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Von Neumann regular semimodule

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Abstract In [11], the author introduced inverse semimodules. Recently, in [8], we further studied properties of inverse semimodules and Clifford semimodules. In this paper, we consider inverse semimodules over a semiring R such that R is a distributive lattice of rings. In [3], the authors introduced von Neumann regular module and studied its properties. In this paper, we introduce von Neumann regular semimodule and study its different interesting properties.

Keywords semimodule · inverse semimodule · Clifford semimodule · von Neumann regular semimodule

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1 Introduction

The study of semimodules over semirings has a long history. In [11], Yusuf introduced the concept of inverse semimodule over a semiring and obtained some analogues to theorems in module theory for inverse semimodules. According to him an inverse semimodule M over a semiring R is an R-semimodule in which the semigroup $(M, +)$ is an inverse semigroup. In [8], we studied congruences, specially the R-module congruences, on inverse semimodules. Semimodules over semirings appear in many areas of mathematics and it has wide applications in the area of theoretical computer science as well as in the cryptography (see [6]). Some properties and characterizations of the bases are discussed and some equivalent conditions for a basis to be a free basis in a finitely generated free semimodule over a semiring are given by Tan in the paper [10]. The different possible cardinalities for a basis in a finitely generated free semimodule over a semiring are considered and some equivalent descriptions are obtained for a commutative semiring R satisfying the property that any two bases for a finitely generated free R-semimodule have the same cardinality. In [3], Jayaram and Tekir defined von Neumann regular modules over commutative ring and studied different interesting properties. In this paper, we define von Neumann regular semimodule and investigate different properties of it. Some basic definitions and preliminaries are discussed in Section 2 and finally in Section 3, we study von Neumann regular semimodule.

2 Definitions and preliminaries

A *semiring* $(R, +, \cdot)$ is a type $(2, 2)$ algebra whose semigroup reducts $(R, +)$ and $(R, \cdot)$ are connected by distributivity, i.e., $a(b+c) = ab+ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in R$. We call a semiring $(R, +, \cdot)$ *additive regular* if for every element $a \in R$ there exists an element $x \in R$ such that $a + x + a = a$. In [12], J. Zeleznekow first studied additive regular semirings. We call a semiring $(R, +, \cdot)$ an *additive inverse semiring* if $(R, +)$ is an inverse semigroup, i.e., for each $a \in R$ there exists a unique element $a' \in R$ such that $a + a' + a = a$ and $a' + a + a' = a'$. In his paper [4], Karvellas studied additive inverse semirings. Throughout the paper, the set of all additive idempotents of the semiring R is denoted by $E^+(R)$. A subsemiring I of a semiring $(R, +, \cdot)$ is called an *ideal* of R if $RI, IR \subseteq I$. An ideal I is called a full ideal if $E^+(R) \subseteq I$. Following [2], a semiring $(R, +, \cdot)$ is called a *skew-ring* if its additive reduct $(R, +)$ is a group. A congruence ρ on a semiring R is called a *distributive lattice* congruence if R/ρ is a distributive lattice. A semiring R is called a *distributive lattice D of skew-rings $R_\alpha(\alpha \in D)$* if R admits a distributive lattice congruence ρ on R such that $D = R/\rho$ and each R_α is a ρ-class.

We need the following two results.

Theorem 2.1 *[5] A semiring R is a distributive lattice of skew-rings if and only if R is an additive inverse semiring satisfying the following conditions:*
(i) $a + a' = a' + a$,
(ii) $a(a + a') = a + a'$,
(iii) $a(b + b') = (b + b')a$,
(iv) $a + a(b + b') = a$, for all $a, b \in R$.

Corollary 2.2 *[5] Let R be an additive commutative semiring. Then R is a distributive lattice of rings if and only if it is an additive inverse semiring satisfying the following conditions:*
(i) $a(a + a') = a + a'$,
(ii) $a(b + b') = (b + b')a$,
(iii) $a + a(b + b') = a$, for all $a, b \in R$.

