h_{ija}. \end{aligned} \tag{7}$$

The previous relation is thus contracted with respect to the indices h and l obtaining:

$$\mu_a C^a_{ijk} = (n-3)\sigma_a C^a_{ijk}. \tag{8}$$

Transvecting (7) with σ_h it is inferred that:

$$\begin{aligned} \mu_l \sigma_h C^h_{ijk} = {}& -\sigma_l \sigma_h C^h_{ijk} - (\sigma^h \sigma_h) C_{lijk} - \sigma_i \sigma_h C^h_{ljk} \\ & - \sigma_j \sigma_h C^h_{ilk} - \sigma_k \sigma_h C^h_{ijl} - g_{lj} T_{ik} + g_{lk} T_{ij}, \end{aligned} \tag{9}$$

where we have defined the symmetric tensor $T_{ik} = \sigma_h \sigma^a C^h_{ika}$ with the obvious property $\sigma^k T_{ik} = 0$. A further multiplication of (9) with σ^k gives:

$$\mu_l T_{ij} = -(\sigma^h \sigma_h)\sigma^k C_{lijk} - (\sigma_h \sigma^h)\sigma^k C_{kijl} - \sigma_i T_{lj} - \sigma_j T_{il}. \tag{10}$$

Now write three versions of the previous equation with indices l, i, j cyclically permutated and sum up; recalling the first Bianchi identity for the Weyl tensor we obtain:

$$(\mu_l + 2\sigma_l)T_{ij} + (\mu_i + 2\sigma_i)T_{lj} + (\mu_j + 2\sigma_j)T_{li} = 0. \tag{11}$$

From Walker's lemma (see [33]) this implies $\mu_i + 2\sigma_i = 0$ or $T_{ij} = 0$; if we suppose that $\mu_i = -2\sigma_i$, then from equation (8) we have simply $\sigma_a C^a_{ijk} = 0$ and thus also $T_{ij} = 0$; from (9) it is thus $(\sigma^h \sigma_h) C_{lijk} = 0$. If $\mu_i + 2\sigma_i \neq 0$, then $T_{ij} = 0$ and from (10) we get $[\sigma^k C_{lijk} + \sigma^k C_{kijl}] = 0$ if $(\sigma^h \sigma_h) \neq 0$. From the first Bianchi identity for the conformal tensor we infer $\sigma^k(2C_{lijk} + C_{ijlk}) = 0$; thus from $\sigma^k C_{lijk} = \sigma^k C_{jlik}$ we conclude that $3\sigma^k C_{lijk} = 0$. Hence in either case $\sigma^k C_{lijk} = 0$ and from (9) $(\sigma^h \sigma_h) C_{lijk} = 0$; since $(\sigma^h \sigma_h) \neq 0$ we have $C_{lijk} = 0$. We have proved the following:

Theorem 2.1. *Let M be an n-dimensional pseudo-Riemannian $(CR)_n$ manifold equipped with a proper conformal vector field. Then either the space is conformally flat or σ^h is a null vector and the following holds*

$$\sigma^k C_{lijk} = 0. \tag{12}$$

Theorem 2.1 is proved by Barua and De [4]. For the sake of completeness we give the proof here. Moreover by taking the covariant derivative of

$$\sigma_m C^m_{jkl} = 0, \tag{13}$$

and from (13) it is also:

$$(\nabla_i \sigma_m) C^m_{jkl} = 0. \tag{14}$$

A further covariant derivative gives $(\nabla_h \nabla_i \sigma_m) C^m_{jkl} = 0$ and consequently

$$[(\nabla_i \nabla_h - \nabla_h \nabla_i)\sigma_m] C^m_{jkl} = 0,$$

and thus by the Ricci identity we infer:

$$R^p_{ihm} \sigma_p C^m_{jkl} = 0. \tag{15}$$

We have thus the following:

Corollary 2.2. *Let M be an n-dimensional pseudo-Riemannian $(CR)_n$ manifold equipped with a proper conformal vector field. Then* (14) *and* (15) *hold.*

