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Original article

An algebraic ordered extension of vector space

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Abstract

In this paper we have discussed an algebraic ordered extension of vector space. This new structure comprises a semigroup structure, a scalar multiplication and a compatible partial order. It is an algebraic axiomatisation of topological hyperspace; also it can be thought of as a generalisation of vector space in the sense that, it always contains a vector space and conversely, every vector space can be embedded maximally into such a structure. Initially the idea of this structure was given by S. Ganguly et al. with the name "quasi-vector space" in "*An Associated Structure Of A Topological Vector Space*; Bull. Cal. Math. Soc; Vol-96, No.6 (2004), 489-498". The axioms of this structure evolve a very rapid growth of its elements with respect to the partial order and also evoke some sort of positiveness in each element. Meanwhile, a vector space is evolved within this structure and positivity of each element of the new structure is judged with respect to the elements of the vector space generated. Considering the exponential behaviour of its elements, we have studied this structure in the present paper with a new nomenclature —"**exponential vector space**" in short '***evs***'. We have developed a quotient structure on an evs by defining 'congruence' on it and have shown that the quotient structure also forms an evs with respect to suitably defined operations and partial order. We have obtained an isomorphism theorem and a correspondence theorem in the context of exponential vector space. Further, we have topologised the quotient evs by defining compatibility of the associated congruence with the topology of the base evs. A necessary and sufficient condition has been deduced so that the order-isomorphism stated under the isomorphism theorem becomes topological. Also, we have constructed order-morphisms on a quotient evs corresponding to that on the base evs.

Keywords: Vector space; Exponential vector space; Congruence

1. Introduction

The algebraic ordered extension of vector space discussed in this paper is a new algebraic structure consisting of a semigroup, a scalar multiplication and a compatible partial order which can be thought of as an algebraic axiomatisation of topological hyperspace. We start with the definition of this new structure.

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Definition 1.1 (*[1]*). Let $(X, \leq)$ be a partially ordered set, '+' be a binary operation on X [called *addition*] and '$\cdot$': $K \times X \longrightarrow X$ be another composition [called *scalar multiplication*, K being a field]. If the operations and the partial order satisfy the following axioms then $(X, +, \cdot, \leq)$ is called an *exponential vector space* (in short *evs*) over K [This structure was initiated with the name *quasi-vector space* or *qvs* by S. Ganguly et al. in [1]].

$A_1 : (X, +)$ is a commutative semigroup with identity θ

$A_2 : x \leq y\ (x, y \in X) \Rightarrow x + z \leq y + z$ and $\alpha \cdot x \leq \alpha \cdot y,\ \forall z \in X, \forall \alpha \in K$

A_3 : (i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$

(iv) $1 \cdot x = x$, where '1' is the multiplicative identity in K,

$\forall\, x, y \in X,\ \forall\, \alpha, \beta \in K$

$A_4 : \alpha \cdot x = \theta$ iff $\alpha = 0$ or $x = \theta$

$A_5 : x + (-1) \cdot x = \theta$ iff $x \in X_0 := \left\{z \in X :\ y \nleq z,\ \forall\, y \in X \smallsetminus \{z\}\right\}$

A_6 : For each $x \in X$, $\exists\, p \in X_0$ such that $p \leq x$.

In the above definition, X_0 is precisely the set of all minimal elements of the evs X with respect to the partial order on X and forms the maximum vector space (within X) over the same field as that of X [1]. We call this vector space X_0 as the '*primitive space*' or '*zero space*' of X and the elements of X_0 as '*primitive elements*'.

First of all, by axiom A_3 (iii), we note that for any non-primitive element $x \in X$, $x + x \geq 2x$ or equivalently $\frac{1}{2}x + \frac{1}{2}x \geq x$ which shows a very rapid growth of the non-primitive elements of X with respect to the partial order of X. Also the axiom A_6 reflects some sense of positivity of each element of X. All these facts emerge some sort of exponential flavour within the structure. Considering the importance and influence of the partial order which prevails an essence of hyperspace in the entire structure we find that the name "*exponential vector space*" should be a suitable nomenclature for the structure previously called "*quasi-vector space*".

Secondly, given any vector space V over some field K, an evs X can be constructed (as shown below) such that V is isomorphic to X_0. In this sense, "exponential vector space" can be considered as an algebraic ordered extension of vector space.

Example 1.2 (*This Example is a Slight Modification of the Example 3.2 in [2]*).

Let $X := \left\{(r, a) \in \mathbb{R} \times V : r \geq 0, a \in V\right\}$, where V is a vector space over some field K. Define operations and partial order on X as follows : for $(r, a), (s, b) \in X$ and $\alpha \in K$,

(i) $(r, a) + (s, b) := (r + s, a + b)$;

(ii) $\alpha(r, a) := (r, \alpha a)$, if $\alpha \neq 0$ and $0(r, a) := (0, \theta)$, θ being the identity in V;

(iii) $(r, a) \leq (s, b)$ iff $r \leq s$ and $a = b$.

Then X becomes an exponential vector space over K with the primitive space $\{0\} \times V$ which is evidently isomorphic to V.

The evs structure was motivated by the following example of hyperspace.

Example 1.3 (*[1]*). Let $\mathscr{C}(\mathscr{X})$ be the topological hyperspace consisting of all non-empty compact subsets of a Hausdörff topological vector space $\mathscr{X}$ over the field $\mathbb{K}$ of real or complex numbers. Then $\mathscr{C}(\mathscr{X})$ becomes an evs with respect to the operations and partial order defined as follows. For $A, B \in \mathscr{C}(\mathscr{X})$ and $\alpha \in \mathbb{K}$,

(i) $A + B := \{a + b : a \in A, b \in B\}$

(ii) $\alpha A := \{\alpha a : a \in A\}$

(iii) The usual set-inclusion as the partial order.

The motivation behind the study of exponential vector space is that it envisages a structure in the line of functional analysis within a variety of mathematical objects; this is reflected through the following examples.

Example 1.4. Let $\mathscr{F}(S)$ be the set of all non-negative functions defined on a nonempty set S. Then $\mathscr{F}(S)$ is an evs with respect to the following operations and partial order : For $f, g \in \mathscr{F}(S)$ and $\alpha \in \mathbb{C}$,

(i) $(f+g)(x) := f(x)+g(x),\ \forall x \in S$
(ii) $(\alpha \cdot f)(x) := |\alpha| f(x),\ \forall x \in S$
(iii) $f \preceq g$ iff $f(x) \le g(x),\ \forall x \in S$

Thus the positive and negative parts of a real-valued function defined on S are members of the above evs. □

Example 1.5 (*[2]*)**.** Let X be a lattice with a least element θ. Then it is an evs with respect to the following operations and partial order : For $a, b \in X$ and $\alpha \in K$ (K being any field)
(i) $a+b := \max\{a, b\}$
(ii) $\alpha \cdot a := a$ if $\alpha \ne 0$ and $0 \cdot a = \theta$
(iii) the linear order of X as the partial order.

Thus the set of natural numbers $\mathbb{N}$, any interval of type $[a, b)$ or $[a, \infty)$ or $[a, b]$ are examples of evs over any field. □

We now topologise an exponential vector space. For this we need the following concept.

Definition 1.6 (*[3]*)**.** Let '$\le$' be a preorder in a topological space Z; the preorder is said to be *closed* if its graph $G_{\le}(Z) := \left\{(x, y) \in Z \times Z : x \le y\right\}$ is closed in $Z \times Z$ (endowed with the product topology).

Theorem 1.7 (*[3]*)**.** *A partial order '$\le$' in a topological space Z will be a closed order iff for any $x, y \in Z$ with $x \nleq y$, $\exists$ open neighbourhoods U, V of x, y respectively in Z such that $(\uparrow U) \cap (\downarrow V) = \emptyset$, where $\uparrow U := \{z \in Z : z \ge u$ for some $u \in U\}$ and $\downarrow V := \{z \in Z : z \le v$ for some $v \in V\}$.*

Definition 1.8. An exponential vector space X over the field $\mathbb{K}$ of real or complex numbers is said to be a *topological exponential vector space* if there exists a topology on X with respect to which the addition and the scalar multiplication are continuous and the partial order '$\le$' is closed (Here $\mathbb{K}$ is equipped with the usual topology).

Remark 1.9. If X is a topological exponential vector space then its primitive space X_0 becomes a topological vector space, since restriction of a continuous function is continuous. Moreover, the closedness of the partial order '$\le$' in a topological exponential vector space X readily implies (in view of Theorem 1.7) that X is Hausdörff and hence X_0 becomes a Hausdörff topological vector space.

Example 1.10 (*[4]*)**.** Let $X := [0, \infty) \times V$, where V is a vector space over the field $\mathbb{K}$ of real or complex numbers. Define operations and partial order on X as follows : for $(r, a), (s, b) \in X$ and $\alpha \in \mathbb{K}$,
(i) $(r, a) + (s, b) := (r+s, a+b)$,
(ii) $\alpha(r, a) := (|\alpha| r, \alpha a)$,
(iii) $(r, a) \le (s, b)$ iff $r \le s$ and $a = b$.
Then $[0, \infty) \times V$ becomes an exponential vector space with the primitive space $\{0\} \times V$ which is clearly isomorphic to V.