Let $(M, +)$ be a commutative semigroup (with or without the zero element) and $(R, +, \cdot)$ be a semiring with identity. Then M is called a *left R-semimodule* or simply an *R-semimodule* if there exists a mapping $R \times M \to M$, written as $(r, a) \mapsto ra$, for all $r \in R$ and for all $a \in M$, satisfying (i) $r(m + n) = rm + rn$, (ii) $(r + s)m = rm + sm$, (iii) $r(sm) = (rs)m$ and

(iv) $1m = m$ for all $r, s \in R$ and $m, n \in M$. An R-semimodule M is said to be an *inverse semimodule* [11], if M is an inverse semigroup. A subset S of an R-semimodule M is said to be a *k-set* if $a, a+b \in S$ imply that $b \in S$. Throughout this paper, all semirings R are assumed to be additive as well as multiplicative commutative which are distributive lattices of rings. This means R denotes an additive commutative and multiplicative commutative additive inverse semiring satisfying the conditions (i) and (iii) of Corollary 2.2. Also, assume that R contains an identity element 1 such that $1 \notin E^+(R)$ and all semimodules are inverse semimodules with $M \neq E(M)$, where $E(M)$ is the set of all idempotents of the semigroup M.

We now state some basic properties of inverse R-semimodule.

Theorem 2.3 *[8] Let M be an inverse R-semimodule. Then*
(i) $(ra)' = ra' = r'a$ and $ra = r'a'$ for all $r \in R$ and for all $a \in M$.
(ii) $ea \in E(M)$ for all $e \in E^+(R)$ and for all $a \in M$.
(iii) $ra \in E(M)$ implies that $ra = ru = eu$ for some $e \in E^+(R), u \in E(M)$.

A subsemimodule N of an R-semimodule M is said to be *full* if $E(M) \subseteq N$. For an R-semimodule M, let $\mathscr{L}(M)$ denote the lattice of all full subsemimodules of M. For any $a \in R$, the principal ideal generated by a is denoted by $\langle a \rangle$ and the M-cyclic subsemimodule generated by a is denoted by aM. It is well known that an element $a \in R$ is called *von Neumann regular* [7] (vn-regular for short) if there exists $x \in R$ such that $a = axa$. In this case, it can be easily verified that $e = ax$ is a multiplicative idempotent in R and $\langle a \rangle = \langle e \rangle$. In [7], we proved that a semiring R is a vn-regular semiring if and only if all elements of R are vn-regular elements if and only if $aR = a^2R$ for all $a \in R$. In this paper, we study vn-regular semimodules, which generalize the study of vn-regular modules. Let N and K be two subsemimodules of M. Then the set $\{a \in R : aK \subseteq N\}$ is denoted by $(N : K)$. It is easy to verify that $(N : K)$ is an ideal of R. Thus $(E(M) : M)$ is a full ideal of R. We say that a semimodule M is a *multiplication semimodule* if every subsemimodule of M is of the form IM, for some ideal I of R. A subsemimodule N of an R-semimodule M is said to be a k-subsemimodule of M if for $a, a+b \in N$ for some $b \in M$ imply that $b \in N$. For any subsemimodule N of an R-semimodule M, the k-closure of N, denoted by $\overline{N}$, is defined by $\overline{N} = \{m \in M : m + s_1 = s_2$ for some $s_1, s_2 \in N\}$. One can easily prove that a subsemimodule N of an R-semimodule M is a k-subsemimodule if and only if $\overline{N} = N$.

3 Von Neumann regular semimodule

In this section, we study vn-regular semimodules. For this purpose we first introduce the following definition.