Another important property of a conformally recurrent space equipped with a proper conformal vector is now elucidated. First we need the following Lemma (see [16], [17]):

Lemma 2.3 (Lovelock's differential identity)**.** *Let M be an n-dimensional pseudo-Riemannian manifold. Then the following identity is fulfilled*:

$$\nabla_i \nabla_m R^m_{jkl} + \nabla_j \nabla_m R^m_{kil} + \nabla_k \nabla_m R^m_{ijl} = -R_{im} R^m_{jkl} - R_{jm} R^m_{kil} - R_{km} R^m_{ijl}. \tag{16}$$

Lovelock's identity is thus written for the conformal curvature tensor (see [17] and [19]):

$$\begin{aligned} &\nabla_i \nabla_m C^m_{jkl} + \nabla_j \nabla_m C^m_{kil} + \nabla_k \nabla_m C^m_{ijl} \\ = &-\frac{n-3}{n-2}\{R_{im} R^m_{jkl} + R_{jm} R^m_{kil} + R_{km} R^m_{ijl}\}. \end{aligned} \tag{17}$$

Assuming that the covector λ_i is closed, i.e., that $\nabla_j \lambda_k = \nabla_k \lambda_j$ from equations (1) and (17) the following theorem is thus proven (see [17] Theorem 3.10 and Corollary 3.11):

Theorem 2.4. *Let M be an $n(n > 3)$ dimensional $(CR)_n$ pseudo-Riemannian manifold. If the covector λ_j is closed, then the following algebraic relation is fulfilled.*

$$R_{im} R^m_{jkl} + R_{jm} R^m_{kil} + R_{km} R^m_{ijl} = 0. \tag{18}$$

If the Ricci tensor satisfies equation (18) it is named *Riemannian compatible* (see [18], [19] and [20]). Geometric and topological consequences of this condition were extensively studied in [19]. If we insert in relation (18) the local form of the Weyl tensor [26] we obtain:

$$R_{im} C^m_{jkl} + R_{jm} C^m_{kil} + R_{km} C^m_{ijl} = 0. \tag{19}$$

The Ricci tensor is thus *Weyl-compatible.* In recent works Weyl compatibility has been extensively investigated in the Riemannian case [19] and in the pseudo-Riemannian case [20]. In Section 3 we will give a deep account of its consequences on the structure of $(CR)_4$ space-times. If we use Einstein's equations in (19) we infer an analogous condition for the stress energy tensor namely:

$$T_{im} C^m_{jkl} + T_{jm} C^m_{kil} + T_{km} C^m_{ijl} = 0. \tag{20}$$

From the above discussion we may state the following:

Theorem 2.5. *Let M be an $n(n > 3)$ dimensional $(CR)_n$ pseudo-Riemannian manifold. If the covector λ_j is closed, then the stress energy tensor is Weyl compatible.*

From equations (19) and (20) transvecting with σ^i and using (12) it is inferred easily that

$$\begin{aligned} \sigma^i R_{im} C^m_{jkl} &= 0, \\ \sigma^i T_{im} C^m_{jkl} &= 0. \end{aligned} \tag{21}$$

We have thus the following:

Theorem 2.6. *Let M be an $n(n>3)$ dimensional $(CR)_n$ pseudo-Riemannian manifold equipped with a proper conformal vector field. If the covector λ_j is closed, then equations* (21) *hold.*

3. Conformally recurrent space-times with proper conformal vector field

In this section we study the properties of $(CR)_4$ Lorentzian manifolds (space-times) equipped with a proper conformal vector field. We begin with some auxiliary lemmas recently reviewed in [21].