In this example, if we consider V as a Hausdörff topological vector space then $[0, \infty) \times V$ becomes a topological exponential vector space with respect to the product topology, where $[0, \infty)$ is equipped with the subspace topology inherited from the real line $\mathbb{R}$.

If instead of V we take the trivial vector space $\{\theta\}$ in this example then, the resulting topological evs is $[0, \infty) \times \{\theta\}$ which can be clearly identified with the half ray $[0, \infty)$ of the real line.

In the present paper we have introduced a quotient structure on an evs. For this we have first defined the concept of congruence on an evs and found some associated results. It has been also proved that the quotient structure again forms an evs with respect to suitably defined operations and partial order. Further, we have defined the compatibility between the congruence associated with the quotient evs and the topology of the base evs in order to make the quotient evs topological. We have obtained an isomorphism theorem and a correspondence theorem in the context of exponential vector space. Finally we have deduced a necessary and sufficient condition so that the order-isomorphism stated under the isomorphism theorem becomes topological.

2. Prerequisites

Definition 2.1 (*[5]*). A subset Y of an exponential vector space X is said to be a *sub exponential vector space* (*subevs* in short) if Y itself is an exponential vector space with respect to the compositions of X being restricted to Y.

Note 2.2 (*[5]*). A subset Y of an exponential vector space X over a field K is a sub exponential vector space iff Y satisfies the following :
(i) $\alpha x + y \in Y,\ \forall \alpha \in K, \forall x, y \in Y$.
(ii) $Y_0 \subseteq X_0 \bigcap Y$, where $Y_0 := \{z \in Y : y \not\leq z, \forall y \in Y \smallsetminus \{z\}\}$
(iii) For any $y \in Y, \exists\, p \in Y_0$ such that $p \leq y$.

If Y is a subevs of X then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$ we have $X_0 \cap Y \subseteq Y_0$. $[0, \infty) \times \{\theta\}$ is clearly a subevs of the evs $[0, \infty) \times V$.

<u>**Arbitrary product of exponential vector spaces :**</u> Let $\{X_i : i \in \Lambda\}$ be an arbitrary family of exponential vector spaces over a common field K and $X := \prod_{i\in\Lambda} X_i$ be the Cartesian product. Then, X becomes an exponential vector space over K with respect to the following operations and partial order (see section 5 of [2]) :

For $x = (x_i)_i$, $y = (y_i)_i \in X$ and $\alpha \in K$ we define (i) $x + y := (x_i + y_i)_i$, (ii) $\alpha x := (\alpha x_i)_i$, (iii) $x \ll y$ if $x_i \leq y_i$, $\forall i \in \Lambda$.
Here the notation $x = (x_i)_i \in X$ means that the point $x \in X$ is the map $x : i \mapsto x_i$ $(i \in \Lambda)$, where $x_i \in X_i$, $\forall i \in \Lambda$. The additive identity of X is given by $\theta = (\theta_i)_i$, θ_i being the additive identity in X_i. Also the primitive space of X is given by $X_0 = \prod_{i\in\Lambda}[X_i]_0$.

This product space X becomes a topological exponential vector space over the field $\mathbb{K}$ whenever each factor space X_i is a topological evs over $\mathbb{K}$ and X is endowed with the product topology, which is the weakest topology on X so that each projection map $p_i : X \longrightarrow X_i$ given by $p_i : x \longmapsto x_i$ is continuous.

Thus for any cardinal number β, $[0, \infty)^\beta$ becomes a topological evs.

Definition 2.3 (*[4]*). A mapping $f : X \longrightarrow Y$ (X, Y being two exponential vector spaces over the field K) is called an *order-morphism* if
(i) $f(x + y) = f(x) + f(y), \forall x, y \in X$
(ii) $f(\alpha x) = \alpha f(x), \forall \alpha \in K, \forall x \in X$
(iii) $x \leq y\ (x, y \in X) \Rightarrow f(x) \leq f(y)$
(iv) $p \leq q\ \big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$.

A bijective (injective, surjective) order-morphism is called an *order-isomorphism* (*order-monomorphism*, *order-epimorphism* respectively).

If X, Y are two topological evs over $\mathbb{K}$ then an order-isomorphism $f : X \longrightarrow Y$ is said to be a *topological order-isomorphism* if f is a homeomorphism.

If $f : X \longrightarrow Y$ is an order-morphism and θ, θ' are the identity elements of X, Y respectively then $f(\theta) = f(0.\theta) = 0.f(\theta) = \theta'$. It is also clear that $f(X_0) \subseteq Y_0$ and hence $X_0 \subseteq f^{-1}(Y_0)$ in general. If $X_0 = f^{-1}(Y_0)$ (which is true for any order-monomorphism) then those order-morphisms are of special interest; so we make the following definition :

Definition 2.4. An order-morphism $f : X \longrightarrow Y$ (X, Y being two evs over a common field) is said to be a *normal order-morphism* if $X_0 = f^{-1}(Y_0)$.

Definition 2.5. A property of an evs is called an *evs property* if it remains invariant under order-isomorphism.

The concept of order-isomorphism is competent enough to extract the structural beauty of an evs by judging the invariance of its various properties. Since the composition of two order-isomorphisms, the inverse of an order-isomorphism and the identity map are again order-isomorphisms, the concept thereby produces a partition on the collection of all evs over some common field; this helps one to distinguish two evs belonging to two different classes under this partition.

Definition 2.6 (*[5]*)**.** In an evs X the *primitive* of $x \in X$ is defined as the set

$$P_x := \{p \in X_\circ : p \leq x\}$$

The axiom A_6 in Definition 1.1 ensures that the primitive of each element of an evs is nonempty.

Definition 2.7 (*[5]*)**.** An evs X is said to be a *single primitive evs* if P_x is a singleton set for each $x \in X$. Also, in a single primitive evs X, $P_{x+y} = P_x + P_y$ and $P_{\alpha x} = \alpha P_x$, $\forall x, y \in X$ and for all scalar α.

Single primitivity is an evs property [5].

Proposition 2.8. *If $f : X \longrightarrow Y$ (X, Y being two exponential vector spaces over the same field K) is an order-morphism then $f(M) := \{f(m) : m \in M\}$ is a subevs of Y, for any subevs M of X.*

Proof. Let $f(m), f(m') \in f(M)$ and $\alpha \in K$. Then $\alpha f(m) + f(m') = f(\alpha m + m') \in f(M)$, since M is a subevs. Now $y \in [f(M)]_0 \Rightarrow \exists\, x \in M$ such that $y = f(x)$. Let $p \in M_0$ such that $p \leq x \Rightarrow f(p) \leq f(x) = y \Rightarrow f(p) = y$, since y is a minimal element of $f(M)$. Again $M_0 = M \cap X_0$ [$\because M$ is a subevs] $\Rightarrow p \in X_0$ and hence $y = f(p) \in f(X_0) \subseteq Y_0$. Thus it follows that $[f(M)]_0 \subseteq f(M) \cap Y_0$ and hence $[f(M)]_0 = f(M) \cap Y_0$. Again for any $m \in M$, $\exists\, q \in M_0$ such that $q \leq m \Rightarrow f(q) \leq f(m)$, where $f(q) \in f(M) \cap Y_0 = [f(M)]_0$. Then, the result follows from Note 2.2. □

Proposition 2.9. *If $f : X \longrightarrow Y$ (X, Y being two exponential vector spaces over the same field K) is a normal order-morphism then $f^{-1}(N) := \{x \in X : f(x) \in N\}$ is a subevs of X, for any subevs N of $f(X)$* [By above Proposition 2.8*,* $f(X)$ is a subevs of Y].

Proof. Let $x, x' \in f^{-1}(N)$ and $\alpha \in K$. Then $f(x), f(x') \in N \Rightarrow f(\alpha x + x') = \alpha f(x) + f(x') \in N$ [$\because N$ is a subevs] $\Rightarrow \alpha x + x' \in f^{-1}(N)$.