Definition 3.1 *Let R be a semiring and M be an R-semimodule. An element e of R is said to be weak idempotent if $e^2m = em$ for all $m \in M$.*

Lemma 3.2 *Let R be a semiring and M be an R-semimodule. An element e of R is a weak idempotent if and only if $e^2 + e' \in (E(M) : M)$.*

Proof. First suppose that e is a weak idempotent of R. Then $e^2m = em$ for all $m \in M$. This implies $e^2m + e'm = em + e'm = em + (em)'$ for all $m \in M$, i.e., $(e^2 + e')m \in E(M)$ for all $m \in M$, i.e., $e^2 + e' \in (E(M) : M)$. Conversely, let $e^2+e' \in (E(M) : M)$. Then $(e^2+e')m \in E(M)$ for all $m \in M$, i.e., $(e^2 + e')m = (e^2 + e')m + ((e^2 + e')m)'$ for all $m \in M$. This leads to $e^2m+e'm = e^2m+e'm+(e^2)'m+em = (e^2+(e^2)')m+em+e'm = em+e'm$ for all $m \in M$, i.e., $e^2m + e'm + em = em + e'm + em$ for all $m \in M$, i.e., $e^2m + (e + e')m = em$ for all $m \in M$, i.e., $e^2m + e(e + e')m = em$ for all $m \in M$, i.e., $e^2m = em$ for all $m \in M$. Hence e is a weak idempotent of R. □

Lemma 3.3 *Let e be a weak idempotent of a semiring R and M be an R-semimodule. Then $E(M) + eM = (1 + 1' + e)M$.*

Proof. Since e is a weak idempotent of R, we have $e^2m = em$ for all $m \in M$. Clearly, $(1 + 1' + e)M \subseteq E(M) + eM$. For the reverse inclusion, let $m_1 + em_2 \in E(M) + eM$. Then $m_1 + em_2 = 1 \cdot m_1 + em_2 + (em_2)' + em_2 = (1+e+e')m_1 + em_2 + e'm_2 + e^2m_2 = m_1 + em_1 + e'm_1 + (1+1'+e)(em_2) = m_1 + m_1' + em_1 + (1 + 1' + e)(em_2) = (1 + 1' + e)m_1 + (1 + 1' + e)em_2 = (1+1'+e)(m_1 + em_2) \in (1+1'+e)M$. Thus $E(M) + eM \subseteq (1 + 1' + e)M$ and hence $E(M) + eM = (1 + 1' + e)M$. □

Definition 3.4 *Let N be a full subsemimodule of an R-semimodule M. We say that N has a complement in $\mathscr{L}(M)$ if there exists a full subsemimodule $K \in \mathscr{L}(M)$ such that $N + K = M$ and $N \cap K = E(M)$, where $\mathscr{L}(M)$ is the set of all full subsemimodules of M. If a full subsemimodule N of an R-semimodule M has a complement K, then we say that N and K are complement to each other.*

Lemma 3.5 *Let e, f be two weak idempotent elements of a semiring R and M be an R-semimodule. Then*
(i) $1 + e', ef, e + f(1 + e')$ are weak idempotent elements of R.
(ii) $(1 + 1' + e)M = (1 + 1' + f)M$ if and only if $\langle e\rangle + (E(M) : M) = \langle f\rangle + (E(M) : M)$.
(iii) $(1 + 1' + e)M$ has a complement in $\mathscr{L}(M)$.

Proof. (i) Let $m \in M$. Now,

$$\begin{aligned}(1 + e')^2m &= (1 + e' + e' + e'e')m \\ &= (1 + e' + e' + e)m \\ &= (1 + e')m.\end{aligned}$$

Hence $1 + e'$ is a weak idempotent of R.

Again, $(ef)^2m = e^2f^2m = e^2fm = fe^2m = fem = efm$ implies ef is a weak idempotent of R.

Also,

$$\begin{aligned}((e+f(1+e'))^2m &= (e^2+2ef(1+e')+f^2(1+e')^2)m\\ &= e^2m+2ef(1+e')m+f(1+e')m\\ &= em+2f(e+ee')m+f(1+e')m\\ &= em+2f(em+(e^2m)')+f(1+e')m\\ &= em+2f(em+e'm)+f(1+e')m\\ &= (e+2f(e+e'))m+f(1+e')m\\ &= (e+f(e+e'))m+f(1+e')m\\ &= (e+e(f+f'))m+f(1+e')m\\ &= em+f(1+e')m\\ &= (e+f(1+e'))m.\end{aligned}$$

Therefore, $e+f(1+e')$ is a weak idempotent of R.