Lemma 3.1 (see [16] page 128 and [21])**.** *Let M be a 4-dimensional pseudo-Riemannian manifold. Then the following identity involving the conformal curvature tensor holds*:

$$\begin{aligned}&\delta^i_r C^{jk}_{st} + \delta^i_t C^{jk}_{rs} + \delta^i_s C^{jk}_{tr} + \delta^k_r C^{ij}_{st} + \delta^k_t C^{ij}_{rs}\\ &+\delta^k_s C^{ij}_{tr} + \delta^j_r C^{ki}_{st} + \delta^j_t C^{ki}_{rs} + \delta^j_s C^{ki}_{tr} = 0.\end{aligned} \tag{22}$$

Remark 3.2 (see [9] page 46 and [21]). Let M be a 4-dimensional pseudo-Riemannian manifold. Let A be a null vector and B a vector orthogonal to A, i.e., $A^i B_i = 0$. Then B is space-like or null and proportional to A, i.e., $B_j = \lambda A_j$ for some $\lambda \in R$.

Now consider a non conformally flat 4-dimensional Lorentzian manifold and two vector fields A_j and B_j satisfying $A^m C_{jklm} = 0$ and $B^m C_{jklm} = 0$. On multiplying equation (22) by $A_j B^s$ we infer $(A_j B^j) C^{ki}_{tr} = 0$; in the same way we have $(A_j A^j) C^{ki}_{tr} = 0$ and $(B_j B^j) C^{ki}_{tr} = 0$ (see [16] page 128). Thus A_j and B_j are orthogonal null vectors. Combining these results with Lemma 3.2 we state:

Theorem 3.3. *Let M be a 4-dimensional non conformally flat manifold with $A^m C_{jklm} = 0$ and $B^m C_{jklm} = 0$. Then $A_j A^j = 0$, $B_j B^j = 0$ and $B_j = \lambda A_j$ for some $\lambda \in R$.*

Hall (p.148 in [12]) also pointed out that if A_i and B_i are orthogonal null vectors, then $B_j = \lambda A_j$.

Now if we consider a non conformally flat $(CR)_4$ Lorentzian manifold equipped with a proper conformal vector, it is $\sigma_m C^m_{jkl} = 0$ and thus we have:

Corollary 3.4. *Let M be a 4-dimensional non conformally flat $(CR)_4$ Lorentzian manifold admitting a proper conformal vector field. Then the fundamental covector σ_j is null and unique up to a scaling.*

Now recall the Bel-Debever version of the Petrov [25] classification of the Weyl tensor on 4 dimensional Lorentzian manifolds (see [3], [7], [12] and [27]); it is based on null vectors k satisfying increasingly restricted conditions as follows:

$$\begin{aligned}&\text{a) type } I \qquad k_{[b}C_{a]rs[q}k_{n]}k^r k^s = 0,\\ &\text{b) type } II, D \qquad k_{[b}C_{a]rsq}k^r k^s = 0,\end{aligned} \tag{23}$$

c) type III $\quad k_{[b}C_{a]rsq}k^r = 0,$

d) type N $\quad C_{arsq}k^r = 0,$

e) type O $\quad C_{arsq} = 0.$

When k satisfies condition b) the Weyl tensor is named algebraically special (see [12], [27], [30] and [31]). Choosing $k_i = \sigma_i$ in the null tetrad formalism we may assert:

Corollary 3.5. *Let M be a 4-dimensional non conformally flat $(CR)_4$ Lorentzian manifold equipped with a proper conformal vector field. Then the Weyl tensor is of Petrov type N with respect to the null vector σ_j.*

As a first application of Theorem 3.3 we may consider equation (12) and (21), that is, $\sigma^m C_{jklm} = 0$ and $\sigma^i R_{im} C^m_{jkl} = 0$. The last defines a vector $B_m = \sigma^i R_{im}$ such that $B^m C_{jklm} = 0$. Thus it is $\sigma^i R_{im} = \lambda\sigma_m$ and σ_j is an eigenvector of the Ricci tensor. Recalling Einstein's equations we infer $k\sigma^i T_{im} = (\lambda - \frac{1}{2}R)\sigma_m$ so that the following theorem holds.