Now we show that $[f^{-1}(N)]_0 \subseteq X_0 \bigcap f^{-1}(N)$. For this let $q \in [f^{-1}(N)]_0 \subseteq f^{-1}(N)$. We claim that $f(q) \in N_0$. If possible let $\exists\, q' \in N$ with $q' \leq f(q)$ but $q' \neq f(q)$. Then $f^{-1}f(q) \subseteq\uparrow f^{-1}(q')\big[\because q', f(q) \in N \subseteq f(X)\big] \Rightarrow q \geq p$, for some $p \in f^{-1}(q') \subseteq f^{-1}(N)$. This implies that $q = p$, since $q \in [f^{-1}(N)]_0 \Rightarrow f(q) = f(p) = q'$ — a contradiction. Therefore $f(q) \in N_0 \Rightarrow q \in f^{-1}(N_0)$. Thus we have $[f^{-1}(N)]_0 \subseteq f^{-1}(N_0)$. Again $f^{-1}(N_0) = f^{-1}\big(N \cap [f(X)]_0\big) = f^{-1}(N \cap f(X) \cap Y_0) = f^{-1}(N \cap Y_0) = f^{-1}(N) \cap f^{-1}(Y_0) = f^{-1}(N) \cap X_0$ $\big[[\because X_0 = f^{-1}(Y_0)$ for, f is normal$\big]$. Also we know that $f^{-1}(N) \cap X_0 \subseteq [f^{-1}(N)]_0$. Thus it follows that

$$[f^{-1}(N)]_0 = f^{-1}(N_0) = f^{-1}(N) \cap X_0$$

Next let $x \in f^{-1}(N)$. Then $f(x) \in N$. So $\exists\, p \in N_0$ such that $p \leq f(x)$. Again $p \in N_0 \subseteq f(X) \Rightarrow f^{-1}f(x) \subseteq\uparrow f^{-1}(p) \Rightarrow \exists\, q \in f^{-1}(p)$ such that $x \geq q$. Now $q \in f^{-1}(p) \subseteq f^{-1}(N_0) = [f^{-1}(N)]_0$. Therefore by Note 2.2 it follows that $f^{-1}(N)$ is a subevs of X. □

3. Quotient structure on exponential vector space

In this section we shall discuss some quotient structure on an exponential vector space. For this we introduce first the concept of congruence on an exponential vector space as follows:

Definition 3.1. Let E be an equivalence relation on an exponential vector space X over a field K. Then E is said to be a *congruence* on X if it satisfies the following :

(i) $(a, b) \in E \Longrightarrow (x + a, x + b) \in E, \ \forall x \in X$

(ii) $(a, b) \in E \Longrightarrow (\alpha a, \alpha b) \in E, \ \forall \alpha \in K$

(iii) $x \leq y \leq z \ \&\ (x, z) \in E \Longrightarrow (x, y) \in E$[and hence $(y, z) \in E$]

(iv) $a \leq x \leq b \ \&\ (x, y) \in E \Longrightarrow \exists\, c, d \in X$ such that $c \leq y \leq d$ and $(a, c) \in E, (b, d) \in E$

Any congruence E on an exponential vector space X (over a field K) produces the quotient set $X/E := \big\{[x] : x \in X\big\}$, where $[x]$ denotes the equivalence class containing x (with respect to E) i.e

$$[x] := \big\{y \in X : (x, y) \in E\big\}$$

We now show that X/E becomes an exponential vector space over K with respect to the following operations and partial order.

(i) $[x]+[y] := [x+y],\ \forall [x],[y] \in X/E$

(ii) $\alpha[x] := [\alpha x],\ \forall [x] \in X/E, \forall \alpha \in K$

(iii) $[x] \preccurlyeq [y] \iff$ for any $x' \in [x]$, $\exists\, y' \in [y]$ such that $x' \leq y'$ and for any $y'' \in [y]$, $\exists\, x'' \in [x]$ such that $x'' \leq y''$.

Thus $[x] \preccurlyeq [y]$ in $X/E \iff [x] \subseteq \downarrow [y]$ and $[y] \subseteq \uparrow [x]$, where for the set-inclusion relation, $[x], [y], \uparrow [x]$ and $\downarrow [y]$ are considered as subsets of X.

We first show that the aforesaid operations are well-defined and the order '$\preccurlyeq$' is actually a partial order. For this let $(x,x'),(y,y') \in E \Rightarrow (x+y, x'+y), (x'+y, x'+y') \in E \Rightarrow (x+y, x'+y') \in E \Rightarrow [x+y]=[x'+y']$. Also $(x,x') \in E \Rightarrow (\alpha x, \alpha x') \in E \Rightarrow [\alpha x] = [\alpha x']$. This justifies that the operations are well-defined.

Clearly the order '$\preccurlyeq$' on X/E is reflexive and transitive. To check that it is anti-symmetric, let $[x] \preccurlyeq [y]$ and $[y] \preccurlyeq [x] \Rightarrow$ for any $x' \in [x], \exists\, y' \in [y]$ such that $x' \leq y'$ and for this $y' \in [y], \exists\, x'' \in [x]$ such that $y' \leq x'' \Rightarrow x' \leq y' \leq x'' \Rightarrow (x',y') \in E$ [by axiom (iii) of Definition 3.1 of congruence] $\Rightarrow [x]=[x']=[y']=[y]$.

To show that X/E is an exponential vector space over K we need the following lemma.

Lemma 3.2. *$[u] \preccurlyeq [v]$, whenever $u \leq v$ $(u, v \in X)$.*

Proof. For each $u' \in [u]$, we have by axiom (iv) of Definition 3.1 of congruence, some $v' \in [v]$ ($\because u \leq v$) such that $u' \leq v'$. Also by same axiom, for each $v'' \in [v], \exists\, u'' \in [u]$ such that $u'' \leq v''$. This justifies that $[u] \preccurlyeq [v]$, whenever $u \leq v$. □

Theorem 3.3. *X/E is an exponential vector space for any congruence E on an evs X.*

Proof. **A**$_1$: Obviously $(X/E, +)$ is a commutative semigroup with identity $[\theta]$, θ being the identity in X.
A$_2$: Let $[x] \preccurlyeq [y]$. Then for $x, \exists\, y' \in [y]$ such that $x \leq y'$ (by definition of '$\preccurlyeq$' in X/E) $\Rightarrow x+z \leq y'+z$, for any $z \in X$. So by Lemma 3.2 we have $[x+z] \preccurlyeq [y'+z]$. Now $(y,y') \in E \Rightarrow (y+z, y'+z) \in E \Rightarrow [x+z] \preccurlyeq [y'+z] = [y+z] \Rightarrow [x]+[z] \preccurlyeq [y]+[z], \forall [z] \in X/E$.

Now let $[x] \preccurlyeq [y]$ and $\alpha \in K$. We show that $[\alpha x] \preccurlyeq [\alpha y]$. For $x, \exists\, y' \in [y]$ such that $x \leq y' \Rightarrow \alpha x \leq \alpha y'$. Then by Lemma 3.2 we have $[\alpha x] \preccurlyeq [\alpha y']$. Again $(y,y') \in E \Rightarrow (\alpha y, \alpha y') \in E \Rightarrow [\alpha x] \preccurlyeq [\alpha y'] = [\alpha y] \Rightarrow \alpha[x] \preccurlyeq \alpha[y]$.
A$_3$: Let $[x],[y] \in X/E$ and $\alpha, \beta \in K$. Then

(i) $\alpha([x]+[y]) = \alpha([x+y]) = [\alpha(x+y)] = [\alpha x + \alpha y] = \alpha[x] + \alpha[y]$.
(ii) $\alpha(\beta[x]) = \alpha[\beta x] = [(\alpha\beta)x] = (\alpha\beta)[x]$.
(iii) $(\alpha+\beta)[x] = [(\alpha+\beta)x]$. Now $(\alpha+\beta)x \leq \alpha x + \beta x \Rightarrow [(\alpha+\beta)x] \preccurlyeq [\alpha x + \beta x] = \alpha[x]+\beta[x]$, by Lemma 3.2.
(iv) Clearly $1[x]=[x]$.

A$_4$: $\alpha[x]=[\theta]$ iff $[\alpha x]=[\theta]$ iff $(\alpha x, \theta) \in E$ iff $(x,\theta) \in E$, whenever $\alpha \neq 0$ iff either $\alpha = 0$ or $[x]=[\theta]$.
A$_5$: $[x]-[x]=[\theta] \Leftrightarrow [x-x]=[\theta] \Leftrightarrow (x-x,\theta) \in E$. Let $Y := \{[x] : (x-x,\theta) \in E\}$. We claim that $Y = [X/E]_0$.

First of all, $[p] \in Y, \forall p \in X_0$. Let $[x] \in Y$ and $[y] \preccurlyeq [x] \Rightarrow$ for $x, \exists\, y' \in [y]$ such that $y' \leq x \Rightarrow y'-x \leq x-x$ $\Rightarrow$ by Lemma 3.2, $[y'-x] \preccurlyeq [x-x] = [\theta]$, by construction of $Y \Rightarrow$ for $\theta, \exists\, z \in [y'-x]$ such that $z \leq \theta \Rightarrow z = \theta$ $\Rightarrow [y'-x]=[z]=[\theta] \Rightarrow [y'-x]+[x] = [\theta]+[x] \Rightarrow [y']+[-x+x]=[x] \Rightarrow [y']+[\theta]=[x] \Rightarrow [y]=[x]$ $(\because (y,y') \in E) \Rightarrow [x] \in [X/E]_0$.

Conversely, if $[x] \in [X/E]_0$ then for any $p \in X_0$ with $p \leq x$ we must have $[p]=[x] \Rightarrow (p,x) \in E \Rightarrow (p-p, x-x) \in E \Rightarrow (\theta, x-x) \in E \Rightarrow [x] \in Y$. Thus we have

$$[X/E]_0 = \{[x] \in X/E : (x-x,\theta) \in E\}$$

and hence $[x]-[x]=[\theta] \iff [x] \in [X/E]_0$.
A$_6$: As X is an exponential vector space so for each $x \in X, \exists\, p \in X_0$ such that $p \leq x \Rightarrow [p] \preccurlyeq [x]$, by Lemma 3.2, where $[p] \in [X/E]_0$.

Therefore X/E is an exponential vector space over K with respect to aforesaid operations and partial order. □

Note 3.4. For any quotient evs X/E, where E is a congruence on an evs X, the primitive space of X/E is given by : $\big[X/E\big]_0 = \big\{[x] \in X/E : (x - x, \theta) \in E\big\}$.