(ii) First suppose that $\langle e\rangle+(E(M):M)=\langle f\rangle+(E(M):M)$. Then clearly $eM+E(M)=fM+E(M)$, i.e., $(1+1'+e)M=(1+1'+f)M$. Conversely, let $(1+1'+e)M=(1+1'+f)M$, i.e., $E(M)+eM=E(M)+fM$. Now, $f=f+f(e+e')=fe+f(1+e')=ef+f(1+e')$. Let $m\in M$. Now $fm\in fM\subseteq E(M)+fM=E(M)+eM$ implies $fm=m_1+em_2$ for some $m_1\in E(M)$ and $m_2\in M$. Again, $f(1+e')m=(1+e')fm=(1+e')(m_1+em_2)=(1+e')m_1+(em_2+(e^2m_2)')=(1+e')m_1+(em_2+e'm_2)\in E(M)$. Therefore, $f=fe+(f+fe')\in\langle e\rangle+(E(M):M)$ and thus $\langle f\rangle\subseteq\langle e\rangle+(E(M):M)$. This implies $\langle f\rangle+(E(M):M)\subseteq\langle e\rangle+(E(M):M)$. Similarly, we can show that $\langle e\rangle+(E(M):M)\subseteq\langle f\rangle+(E(M):M)$. Hence $\langle e\rangle+(E(M):M)=\langle f\rangle+(E(M):M)$.

(iii) It is easy to verify that $(1+1'+e)M+(1+e')M=M$. Also, $(1+1'+e)M\cap(1+e')M=E(M)$. Therefore, $(1+1'+e)M$ has a complement $(1+e')M$ in $\mathscr{L}(M)$.

□

Lemma 3.6 *For an R-semimodule M, $\mathscr{L}(M)$ is a modular lattice.*

Proof. Let $A,B,C\in\mathscr{L}(M)$ such that $C\subseteq A$. Clearly, $(A\cap B)+C\subseteq A\cap(B+C)$.

For the reverse inclusion, let $x\in A\cap(B+C)$. Then $x\in A$ and $x=b+c$ for some $b\in B$ and $c\in C$. This implies $c'\in C\subseteq A$ and thus $x+c'=b+c+c'\in A\cap B$. Hence $x=b+c=(b+c+c')+c=(x+c')+c\in(A\cap B)+C$. Therefore, $A\cap(B+C)\subseteq(A\cap B)+C$. Consequently, $A\cap(B+C)=(A\cap B)+C$. Hence $\mathscr{L}(M)$ is a modular lattice.

□

Lemma 3.7 *Let M be a colon distributive R-semimodule. If $N\in\mathscr{L}(M)$ has a complement K, then there exists a weak idempotent e of R such that $N=(1+1'+e)M$ and $K=(1+e')M$.*

Proof. Since $N\in\mathscr{L}(M)$ has a complement, so there exists $K\in\mathscr{L}(M)$ such that $N+K=M$ and $N\cap K=E(M)$. Now $R=(M:M)=(N+K:M)=(N:M)+(K:M)$. Let $A=(N:M)$ and $B=(K:M)$. Then A

and B are full ideals of R such that $R = A + B$ and $A \cap B = (E(M) : M)$. Now $A + B = R$ implies there exists $x \in A$ and $y \in B$ such that $x + y = 1$. This implies $x^2 + xy = x$, i.e., $x^2 + (x^2)' + xy = (x^2)' + x$, i.e., $x + (x^2)' = x^2 + (x^2)' + xy \in A \cap B = (E(M) : M)$. This implies $(x^2)'m + xm \in E(M)$ for all $m \in M$, i.e., $x^2m + x'm = xm + x'm$ for all $m \in M$. Therefore, $x^2m = (x^2 + (x^2)' + x^2)m = (x + x' + x^2)m = xm + (x'm + x^2m) = xm+(x'm+xm) = xm$ for all $m \in M$. Thus x is a weak idempotent of R. Let $a \in A$. Then $ay \in AB \subseteq A \cap B = (E(M) : M)$ implies $aym \in E(M)$ for all $m \in M$. Again, $axm \in AM \subseteq N$ for all $m \in M$. Then $a = a(x+y) = ax+ay$ implies $am = axm + aym \in \langle x\rangle M + E(M) \subseteq xM + E(M) = (1 + 1' + x)M$ for all $m \in M$. Hence $AM \subseteq (1 + 1' + x)M$.