Theorem 3.6. *Let M be a 4-dimensional non conformally flat $(CR)_4$ Lorentzian manifold with $\nabla_j\lambda_k = \nabla_k\lambda_j$ and equipped with a proper conformal vector. Then the covector σ_j is an eigenvector of the Ricci and the stress-energy tensors.*

As a second, consider equation (13) that is the condition $(\nabla_i\sigma_m)C^m_{jkl} = 0$. Equation (22) is then multiplied by $\nabla^p\sigma_j$ to infer

$$(\nabla^p\sigma_r)C^{ki}_{st} + (\nabla^p\sigma_t)C^{ki}_{rs} + (\nabla^p\sigma_s)C^{ki}_{tr} = 0. \tag{24}$$

Contraction of s and p and the closedness of σ_j gives immediately

$$(\nabla^s\sigma_s)C^{ki}_{tr} = 0. \tag{25}$$

We have thus proved:

Theorem 3.7. *Let M be a 4-dimensional non conformally flat $(CR)_4$ Lorentzian manifold equipped with a proper conformal vector field. Then the divergence of σ_j vanishes, that is, $\nabla^j\sigma_j = 0$.*

Finally, from $\sigma^m C_{jklm} = 0$ and $\sigma^i R_{im} = \lambda\sigma_m$ a direct calculations brings (see also Hall's theorem in [20] and [31]) $\sigma^m\sigma^j R_{jklm} = (\lambda - \frac{R}{6})\sigma_k\sigma_l$ from which it is $\sigma_{[p}R_{k]jlm}\sigma^m\sigma^j = 0$ and the Riemannian tensor is algebraically special.

Theorem 3.8. *Let M be a 4-dimensional non conformally flat $(CR)_4$ Lorentzian manifold with $\nabla_j\lambda_k = \nabla_k\lambda_j$ and equipped with a proper conformal vector field. Then $\sigma_{[p}R_{k]jlm}\sigma^m\sigma^j = 0$ holds.*

Now, multiplying equation (22) by σ^j and recalling that $\sigma^m C_{jklm} = 0$ in a 4-dimensional metric of any signature we get:

$$\sigma_r C^{ki}_{st} + \sigma_t C^{ki}_{rs} + \sigma_s C^{ki}_{tr} = 0. \tag{26}$$

Theorem 3.9. *Let M be a 4-dimensional non conformally flat $(CR)_4$ pseudo Riemannian manifold admitting a proper conformal vector field. Then* $\sigma_r C^{ki}_{st} + \sigma_t C^{ki}_{rs} + \sigma_s C^{ki}_{tr} = 0$.

In geometric literature a deep study is devoted to n-dimensional pseudo Riemannian manifolds satisfying the following condition (see for example [8], [21])

$$A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0, \tag{27}$$

being A_j a covector. Here we prove the following for sake of completeness (see [8]):

Lemma 3.10. *Let M be an n-dimensional non-conformally flat pseudo-Riemannian manifold. If* $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$ *then*:
1) $A^i A_i = 0$.
2) $C_{jklm} C^{jklm} = 0$.
3) $C^k_{lmj} C^j_{pqk} = 0$.
4) $C_{jklm} = A_j A_m T_{kl} - A_j A_l T_{mk} - A_k A_m T_{jl} + A_k A_l T_{jm}$ *being T_{kl} a symmetric* $(0,2)$ *tensor.*

Proof. Multiplying (27) with g^{im} we get $A^m C_{jklm} = 0$ and contracting with A^i one obtains $(A^i A_i) C_{jklm} = 0$ from which we infer 1); on the other hand contracting with C^{jklm} one obtains $A_i C_{jklm} C^{jklm} = 0$ from which we infer 2). Finally, contracting with C^{kj}_{pq} and using $A_m C^m_{jkl} = 0$ we get $A_i C_{jklm} C^{kj}_{pq} = 0$ from which 3) follows immediately. Now let θ^i be a unit vector such that $\theta^i A_i = 1$: then contracting the condition $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$ with θ^i a first time we infer:

$$C_{jklm} = A_j(\theta^i C_{iklm}) - A_k(\theta^i C_{ijlm}). \tag{28}$$

Contracting again the previous result with θ^m we get:

$$\theta^m C_{mlkj} = A_j T_{kl} - A_k T_{jl},$$

being $T_{kl} = \theta^i \theta^m C_{iklm}$ a symmetric (0,2) tensor. Inserting back in equation (28) we get the result 4). □

The third relation in Lemma 3.10 was recently obtained by one of the present authors in [21]; it has some importance in the study of Pontryagin forms on a pseudo Riemannian manifolds satisfying condition (27).