To proceed further let us first consider the canonical map $\pi : X \longrightarrow X/E$ defined by $\pi(x) = [x], \forall x \in X$. Then
(i) $\pi(x + y) = [x + y] = [x] + [y] = \pi(x) + \pi(y), \forall x, y \in X$
(ii) $\pi(\alpha x) = [\alpha x] = \alpha[x] = \alpha\pi(x), \forall \alpha \in K, \forall x \in X$
(iii) $x \leq y\ (x, y \in X) \Longrightarrow \pi(x) = [x] \preccurlyeq [y] = \pi(y)$ $\big[$by Lemma 3.2$\big]$
(iv) $\pi(x) \preccurlyeq \pi(y)$ in $X/E \Longrightarrow [x] \preccurlyeq [y]$ in $X/E \Longrightarrow [x] \subseteq\downarrow [y]$ and $[y] \subseteq\uparrow [x]$ which implies $\pi^{-1}\pi(x) \subseteq\downarrow \pi^{-1}\pi(y)$ and $\pi^{-1}\pi(y) \subseteq\uparrow \pi^{-1}\pi(x)$, since $\pi^{-1}\pi(x) = [x], \forall x \in X$.

Above discussion readily gives the following result :

Result 3.5. *The canonical map $\pi : X \longrightarrow X/E$ is a surjective order-morphism.*

Theorem 3.6. *Let E be a congruence on an exponential vector space X over a field K and Y be any subevs of X. Then $Y/E := \big\{[y] : y \in Y\big\}$ is a subevs of X/E.*

Proof. Clearly $Y/E = \pi(Y)$. Since π is an order-morphism (by Result 3.5), the theorem follows from Proposition 2.8. ☐

We now discuss some methods for constructing congruence on an evs. The following concept is useful.

Definition 3.7. Let $\phi : X \to Y$ be an order-morphism, where X and Y are two evs over the same field K. We define the kernel of ϕ as $\ker\phi := \big\{(x, y) \in X \times X : \phi(x) = \phi(y)\big\}$.

Clearly, $\Delta := \big\{(x, x) : x \in X\big\} \subseteq \ker\phi$ and ϕ is injective iff $\Delta = \ker\phi$.

The following result provides a convenient way to construct a congruence on a given evs.

Result 3.8. *For any order-morphism $\phi : X \longrightarrow Y$, $\ker\phi$ is a congruence on X, where X and Y are two evs over the same field K.*

Proof. Let $E := \ker\phi$. Clearly, E is an equivalence relation on X.

(i) Let, $(x, y) \in E$. So $\phi(x) = \phi(y) \Rightarrow \phi(x) + \phi(z) = \phi(y) + \phi(z) \Rightarrow \phi(x + z) = \phi(y + z), \forall z \in X \Rightarrow (x + z, y + z) \in E, \forall z \in X$.

(ii) For any $\alpha \in K$, $(x, y) \in E \Rightarrow \phi(x) = \phi(y) \Rightarrow \alpha\phi(x) = \alpha\phi(y) \Rightarrow \phi(\alpha x) = \phi(\alpha y) \Rightarrow (\alpha x, \alpha y) \in E$.

(iii) Let $x \leq y \leq z$ and $(x, z) \in E$. Therefore $\phi(x) = \phi(z)$ and $\phi(x) \leq \phi(y) \leq \phi(z)$. Hence $\phi(x) = \phi(y) = \phi(z)$ $\Rightarrow (x, y), (y, z) \in E$.

(iv) Let $a \leq x \leq b$ and $(x, y) \in E$. Therefore $\phi(x) = \phi(y)$ and $\phi(a) \leq \phi(x) \leq \phi(b) \Rightarrow \phi(a) \leq \phi(y) \leq \phi(b)$. Since ϕ is an order-morphism so, $\phi^{-1}(\phi(y)) \subseteq\uparrow \phi^{-1}(\phi(a))$ and $\phi^{-1}(\phi(y)) \subseteq\downarrow \phi^{-1}(\phi(b))$.
Therefore $y \in \phi^{-1}(\phi(y)) \Rightarrow \exists\, c \in \phi^{-1}(\phi(a))$ and $d \in \phi^{-1}(\phi(b))$ such that $c \leq y \leq d$. Now $\phi(c) = \phi(a)$ and $\phi(d) = \phi(b)$. So $(a, c), (b, d) \in E$.

Therefore $E = \ker\phi$ is a congruence on X [in view of Definition 3.1]. ☐

Example 3.9. Consider the evs $X := [0, \infty)^2$ over the field $\mathbb{K}$. Let $\phi : X \to [0, \infty)$ be defined by : $\phi(x, y) = x,\ \forall (x, y) \in X$. Clearly ϕ is an order-morphism on X. So $\ker\phi = \Big\{\big((x_1, y_1), (x_2, y_2)\big) : x_1 = x_2\Big\}$ is a congruence on X [by Result 3.8]. For any $(a, b) \in X$ the class containing (a, b) is given by : $\big[(a, b)\big] = \big\{(x, y) \in X : x = a\big\} = \{a\} \times [0, \infty)$ which represents the straight line $x = a$ lying in the first quadrant.

The following result states a sufficient condition under which a subset of the evs $X \times X$ becomes a congruence on the evs X.

Result 3.10. *Let G be a subset of the evs $X \times X$ over a field K satisfying the following conditions :*
(i) $\Delta \subseteq G$
(ii) $G^{-1} := \big\{(a, b) : (b, a) \in G\big\} \subseteq G$
(iii) $\downarrow G \subseteq G$
(iv) $\alpha x + \beta y \in G, \forall \alpha, \beta \in K, \forall x, y \in G$
Then G is a congruence on X so that X/G forms a quotient evs.

Proof. From conditions (i) and (ii) it follows that G is a reflexive and symmetric relation on X. Let $(a, b), (b, c) \in G$. Therefore $-1(b, b) = (-b, -b) \in G \Rightarrow (a + b - b, b + c - b) = (a, b) + (b, c) + (-b, -b) \in G$ [by condition (iv)]. Now $\theta \leq (b - b), \forall b \in X \Rightarrow a \leq (a + b - b)$ and $c \leq (c + b - b) \Rightarrow (a, c) \in \downarrow G \subseteq G$ [by condition (iii)]. Therefore G is transitive and hence an equivalence relation on X.

Let $(a, b) \in G$. Then for any $x \in X$, $(x, x) \in G \Rightarrow (a + x, b + x) \in G, \forall x \in X$ [by condition (iv)].

For any $\alpha \in K$ and $(x, y) \in G$, $\alpha(x, y) = (\alpha x, \alpha y) \in G$ [by condition (iv)].

Let $x \leq y \leq z$ and $(x, z) \in G$. Then $(x, y) \ll (x, z) \Rightarrow (x, y) \in \downarrow G \subseteq G$ [by condition (iii)]. Also $(y, x), (x, z) \in G \Rightarrow (y, z) \in G$.

Let $(x, y) \in G$ and $a \leq x \leq b$. Then $(a, y) \ll (x, y) \Rightarrow (a, y) \in \downarrow G \subseteq G$. Also $(x, y), (b, b), (-x, -x) \in G \Rightarrow (x + b - x, y + b - x) \in G$. Again, $x \leq b \Rightarrow \theta \leq x - x \leq b - x \Rightarrow b \leq b + x - x$ and $y \leq y + b - x$. Therefore $(b, y) \in \downarrow G \subseteq G$.

Hence, in view of Definition 3.1, G is a congruence on X. □

We now prove in the following theorem that every order-morphic image of an evs X is a quotient evs for some suitable choice of congruence E on X.

Theorem 3.11 (*First Isomorphism Theorem*). *For any order-morphism $\phi : X \longrightarrow Y$ (X, Y being two evs over the same field K), the quotient evs $X/\ker\phi$ is order-isomorphic to the evs $\phi(X)$.*

Proof. Clearly, $\phi(X)$ is an evs [by Proposition 2.8] and $X/\ker\phi$ is an evs [by Result 3.8 and Theorem 3.3]. Define

$$\left.\begin{array}{r} \Psi : X/\ker\phi \longrightarrow \phi(X) \\ [x] \longmapsto \phi(x) \end{array}\right\}$$

Then $\Psi([x]) = \Psi([y]) \Longleftrightarrow \phi(x) = \phi(y) \Longleftrightarrow (x, y) \in \ker\phi \Longleftrightarrow [x] = [y], \forall [x], [y] \in X/\ker\phi$. Hence Ψ is well-defined and injective. Clearly Ψ is onto and hence a bijection.

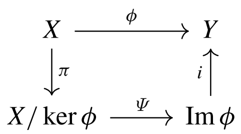
Now $\Psi(\alpha[x] + [y]) = \Psi([\alpha x + y]) = \phi(\alpha x + y) = \alpha\phi(x) + \phi(y) = \alpha\Psi([x]) + \Psi([y]), \forall \alpha \in K$, $\forall [x], [y] \in X/\ker\phi$.