Now $M = RM = ((N : M) + (K : M))M = (N : M)M + (K : M)M$. Also, $(K : M)M \subseteq K$ implies $(N : M)M + K = M$. Therefore, $N = N \cap M = N \cap ((N : M)M + K)$. By the modular law, we have $N = (N : M)M + (N \cap K) = (N : M)M + E(M) = AM + E(M) = (1 + 1' + x)M$. Similarly, we can prove that $K = (1 + x')M$.

□

Definition 3.8 *Let M be an R-semimodule. An element a of R is said to be M-vn-regular if $aM + E(M) = a^2M + E(M)$.*

Definition 3.9 *An R-semimodule M is said to be a vn-regular semimodule if for any $m \in M$, $Rm + E(M) = aM + E(M)$ for some M-vn-regular element a of R.*

Example 3.1 A semiring R is a vn-regular R-semimodule if and only if it is a vn-regular semiring.

Example 3.2 Evey vn-regular R-module is always a vn-regular R-semimodule.

Example 3.3 Let R be a semiring and I be a full maximal ideal of R. It can be easily verified that $M = R/I$ is a vn-regular R-semimodule.

For further study, we need some basic results on determinant of a square matrix over a semiring. Let A be an $n \times n$-matrix over a semiring R. The positive determinant (see [1]), denoted by $|A|^+$, of A is defined by $|A|^+ = \sum_{\sigma \in A_n} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ and the negative determinant (see [1]), denoted by $|A|^-$, of A is defined by $|A|^- = \sum_{\sigma \in S_n \setminus A_n} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where S_n and A_n respectively denote the symmetric group and the alternating group on the set $\{1, 2, \ldots, n\}$. In the paper [9], the author defined determinant, called ε-determinant, of a square matrix over a semiring with an ε-function ε. According to them an element x in an additive commutative semiring R with 0 is called additively invertible in R if $x + y = 0$ for some $y \in R$. Such an element y is obviously unique and denoted by $-x$. Let

$V(R)$ denote the set of all additively invertible elements in R. Let R be a semiring, not necessarily additive inverse. A bijection ε on R is called an ε-function of R if $\varepsilon(\varepsilon(a)) = a$, $\varepsilon(a+b) = \varepsilon(a)+\varepsilon(b)$ and $\varepsilon(ab) = a\varepsilon(b) = \varepsilon(a)b$ for all $a, b \in R$ and $\varepsilon(a) = -a$ for all $a \in V(R)$. Let R be a commutative semiring with an ε-function ε and A be an $n \times n$-matrix. In [9], the author defined ε-determinant of a square matrix A, denoted by $\det_\varepsilon(A)$, by $\det_\varepsilon(A) = \sum_{\sigma \in S_n} \varepsilon^{(t(\sigma))} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where $t(\sigma)$ is the number of inversions in the permutation σ and $\varepsilon^{(k)}$ is defined by $\varepsilon^{(0)}(a) = a$ and $\varepsilon^{(r)}(a) = \varepsilon^{(r-1)}(\varepsilon(a))$ for all positive integers r. Since $\varepsilon^{(2)}(a) = a$, it follows that $\det_\varepsilon(A)$ can be rewritten as $\det_\varepsilon(A) = (|A|^+) + \varepsilon(|A|^-)$. In the same paper the author defined an ε-minor, the ε-cofactor and the ε-adjoint of an $n \times n$-matrix. According to him, an ε-minor of order $n-1$ of A is defined to be the ε-determinant of a submatrix of A obtained by striking out one row and one column from A. The ε-minor obtained by striking out the i-th row and the j-th column is written as $M_{ij} (1 \leq i, j \leq n)$. Here $\varepsilon^{(i+j)} M_{ij}$ is called the ε-cofactor of the element a_{ij} and is denoted by A_{ij}. The ε-adjoint matrix of A, denoted by $adj_\varepsilon(A)$, is defined to be the transposed matrix of ε-cofactors of A, i.e., $adj_\varepsilon(A) = ((A_{ij})_{n \times n})^T$.