Consider the following $4k$ forms ω_k on an orthonormal basis of tangent vectors built with the Riemann tensor (see [10], [11], [19], [21] and [24]):

$$\begin{aligned} &\omega_1(X_1 \cdots X_4) = R^b_{ija} R^a_{klb} (X^i_1 \wedge X^j_2)(X^k_3 \wedge X^l_4), \\ &\omega_2(X_1 \cdots X_8) = R^b_{ija} R^c_{klb} R^d_{mnc} R^a_{pqd} (X^i_1 \wedge X^j_2) \cdots (X^p_7 \wedge X^q_8), \\ &\cdots\cdots \end{aligned} \tag{29}$$

The *Pontryagin forms* (see [19], [21], [24] and also [26] pages 317-318) p_k result from total antisymmetrization of ω_k: $p_k(X_1 \cdots X_{4k}) = \Sigma_p(-1)^p \omega_k(X_1 \cdots X_{4k})$ where P is the permutation taking $(1 \cdots 4k)$ to $(i_1 \cdots i_{4k})$.

In ref [10] the authors considered compact manifolds admitting indefinite metrics with $\nabla_i C^m_{jkl} = 0$: they showed that in such case all the Pontryagin forms vanish. Topological consequences originating from an n-dimensional pseudo Riemannian manifold satisfying condition (27) were studied recently in [21]: we reproduce them here for completeness. First from Lemma 3.10 it is $C^k_{lmj} C^j_{pqk} = 0$. Now as shown by Avez [2] (see also [10]) in the definition of the forms ω_k one may replace the Riemannian curvature tensor with the conformal curvature tensor, i.e., for example:

$$\omega_1(X_1 \cdots X_4) = C^b_{ija} C^a_{klb} (X^i_1 \wedge X^j_2)(X^k_3 \wedge X^l_4). \tag{30}$$

In the case of an n-dimensional pseudo Riemannian $(CQR)_n$ manifold which fulfils (27) it is thus $\omega_1 = 0$. The following theorem was thus recently proven in [21] by one of the present authors.

Theorem 3.11 ([21])**.** *Let M be an n-dimensional pseudo-Riemannian manifold. If $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$, then the first Pontryagin form vanishes, that is, $p_1 = 0$.*

In view of Theorem 3.9 (equation (26)), Lemma 3.10 we infer:

Corollary 3.12. *Let M be a 4-dimensional non conformally flat $(CR)_4$ pseudo Riemannian manifold admitting a proper conformal vector field. Then $C_{jklm} = \sigma_j \sigma_m T_{kl} - \sigma_j \sigma_l T_{mk} - \sigma_k \sigma_m T_{jl} + \sigma_k \sigma_l T_{jm}$ being T_{kl} a symmetric tensor.*

Moreover from Theorem 3.11 we are able to state also:

Corollary 3.13. *Let M be a 4-dimensional non-conformally flat $(CR)_4$ pseudo Riemannian manifold admitting a proper conformal vector field. Then the first Pontryagin form vanishes, that is, $p_1 = 0$.*

Let's now consider a compact orientable 4-dimensional pseudo-Riemannian manifold. The vanishing of the first Pontryagin form has a deep topological consequence. In fact according to Hirzebruch's *signature theorem* (see [13] and [26] pages 229–230) the following holds:

$$3\tau(M) = \int_M p_1. \tag{31}$$

In the previous expression $\tau(M)$ is the Hirzebruch's signature: it is a topological invariant that is linked to the Euler's index by the relation $\tau = \chi \mod 2$ (see [24] page 465). We conclude that:

Theorem 3.14. *Let M be a 4-dimensional compact orientable $(CR)_4$ pseudo-Riemannian manifold admitting a proper conformal vector field. Then the Hirzebruch's signature is null.*

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