Let $[x] \preccurlyeq [y] \Longrightarrow x \leq y'$ for some $y' \in [y] \Longrightarrow \phi(x) \leq \phi(y') = \phi(y)$ [$\because (y', y) \in \ker\phi$] $\Rightarrow \Psi([x]) \leq \Psi([y])$.

Conversely, let $\Psi([x]) \leq \Psi([y])$ i.e. $\phi(x) \leq \phi(y)$. Since ϕ is an order-morphism, $\phi^{-1}(\phi(x)) \subseteq \downarrow \phi^{-1}(\phi(y))$. So $x \leq y'$ for some $y' \in \phi^{-1}(\phi(y)) \Longrightarrow [x] \preccurlyeq [y'] = [y]$ (by Lemma 3.2), since $\phi(y') = \phi(y)$. Hence Ψ is an order-isomorphism i.e. $X/\ker\phi$ is order-isomorphic to $\phi(X)$. □

Corollary 3.12 (*Canonical Decomposition of Order-morphism*). *Every order-morphism $\phi : X \longrightarrow Y$ (X, Y being two evs over the same field K) can be expressed as the composition of an order-epimorphism, an order-isomorphism and an order-monomorphism.*

Proof. Consider the following diagram :



where $\pi : X \longrightarrow X/\ker\phi$ is the canonical map, $\Psi : X/\ker\phi \longrightarrow \operatorname{Im}\phi$ is the map as stated in Theorem 3.11, $i : \operatorname{Im}\phi \longrightarrow Y$ is the inclusion map. Now $(i \circ \Psi \circ \pi)(x) = (i \circ \Psi)([x]) = i(\phi(x)) = \phi(x), \forall x \in X$.
Therefore $\phi = i \circ \Psi \circ \pi$, where π is an order-epimorphism, Ψ is an order-isomorphism and i is an order-monomorphism. □

Note 3.13. From Result 3.8 and Theorem 3.11 it follows that, for any congruence E on an evs X there corresponds a surjective order-morphism $\pi : X \longrightarrow X/E$ such that $\ker\pi = E$ and for any order-morphism ϕ on X, there corresponds a congruence E on X such that $E = \ker\phi$. Hence we can state :

"***The possible number of order-morphisms on an evs X is the number of different canonical order-morphisms on X i.e. the number of different congruences on X.***"

Result 3.14. *Let X be a single primitive evs over the field K. Then there exists a congruence E on X such that X/E becomes a vector space which is isomorphic to the primitive space X_0.*

Proof. Let X be a single primitive evs over K and P_x denotes the primitive of $x \in X$. Define a relation E on X by :

$$E := \{(x, y) \in X \times X : P_x = P_y\}$$

Clearly E is an equivalence relation on X. Also

(i) $(a, b) \in E \Longrightarrow P_a = P_b \Longrightarrow P_x + P_a = P_x + P_b, \forall x \in X \Longrightarrow P_{x+a} = P_{x+b}, \forall x \in X$ [$\because X$ is a single primitive evs] $\Longrightarrow (x + a, x + b) \in E, \forall x \in X$.

(ii) $(a, b) \in E \Longrightarrow P_a = P_b \Longrightarrow \alpha P_a = \alpha P_b, \forall \alpha \in K \Longrightarrow P_{\alpha a} = P_{\alpha b}, \forall \alpha \in K \Longrightarrow (\alpha a, \alpha b) \in E, \forall \alpha \in K$.

(iii) Let $x \le y \le z$ & $(x, z) \in E$. Then $P_x = P_z$. Now $x \le y \Longrightarrow P_x \subseteq P_y \Longrightarrow P_x = P_y$ [$\because X$ is single primitive]. So $P_y = P_z \Longrightarrow (y, z) \in E$.

(iv) Let $a \le x \le b$ & $(x, y) \in E$. Then $P_x = P_y$ & $P_a \subseteq P_x \subseteq P_b \Longrightarrow P_a = P_x = P_b$ [$\because X$ is single primitive] $\Longrightarrow P_a = P_y = P_b \Longrightarrow (a, y), (b, y) \in E$.

Hence in view of Definition 3.1, E is a congruence on X. Now $x \in X \Longrightarrow p \le x$ for some $p \in X_0 \Longrightarrow P_x = P_p \Longrightarrow [x] = [p]$, where $p \in X_0$. Thus

$X/E = \{[p] : p \in X_0\}$

By Theorem 3.3, X/E is an exponential vector space over K with the primitive space $[X/E]_0 = \{[x] \in X/E : (x - x, \theta) \in E\}$. Now for any $[p] \in X/E$, $[p] - [p] = [p] + [-p] = [p - p] = [\theta] \Longrightarrow [p] \in [X/E]_0$. Thus $X/E = [X/E]_0$ and hence X/E is a vector space over K.

Let us now define a map $\Psi : X_0 \longrightarrow X/E$ by $\Psi(p) := [p], \forall p \in X_0$. Then $\Psi(p) = \Psi(q) \Longleftrightarrow [p] = [q] \Longleftrightarrow (p, q) \in E \Longleftrightarrow P_p = P_q \Longleftrightarrow p = q$ [$\because P_p = \{p\}$ and $P_q = \{q\}$, as $p, q \in X_0$]. So Ψ is injective. Also Ψ is onto and for any $p, q \in X_0, \alpha \in K$, $\Psi(p+q) = [p+q] = [p]+[q] = \Psi(p)+\Psi(q)$ and $\Psi(\alpha p) = [\alpha p] = \alpha[p] = \alpha \Psi(p)$. Therefore X_0 is isomorphic to X/E as a vector space, Ψ being an isomorphism. □

Note 3.15. In the above result, $X/E = \{[p] : p \in X_0\}$, where $[p] = \uparrow p, \forall p \in X_0$.

Example 3.16. Consider the single-primitive evs $X = [0, \infty) \times \mathbb{R}$, where $\mathbb{R}$ is the real vector space of real numbers. Then $X_0 = \{(0, b) : b \in \mathbb{R}\}$. By Result 3.14, there exists a congruence E on X such that X/E is isomorphic to X_0. Also by Note 3.15, $X/E = \{[(0, b)] : b \in \mathbb{R}\}$, where $[(0, b)] = \uparrow (0, b) = \{(x, y) \in X : y = b\}$ which represents the straight line $y = b$ lying in the closed right half of the xy plane.

4. Normal congruence and associated results

In this section we shall introduce ‘normal congruence’ and discuss some associated results.

Definition 4.1. A congruence E on an evs X is said to be *normal* if the canonical map $\pi : X \longrightarrow X/E$ is a normal order-morphism.

By Definition 2.4 of normal order-morphism, π will be normal iff $\pi^{-1}([X/E]_0) = X_0$ iff $\bigcup\{[x] : (x - x, \theta) \in E\} = X_0$. This leads to the following theorem.

Theorem 4.2. *For any congruence E on an exponential vector space X the following are equivalent :*
(i) *E is a normal congruence;*
(ii) The canonical map $\pi : X \longrightarrow X/E$ *is normal;*
(iii) $(x - x, \theta) \in E \Longleftrightarrow x - x = \theta$;
(iv) $(x - x, \theta) \in E \Longleftrightarrow x \in X_0$;
(v) $(x, y) \in E$ *with* $y \in X_0 \Longrightarrow x \in X_0$;
(vi) $\bigcup_{p \in X_0}[p] = X_0$

Definition 4.3. Let E be a congruence on an exponential vector space X. Then a subevs Y of X is called *E-compatible* if $(x, y) \in E$ and $x \in Y$ implies $y \in Y$.

Thus Y is E-compatible iff $Y = \bigcup_{y \in Y}[y]$, where $[y] := \{x \in X : (x, y) \in E\}$.

Theorem 4.4 (*Correspondence Theorem*)**.** *Let E be a normal congruence on an exponential vector space X. Then there exists an inclusion-preserving bijection between the collection of all E-compatible subevs of X and the collection of all subevs of X/E.*

Proof. For any E-compatible subevs Y of X, $Y/E := \{[y] : y \in Y\}$ is a subevs of X/E [by Theorem 3.6]. Conversely let $\mathscr{K}$ be any subevs of X/E and $M := \{x \in X : [x] \in \mathscr{K}\}$. We claim that M is an E-compatible subevs of X and $\mathscr{K} = M/E$.

Clearly $M \neq \emptyset$ and $\theta \in M[\because [\theta] \in \mathscr{K}]$. Let $x, y \in M \Rightarrow [x], [y] \in \mathscr{K} \Rightarrow$ for all $\alpha, \beta \in K$ (the field of X), $[\alpha x + \beta y] = \alpha[x] + \beta[y] \in \mathscr{K}$ [$\because \mathscr{K}$ is a subevs of X/E] $\Rightarrow \alpha x + \beta y \in M$.

We now show that $M_0 = M \cap X_0$, where $M_0 := \{x \in M : y \not\leq x, \forall\, y \in M \smallsetminus \{x\}\}$. Clearly, it is enough to prove that $M_0 \subseteq M \cap X_0$.