Lemma 3.10 *[9] Let R be a commutative semiring with an ε-function and $A \in M_n(R)$. Then*

(1) $\det_\varepsilon(A) = \det_\varepsilon(A^T)$*;*

(2) if two rows (or columns) of A are identical, then $\det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^+)$*;*

(3) if $B \in M_n(R)$ is obtained by interchanging two rows (or columns) of A, then $\det_\varepsilon(B) = \varepsilon(\det_\varepsilon(A))$.

Remark 3.1 If R is an additive commutative, multiplicative commutative and additive inverse semiring with identity satisfying the conditions in Corollary 2.2, then for each $a \in R$ there exists unique element a' in R such that $a + a' + a = a$. So in this paper considering the ε-function as $\varepsilon(a) = a' = 1'a$, for all $a \in R$, we get by [9, Remark 2.2] $\det_\varepsilon(A) = (|A|^+) + (|A|^-)'$. In this case, we denote $\det_\varepsilon(A)$ and $adj_\varepsilon(A)$ by $\det A$ and $adj A$ respectively. Moreover, we have

(1)
$$\det A = (|A|^+) + (|A|^-)' = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{\sigma \in S_n} (1')^{(t(\sigma))} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)};$$

(2) $\det A = \det A^T$;

(3) $\det A = |A|^+ + (|A|^+)' \in E^+(R)$, whenever two rows (or columns) of A are identical;

(4) $\det B = (\det A)'$, where B is obtained by interchanging two rows (or columns) of A.

Lemma 3.11 *Let $A = (a_{ij})_{n\times n}$ be an $n \times n$-matrix over a semiring R and let $adjA \cdot A = (b_{ij})_{n\times n}$. Then $b_{ii} = \det A$ and $b_{ij} \in E^+(R)$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.*

Proof. By *(1)* of Remark 3.1, we have $\det A = \sum_{j=1}^{n} a_{ij}A_{ij}$. Also $\det A = \det A^T$
$= \sum_{j=1}^{n} a_{ji}A_{ji}$.

Suppose $r \in \{1, 2, \ldots, n\}$ such that $r \neq i$. We consider the matrix P by replacing the i-th column of A with r-th column of A. Then $a_{ji} = a_{jr}$ for all $j = 1, 2, \ldots, n$. Also P has two identical columns and thus by Remark 3.1, we must have $\det P = \det P^T \in E^+(R)$. Now, expanding the determinant of P^T by i-th column, we have $\det P^T = \sum_{j=1}^{n} a_{ji}A_{ji} = \sum_{j=1}^{n} a_{jr}A_{ji}$. Thus, for $i \neq r$, we have $\sum_{j=1}^{n} a_{jr}A_{ji} = \det P \in E^+(R)$. Now $(A_{ij})^T = adjA = D = (d_{ij})_{n\times n}$ (say) and thus $d_{ij} = A_{ji}$ for all $i, j \in \{1, 2, \ldots, n\}$. Let $(adjA)A = B = (b_{ij})_{n\times n}$. Then $(b_{ij})_{n\times n} = DA = (d_{ij})_{n\times n}(a_{ij})_{n\times n}$. Therefore, $b_{ii} = \sum_{p=1}^{n} d_{ip}a_{pi} = \sum_{p=1}^{n} A_{pi}a_{pi} = \sum_{p=1}^{n} a_{pi}A_{pi} = \det A$ and for $i \neq j$, we have $b_{ij} = \sum_{p=1}^{n} d_{ip}a_{pj} = \sum_{p=1}^{n} A_{pi}a_{pj} = \sum_{p=1}^{n} a_{pj}A_{pi} \in E^+(R)$.