Let $x \in M$. Therefore $[x] \in \mathscr{K}$. So $\exists\, [q] \in [\mathscr{K}]_0 = \mathscr{K} \cap [X/E]_0$ such that $[q] \preccurlyeq [x]$ [$\because \mathscr{K}$ is a subevs of X/E]. Now $[q] \in \mathscr{K} \cap [X/E]_0 \Rightarrow q \in M$ and $(q - q, \theta) \in E$ [by Note 3.4]. As E is normal and $\theta \in X_0$, so $q - q \in X_0$. Also $\theta \leq q - q \Rightarrow q - q = \theta \Rightarrow q \in X_0$. Now $[q] \preccurlyeq [x] \Rightarrow \exists\, q' \in [q]$ such that $q' \leq x$, where $q' \in X_0$ [$\because E$ is normal and $q \in X_0$]. If in particular, $x \in M_0$ then $q' \leq x \Rightarrow x = q' \in X_0$ $[\because [q'] = [q] \in \mathscr{K} \Rightarrow q' \in M]$. Thus $x \in M_0 \Rightarrow x \in M \cap X_0$ i.e. $M_0 \subseteq M \cap X_0$. Therefore $M_0 = M \cap X_0$.

Also, for any $x \in M$, $\exists\, q' \in X_0 \cap M = M_0$ such that $q' \leq x$ [as proved above]. Hence, M is a subevs of X.

Clearly M is E-compatible, since $(x, y) \in E$ with $y \in M \Rightarrow [x] = [y] \in \mathscr{K} \Rightarrow x \in M$. Also $[k] \in \mathscr{K} \Leftrightarrow k \in M$. Therefore $\mathscr{K} = \{[x] : x \in M\} = M/E$.

Thus any subevs of X/E is of the form : $Y/E := \{[x] : x \in Y\}$, where Y is some E-compatible subevs of X.

Let

$\mathfrak{X} :=$ the set of all E-compatible subevs of X
and $\mathfrak{X}_E :=$ the set of all subevs of X/E
Define $\Phi : \mathfrak{X} \longrightarrow \mathfrak{X}_E$ by
$\Phi(Y) := Y/E$, for all $Y \in \mathfrak{X}$

In view of Theorem 3.6 Φ is well-defined. Also by above discussion, Φ is onto. Now let $Y_1, Y_2 \in \mathfrak{X}$ such that $\Phi(Y_1) = \Phi(Y_2)$ i.e. $Y_1/E = Y_2/E$. Now Y_1 being E-compatible, $x \in Y_1 \Leftrightarrow [x] \in Y_1/E$. So $Y_1/E = Y_2/E$ implies $x \in Y_1 \Leftrightarrow [x] \in Y_1/E = Y_2/E \Leftrightarrow x \in Y_2$. Hence $Y_1 = Y_2$. Thus Φ is injective.

Again let $Y_1 \subseteq Y_2$, where $Y_1, Y_2 \in \mathfrak{X}$. Then $[x] \in Y_1/E \Rightarrow x \in Y_1 \subseteq Y_2 \Rightarrow [x] \in Y_2/E$. Thus $Y_1 \subseteq Y_2 \Rightarrow \Phi(Y_1) \subseteq \Phi(Y_2)$. Therefore Φ is an inclusion-preserving bijection. □

Corollary 4.5. *If E is a normal congruence on an evs X then any subevs of the quotient evs X/E is of the form $Y/E := \{[x] : x \in Y\}$, where Y is some E-compatible subevs of X.*

Theorem 4.6. *Any normal congruence E on an evs X is a subevs of the evs $X \times X$.*

Proof. Let E be a normal congruence on an evs X. As $\Delta \subseteq E$ so $E \neq \emptyset$. Let $(x_1, y_1), (x_2, y_2) \in E$. Then by Definition 3.1(i), $(x_1 + x_2, y_1 + x_2), (y_1 + x_2, y_1 + y_2) \in E \Rightarrow (x_1 + x_2, y_1 + y_2) \in E$ i.e. $(x_1, y_1) + (x_2, y_2) \in E$. Also for any $\alpha \in K$ and $(x, y) \in E$, $\alpha(x, y) = (\alpha x, \alpha y) \in E$ [by Definition 3.1 (ii)].

We claim: $E_0 = E \cap [X \times X]_0$ where,

$$E_0 := \Big\{(x, y) \in E : (x_1, y_1) \not\leq (x, y), \forall\, (x_1, y_1) \in E \smallsetminus \{(x, y)\}\Big\}$$

For this it is enough to prove that $E_0 \subseteq E \cap [X \times X]_0$.

Let $(x, y) \in E_0$. Since $x, y \in X, \exists\, x_0, y_0 \in X_0$ such that $(x_0, y_0) \leq (x, y)$. As $(x, y) \in E$ and $x_0 \leq x$ so by Definition 3.1 (iv), $\exists\, y_0' \in X$ such that $y_0' \leq y$ and $(x_0, y_0') \in E$. Now E is normal and $x_0 \in X_0 \Rightarrow y_0' \in X_0 \Rightarrow (x_0, y_0') \in X_0 \times X_0 = [X \times X]_0$. Now $(x_0, y_0') \leq (x, y)$, where $(x_0, y_0') \in E$ and $(x, y) \in E_0$. So we must have $(x, y) = (x_0, y_0') \Rightarrow (x, y) \in E \cap [X \times X]_0$. Hence $E_0 \subseteq E \cap [X \times X]_0$. Consequently, $E_0 = E \cap [X \times X]_0$.

Let $(x, y) \in E$. So $\exists\, (x_0, y_0) \in X_0 \times X_0$ such that $(x_0, y_0) \leq (x, y)$. Then proceeding as above, we get $(x_0, y_0') \leq (x, y)$, for some $(x_0, y_0') \in E_0$.

Hence E is a subevs of $X \times X$. □

5. Topologisation of quotient evs

In this section we shall discuss some conditions under which a quotient evs X/E becomes a topological evs. For this we give the following definitions:

Definition 5.1. For a topological evs (X, τ), the closedness of the partial order in X guarantees the existence of $(U, V) \in \mathscr{U}_x \times \mathscr{U}_y$ $\big[\mathscr{U}_x$ being a local base at x consisting of open nbds. of x, $\forall x \in X\big]$ such that $\uparrow U \cap \downarrow V = \emptyset$, whenever $x, y \in X$ with $x \not\leq y$. Let us call such $(U, V) \in \mathscr{U}_x \times \mathscr{U}_y$, a '*separating open pair for* (x, y)' $\big[$in short '*SOP for* (x, y)'$\big]$.

Definition 5.2. A congruence E on a topological evs (X, τ) is said to be τ-*compatible* if

(i) $U \in \tau \Longrightarrow [U] := \bigcup_{u \in U}[u] \in \tau$.

(ii) If $[x] \not\preccurlyeq [y]$ in E then $\exists (x', y') \in [x] \times [y]$ having a separating open pair (U, V) such that at least one of U and V is a π-saturated set $\big($A set $A \subseteq X$ is called π-*saturated* or E-*saturated* if $A = \pi^{-1}\pi(A) \equiv [A]\big)$, where $\pi : X \longrightarrow X/E$ is the canonical map.

Note 5.3. Let (X, τ) be a topological evs and E be a τ-compatible congruence on X. Then by condition (i) of Definition 5.2 it follows that for any τ-open subset U of X its π-load $\pi^{-1}(\pi(U)) = \bigcup_{u \in U}[u]$ is τ-open. Hence the canonical map $\pi : X \longrightarrow X/E$ is an open quotient map where, X/E is given the quotient topology τ_π relative to π.

Theorem 5.4. *If $f : X \longrightarrow Y$ is an open continuous order-morphism (X, Y being two topological evs over the same field $\mathbb{K}$) then the congruence* $\ker f$ *is compatible with the topology of X.*

Proof. Let $f : X \longrightarrow Y$ be an open continuous order-morphism. Then for any open set U in X, $f^{-1}\big(f(U)\big)$ is also an open subset of X. Let $\pi : X \longrightarrow X/\ker f$ be the canonical map. Then $\pi^{-1}\big(\pi(U)\big) = f^{-1}\big(f(U)\big)\big[\because y \in f^{-1}\big(f(U)\big) \Longleftrightarrow f(y) = f(u)$ for some $u \in U \Longleftrightarrow [y] = [u]$ where $u \in U \Longleftrightarrow y \in \pi^{-1}\big(\pi(U)\big)\big]$. Hence $\bigcup_{u \in U}[u]$ is an open subset of X for any open subset U of X $\big[\because \pi^{-1}(\pi(U)) = \bigcup_{u \in U}[u]\big]$.