□

Lemma 3.12 *Let M be a finitely generated R-semimodule such that $E(M)$ is a k-set. Then an element a of R is M-vn-regular if and only if $\langle a\rangle + (E(M) : M) = \langle a^2\rangle + (E(M) : M)$.*

Proof. If $\langle a\rangle + (E(M) : M) = \langle a^2\rangle + (E(M) : M)$, then $a = a + a' + a \in \langle a\rangle + (E(M) : M) = \langle a^2\rangle + (E(M) : M)$ implies $a = ra^2 + r_1$ for some $r_1 \in (E(M) : M)$ and $r \in R$. Thus $aM + E(M) \subseteq a^2M + E(M)$. Since $a^2M + E(M) \subseteq aM + E(M)$ holds trivially, we have $aM + E(M) = a^2M + E(M)$ and hence a is M-vn-regular.

Conversely, let an element a of R be a M-vn-regular element, i.e., $aM + E(M) = a^2M + E(M)$. Then we have $\langle a\rangle M + E(M) = \langle a\rangle\langle a\rangle M + E(M)$. Since M is finitely generated, it follows that $\langle a\rangle M$ is also finitely generated. Let $\langle a\rangle M$ be generated by $x_1, x_2, \ldots, x_n$. Now for all $i = 1, 2, \ldots, n$; $x_i \in \langle a\rangle M \subseteq \langle a\rangle M + E(M) = \langle a\rangle\langle a\rangle M + E(M)$ implies

$$\begin{array}{lcl} x_1 & = & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + g_1, \\ x_2 & = & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + g_2, \\ \vdots & \vdots & \vdots \\ x_n & = & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + g_n, \end{array}$$

for some $a_{ij} \in \langle a \rangle$ and $g_i \in E(M)$ for $i, j \in \{1, 2, \dots, n\}$,
i.e.,

$$\begin{array}{rcl} (1+a'_{11})x_1+a'_{12}x_2+\cdots+a'_{1n}x_n & = & (a_{11}+a'_{11})x_1+(a_{12}+a'_{12})x_2+\cdots+(a_{1n}+a'_{1n})x_n+g_1, \\ a'_{21}x_1+(1+a'_{22})x_2+\cdots+a'_{2n}x_n & = & (a_{21}+a'_{21})x_1+(a_{22}+a'_{22})x_2+\cdots+(a_{2n}+a'_{2n})x_n+g_2, \\ \vdots & \vdots & \vdots \\ a'_{n1}x_1+a'_{n2}x_2+\cdots+(1+a'_{nn})x_n & = & (a_{n1}+a'_{n1})x_1+(a_{n2}+a'_{n2})x_2+\cdots+(a_{nn}+a'_{nn})x_n+g_n. \end{array}$$

This implies

$$\begin{array}{rcl} (1+a'_{11})x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n & = & e_1, \\ a'_{21}x_1 + (1+a'_{22})x_2 + \cdots + a'_{2n}x_n & = & e_2, \\ \vdots & \vdots & \vdots \\ a'_{n1}x_1 + a'_{n2}x_2 + \cdots + (1+a'_{nn})x_n & = & e_n, \end{array}$$

where $e_i = (a_{i1} + a'_{i1})x_1 + (a_{i2} + a'_{i2})x_2 + \cdots + (a_{in} + a'_{in})x_n + g_i \in E(M)$, for each $i = 1, 2, \dots, n$.