Let $[x] \not\preccurlyeq [y]$ in $X/\ker f$. Then $f(x) \not\leq f(y)$ as $f(x) \leq f(y) \Longrightarrow f^{-1}\big(f(x)\big) \subseteq\downarrow f^{-1}\big(f(y)\big)$ & $f^{-1}\big(f(y)\big) \subseteq\uparrow f^{-1}\big(f(x)\big)\big[\because f$ is an order-morphism $\big]$ i.e. $[x] \subseteq\downarrow [y]$ & $[y] \subseteq\uparrow [x] \Longrightarrow [x] \preccurlyeq [y]$ which is a contradiction. Now, Y is a topological evs in which $f(x) \not\leq f(y) \Longrightarrow \exists \big(W_x, W_y\big) \in \mathscr{U}_{f(x)} \times \mathscr{U}_{f(y)}$ $\big[\mathscr{U}_y$ being a local base at y consisting of open nbds. of y, $\forall\, y \in Y\big]$ such that $\uparrow W_x \cap \downarrow W_y = \emptyset \cdots\cdots (*)$. Now, f is continuous so $\big(f^{-1}(W_x), f^{-1}(W_y)\big) \in \mathscr{U}_x \times \mathscr{U}_y$ $\big[\mathscr{U}_x$ being a local base at x consisting of open nbds. of x, $\forall\, x \in X\big]$ such that $\uparrow f^{-1}(W_x) \cap \downarrow f^{-1}(W_y) = \emptyset$ $\big[\because z \in\uparrow f^{-1}(W_x) \cap \downarrow f^{-1}(W_y) \Longrightarrow y' \leq z, z \leq y''$ for some $y' \in f^{-1}(W_x)$ and $y'' \in f^{-1}(W_y) \Longrightarrow y' \leq y'' \Longrightarrow f(y') \leq f(y'')$ where, $f(y') \in W_x$ and $f(y'') \in W_y \Longrightarrow f(y') \in W_x \cap \downarrow W_y \subseteq\uparrow W_x \cap \downarrow W_y \neq \emptyset$ which is a contradiction to $(*)\big]$. Thus $\big(f^{-1}(W_x), f^{-1}(W_y)\big)$ is an SOP for $(x, y) \in [x] \times [y]$ such that both $f^{-1}(W_x)$ and $f^{-1}(W_y)$ are π-saturated subsets of X $\big[\because \pi^{-1}\big(\pi(f^{-1}(W_x))\big) = f^{-1}\big(f(f^{-1}(W_x))\big) = f^{-1}(W_x)\big]$.

Hence in view of Definition 5.2, $\ker f$ is compatible with the topology of X. □

Result 5.5. *For any subset U of X, $\pi(\uparrow U) =\uparrow \pi(U)$ and $\pi(\downarrow U) =\downarrow \pi(U)$.*

Proof. Let $\pi(x) \in \pi(\uparrow U) \Rightarrow \pi(x) = \pi(y)$ for some $y \in\uparrow U \Rightarrow u \leq y$ for some $u \in U \Rightarrow [u] \preccurlyeq [y] = [x]$ $\big($by Lemma 3.2$\big) \Rightarrow \pi(x) \in\uparrow \pi(U)$. Conversely, let $\pi(x) \in\uparrow \pi(U) \Rightarrow [u] \preccurlyeq [x]$ for some $u \in U \Rightarrow u \leq x'$ for some $x' \in [x] \Rightarrow x' \in\uparrow U \Rightarrow \pi(x) = \pi(x') \in \pi(\uparrow U)$. Hence $\pi(\uparrow U) =\uparrow \pi(U)$. Similarly it can be proved that $\pi(\downarrow U) =\downarrow \pi(U)$. □

Lemma 5.6. *If U is a π-saturated subset of X then $\uparrow U$ and $\downarrow U$ are also so.*

Proof. Let U be a π-saturated subset of X. Then $\pi^{-1}(\pi(U)) = U$. Let $y \in \pi^{-1}(\pi(\uparrow U)) \Rightarrow \pi(y) \in \pi(\uparrow U) =\uparrow \pi(U)$ $\big[$by Result 5.5$\big]$. So $\exists\, u \in U$ such that $\pi(u) \preccurlyeq \pi(y)$ i.e. $[u] \preccurlyeq [y] \Rightarrow \exists\, u' \in [u]$ such that $u' \leq y$. Now $\pi(u') = \pi(u) \in \pi(U) \Rightarrow u' \in \pi^{-1}(\pi(U)) = U \Rightarrow y \in\uparrow U \Rightarrow \pi^{-1}(\pi(\uparrow U)) \subseteq\uparrow U \Rightarrow \pi^{-1}(\pi(\uparrow U)) =\uparrow U$. Therefore $\uparrow U$ is π-saturated. Similarly it can be shown that $\downarrow U$ is also π-saturated. □

Theorem 5.7. *Let (X, τ) be a topological evs over $\mathbb{K}$ and E be a τ-compatible congruence on X. Then $(X/E, \tau_\pi)$ is a topological evs over $\mathbb{K}$, where τ_π denotes the quotient topology on X/E relative to the canonical map $\pi : X \longrightarrow X/E$.*

Proof. Let A_X and $A_{X/E}$ denote the addition in X and X/E respectively. Given, E is a τ-compatible congruence on X. So π is an open quotient map [by Note 5.3]. Thus π is an open continuous surjection which implies that $\pi \times \pi$ is also an open continuous surjection and hence a quotient map. For any $x, y \in X$ we have,

$(\pi \circ A_X)(x, y) = \pi(x + y) = \pi(x) + \pi(y)$ [By Result 3.5] and

$\big(A_{X/E} \circ (\pi \times \pi)\big)(x, y) = A_{X/E}\big(\pi(x), \pi(y)\big) = \pi(x) + \pi(y)$

Therefore $\pi \circ A_X = A_{X/E} \circ (\pi \times \pi)$. The following commutative diagram explains this fact clearly.

$$\begin{array}{ccc} X \times X & \xrightarrow{A_X} & X \\ \big\downarrow{\scriptstyle \pi\times\pi} & & \big\downarrow{\scriptstyle \pi} \\ X/E \times X/E & \xrightarrow{A_{X/E}} & X/E \end{array}$$

So $\pi \circ A_X$ is continuous $\Rightarrow A_{X/E} \circ (\pi \times \pi)$ is continuous $\Rightarrow A_{X/E}$ is continuous [$\because \pi \times \pi$ is a quotient map].

Again, let S_X and $S_{X/E}$ denote the scalar multiplication on X and X/E respectively. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{K} \times X & \xrightarrow{S_X} & X \\ \big\downarrow{\scriptstyle i\times\pi} & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{K} \times X/E & \xrightarrow{S_{X/E}} & X/E \end{array}$$

For any $x \in X$ and $\alpha \in \mathbb{K}$ we have,

$(\pi \circ S_X)(\alpha, x) = \pi(\alpha x) = \alpha\pi(x)$ [By Result 3.5] and

$\big(S_{X/E} \circ (i \times \pi)\big)(\alpha, x) = S_{X/E}\big(\alpha, \pi(x)\big) = \alpha\pi(x)$. Hence $\pi \circ S_X = S_{X/E} \circ (i \times \pi)$. Therefore the above diagram is commutative. Now π is an open continuous surjection and the identity map $i : \mathbb{K} \to \mathbb{K}$ is a homeomorphism. So $i \times \pi$ is an open continuous surjection and hence a quotient map. Therefore $\pi \circ S_X$ is continuous $\Rightarrow S_{X/E} \circ (i \times \pi)$ is continuous $\Rightarrow S_{X/E}$ is continuous [$\because i \times \pi$ is a quotient map].

Let $[x] \not\preccurlyeq [y]$ in X/E. As E is τ-compatible, so $\exists (x', y') \in [x] \times [y]$ having a separating open pair (U, V) such that at least one of U and V is a π-saturated set. i.e. $(U, V) \in \mathscr{U}_{x'} \times \mathscr{U}_{y'}$ $\big[\mathscr{U}_x$ being a local base at x consisting of open nbds. of x, $\forall\, x \in X\big]$, with at least one of U and V π-saturated and $\uparrow U \cap \downarrow V = \emptyset \cdots\cdots (*)$

Let U be a π-saturated set. Then $\uparrow U$ is also a π-saturated set [by Lemma 5.6]. If possible, let $\uparrow \pi(U) \cap \downarrow \pi(V) \neq \emptyset$ and $\pi(z) \in \uparrow \pi(U) \cap \downarrow \pi(V)$. Then $\pi(u) \leq \pi(z)$ and $\pi(z) \leq \pi(v)$, for some $u \in U$ and $v \in V \Rightarrow \pi(u) \leq \pi(v)$ $\Rightarrow \pi(v) \in \pi(V) \cap \uparrow \pi(U) = \pi(V) \cap \pi(\uparrow U) \Rightarrow v \in V \cap \uparrow U$ [$\because$ $\uparrow U$ is π-saturated] $\Rightarrow V \cap \uparrow U \neq \emptyset \Rightarrow$ $\uparrow U \cap \downarrow V \neq \emptyset$ which is a contradiction to $(*)$. Hence $\uparrow \pi(U) \cap \downarrow \pi(V) = \emptyset$. Similarly, if V is a π-saturated set then we can show that $\uparrow \pi(U) \cap \downarrow \pi(V) = \emptyset \cdots\cdots (**)$

Since π is an open map and U, V are open nbds of x', y' respectively, so $\pi(U)$ and $\pi(V)$ are that of $[x'] = [x]$ and $[y'] = [y]$ respectively in X/E. As $[x], [y] \in X/E$ are arbitrary so by $(**)$ it follows that the partial order '$\preccurlyeq$' is closed in X/E [by Theorem 1.7].

Therefore $(X/E, \tau_\pi)$ is a topological evs over $\mathbb{K}$. □

Corollary 5.8. *If (X, τ) is a topological evs over $\mathbb{K}$ and E is a τ-compatible congruence on X, then $[x]$ is a closed subset of (X, τ), $\forall\, x \in X$ and E is a closed subset of the topological space $X \times X$.*

Proof. Under the given conditions, the partial order '$\preccurlyeq$' is closed in X/E. So $(X/E, \tau_\pi)$ is a T_2-space. Hence the corollary follows. □

Result 5.9. *Let (X, τ) be a topological evs in which $\uparrow U$, $\downarrow U \in \tau$, for any $U \in \tau$. Then $\uparrow \mathscr{Y}$, $\downarrow \mathscr{Y} \in \tau_\pi$, for any $\mathscr{Y} \in \tau_\pi$, where τ_π is the quotient topology on the evs X/E relative to the canonical map $\pi : X \to X/E$, E being a τ-compatible congruence on X.*

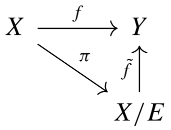
Proof. Let $\mathscr{Y}$ be any open set in $(X/E, \tau_\pi) \Rightarrow \mathscr{Y} = \pi(U)$, for some π-saturated open set U in X. Therefore $\uparrow \mathscr{Y} = \uparrow \pi(U) = \pi(\uparrow U)$ and $\downarrow \mathscr{Y} = \downarrow \pi(U) = \pi(\downarrow U)$ [by Result 5.5]. Now $\uparrow U$ and $\downarrow U$ are open sets [by hypothesis, since U is open] and π is an open map [$\because$ E is τ-compatible]. So $\uparrow \mathscr{Y}, \downarrow \mathscr{Y} \in \tau_\pi$. □

We know by **First Isomorphism Theorem** 3.11 that for any order-morphism $\phi : X \longrightarrow Y$, the quotient evs $X/\ker\phi$ is order-isomorphic to $\phi(X)$. Now we deduce sufficient conditions so that $X/\ker\phi$ becomes topologically order-isomorphic to $\phi(X)$.

Theorem 5.10. *For any order-morphism $\phi : X \longrightarrow Y$ (X, Y being two evs over the same field $\mathbb{K}$), the quotient evs $X/\ker\phi$ is topologically order-isomorphic to $\phi(X)$ provided Y is a topological evs.*

Proof. Let Y be a topological evs. Then $\phi(X)$ being a subevs of Y [in view of Proposition 2.8] is also a topological evs. Again, we know that the property of an evs to be topological is an evs property. Hence $X/\ker\phi$ being order-isomorphic to $\phi(X)$, is a topological evs. Moreover, the order-isomorphism $\Psi : X/\ker\phi \longrightarrow \phi(X)$ (mentioned in Theorem 3.11) is a topological order-isomorphism where $X/\ker\phi$ is provided with the topology $\{\Psi^{-1}(U) : U$ is open in $\phi(X)\}$. □

Theorem 5.11. *Let $f : X \to Y$ be an order-morphism (X, Y being two evs over the field $\mathbb{K}$) and E be a congruence on X such that $E \subseteq \ker f$. Then there exists a unique order-morphism $\tilde{f} : X/E \to Y$ such that the following diagram commutes :*



where, $\pi : X \to X/E$ is the canonical map. Further, if X, Y are topological evs and E is compatible with the topology of X then,
(i) *$\tilde{f}$ is continuous iff f is so.*
(ii) *$\tilde{f}$ is open iff f is so.*
(iii) *$\tilde{f}$ is injective iff $E = \ker f$.*

Proof. Let us define $\tilde{f} : X/E \longrightarrow Y$ by $\tilde{f}([x]) := f(x), \forall [x] \in X/E$. Now $[x] = [y]$ in $X/E \Rightarrow (x, y) \in E \subseteq \ker f \Rightarrow f(x) = f(y) \Rightarrow \tilde{f}([x]) = \tilde{f}([y])$. Hence $\tilde{f}$ is well-defined. Clearly, $\tilde{f} \circ \pi = f$.

For any $[x], [y] \in X/E$ and $\alpha \in \mathbb{K}$ we have, $\tilde{f}\big([x]+[y]\big) = \tilde{f}\big([x+y]\big) = f(x+y) = f(x)+f(y) = \tilde{f}([x])+\tilde{f}([y])$ and $\tilde{f}\big(\alpha[x]\big) = \tilde{f}\big([\alpha x]\big) = f(\alpha x) = \alpha f(x) = \alpha\tilde{f}([x])$. Again, $[x] \preccurlyeq [y] \Rightarrow x \le y'$ for some $y' \in [y] \Rightarrow (y, y') \in E \subseteq \ker f \Rightarrow f(y') = f(y)$. Now $x \le y' \Rightarrow f(x) \le f(y') = f(y) \Rightarrow \tilde{f}([x]) \le \tilde{f}([y])$.

Conversely, let $\tilde{f}([x]) \le \tilde{f}([y])$ and $[z] \in \tilde{f}^{-1}\big(\tilde{f}([x])\big)$. So, $\tilde{f}([z]) = \tilde{f}([x])$ i.e. $f(z) = f(x)$ and $f(x) \le f(y)$. Now $f(z) = f(x) \le f(y) \Longrightarrow f^{-1}\big(f(z)\big) \subseteq \downarrow f^{-1}\big(f(y)\big)$ [$\because$ f is an order-morphism] $\Longrightarrow z \le y''$ for some $y'' \in f^{-1}\big(f(y)\big) \Longrightarrow [z] \preccurlyeq [y'']$ $\big($by Lemma 3.2$\big)$ and $\tilde{f}([y'']) = f(y'') = f(y) = \tilde{f}([y])$ i.e. $[y''] \in \tilde{f}^{-1}\big(\tilde{f}([y])\big)$. Hence $\tilde{f}^{-1}\big(\tilde{f}([x])\big) \subseteq \downarrow \tilde{f}^{-1}\big(\tilde{f}([y])\big)$. Similarly, it can be shown that $\tilde{f}^{-1}\big(\tilde{f}([y])\big) \subseteq \uparrow \tilde{f}^{-1}\big(\tilde{f}([x])\big)$. Hence in view of Definition 2.3, $\tilde{f}$ is an order-morphism.

We now prove the uniqueness of $\tilde{f}$. If possible let $\tilde{g} : X/E \longrightarrow Y$ be another order-morphism such that $\tilde{g} \circ \pi = f$. Then $(\tilde{g} \circ \pi)(x) = f(x), \forall x \in X \Rightarrow \tilde{g}(\pi(x)) = f(x), \forall x \in X \Rightarrow \tilde{g}([x]) = f(x), \forall [x] \in X/E \Rightarrow \tilde{g} = \tilde{f}$. Hence $\tilde{f}$ is the unique order-morphism such that the above diagram commutes.

Now let X, Y be topological evs and E be compatible with the topology of X. Then X/E is a topological evs with the quotient topology τ_π relative to the map π (Theorem 5.7) and the canonical map $\pi : X \longrightarrow X/E$ is an open quotient map (Note 5.3). Thus,

(i) $\tilde{f} \circ \pi = f$ and π is a quotient map $\Longrightarrow \tilde{f}$ is continuous iff f is so.

(ii) π being an open map, $\tilde{f}$ is open $\Longrightarrow f$ is open. Let f be an open map and $\mathscr{G}$ be an open set in $X/E \Longrightarrow \pi^{-1}(\mathscr{G})$ is an open set in X [$\because$ π is continuous] $\Longrightarrow f\big(\pi^{-1}(\mathscr{G})\big)$ is an open set in Y [$\because$ f is open]. Hence $\tilde{f}(\mathscr{G}) = f\big(\pi^{-1}(\mathscr{G})\big)$ is an open set in $Y \Longrightarrow \tilde{f}$ is an open map.

(iii) Let $E = \ker f$. So for

$$(x, y) \in X \times X,\ f(x) = f(y) \Longleftrightarrow (x, y) \in E, \tag{1}$$

Therefore $\tilde{f}([x]) = \tilde{f}([y]) \Longrightarrow f(x) = f(y) \Longrightarrow (x, y) \in E$ [by (1)] $\Longrightarrow [x] = [y] \Longrightarrow \tilde{f}$ is injective.

Conversely, let $\tilde{f}$ be injective. Then $(x, y) \in \ker f \Longleftrightarrow f(x) = f(y) \Longleftrightarrow \tilde{f}([x]) = \tilde{f}([y]) \Longleftrightarrow [x] = [y] \Longleftrightarrow (x, y) \in E$. Hence $E = \ker f$. $\square$

Corollary 5.12. *Let $f : X \longrightarrow Y$ be an order-morphism (X, Y being two topological evs over the same field $\mathbb{K}$) such that the congruence $\ker f$ is compatible with the topology of X. Then the induced order-isomorphism $\tilde{f} : X/\ker f \longrightarrow f(X)(\subseteq Y)$ is topological iff f is open and continuous where, $X/\ker f$ is provided with the quotient topology τ_π.*

Corollary 5.13. *Let E be a congruence on an evs X. Then for any order-morphism f on X, the induced map $\tilde{f}$ is an order-morphism on X/E provided $E \subseteq \ker f$.*

Note 5.14. Corollary 5.13 states a way to construct an order-morphism on the quotient evs X/E corresponding to an order-morphism f on the evs X, provided $E \subseteq \ker f$.

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Further reading

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