This implies

$$\begin{pmatrix} 1+a'_{11} & a'_{12} & \dots & a'_{1n} \\ a'_{21} & 1+a'_{22} & \dots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ a'_{n1} & a'_{n2} & \dots & 1+a'_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1+a'_{11} & a'_{12} & \dots & a'_{1n} \\ a'_{21} & 1+a'_{22} & \dots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ a'_{n1} & a'_{n2} & \dots & 1+a'_{nn} \end{pmatrix}$ and $(adjA)A = B = (b_{ij})_{n\times n}$,
where $b_{ii} = \det A$ and $b_{ij} \in E^+(R)$ for all $i \neq j$. Then we have

$$(adjA)A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (ajdA) \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}, \text{ i.e., } B \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (adjA) \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix},$$

i.e., $$\begin{pmatrix} \det A & b_{12} & \dots & b_{1n} \\ b_{21} & \det A & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & \det A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}.$$

This implies

$$\begin{pmatrix} (\det A)x_1 + b_{12}x_2 + \cdots + b_{1n}x_n \\ b_{21}x_1 + (\det A)x_2 + b_{23}x_3 + \cdots + b_{2n}x_n \\ \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn-1}x_{n-1} + (\det A)x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} A_{i1}e_i \\ \sum_{i=1}^{n} A_{i2}e_i \\ \vdots \\ \sum_{i=1}^{n} A_{in}e_i \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},$$

where for $j = 1, 2, \dots, n$; $f_j = \sum_{i=1}^{n} A_{ij}e_i \in E(M)$. Since $E(M)$ is k-set, we must have $(\det A)x_i \in E(M)$, for all $i = 1, 2, \dots, n$. Again, since each $a_{ij} \in \langle a \rangle$ and $\langle a \rangle$ is an ideal, it follows that $a'_{ij} \in \langle a \rangle$ and thus we have $a'_{ij} = r_{ij}a$ for some $r_{ij} \in R$, for all $i, j \in \{1, 2, \dots, n\}$. Therefore, $\det A = (1+a'_{11})(1+a'_{22})\cdots(1+a'_{nn}) + \sum_{\sigma \in S_n \setminus \{id\}} (1')^{(t(\sigma))} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} = 1+ra$ for some $r \in R$. Thus, $(1+ra)aM \subseteq E(M)$. So $(1+ra)a \in (E(M) : M)$. This implies that $(1 + ra)a = r_1$ for some $r_1 \in (E(M) : M)$. Then $a + ra^2 = r_1$, i.e., $a + ra^2 + (ra^2)' = r_1 + (ra^2)'$, i.e., $a = r'a^2 + r_1 \in \langle a^2 \rangle + (E(M) : M)$. Therefore $\langle a \rangle + (E(M) : M) \subseteq \langle a^2 \rangle + (E(M) : M)$ and hence it follows that$\langle a \rangle + (E(M) : M) = \langle a^2 \rangle + (E(M) : M)$.

□

Theorem 3.13 *Suppose M is a finitely generated R-semimodule such that $E(M)$ is a k-set. Then an element a of R is M-vn-regular if and only if $aM + E(M) = eM + E(M)$ for some weak idempotent e of R.*

Proof. First suppose that the element a of R is M-vn-regular. Then by definition, we have $aM + E(M) = a^2M + E(M)$ and so by Lemma 3.12 $\langle a \rangle + (E(M) : M) = \langle a^2 \rangle + (E(M) : M)$. This implies $a = ra^2 + r_1$ for some $r \in R$ and $r_1 \in (E(M) : M)$. Let $e = ar$. Then $e = (ra^2 + r_1)r$, i.e., $e = e^2 + rr_1$. Now $(e + (e^2)')m = (e^2 + (e^2)' + rr_1)m \in E(M)$ for all m. This implies $e + (e^2)' \in (E(M) : M)$, i.e., $e^2 + e' \in (E(M) : M)$ and hence by Lemma 3.2, we have that e is a weak idempotent of R. We now show that $\langle a \rangle + (E(M) : M) = \langle e \rangle + (E(M) : M)$. As $e = ar$, it follows that $\langle e \rangle \subseteq \langle a \rangle$. As $a = ra^2 + r_1$, we have $a = ea + r_1$, so $a \in \langle e \rangle + (E(M) : M)$, and therefore $\langle a \rangle + (E(M) : M) = \langle e \rangle + (E(M) : M)$. Hence $aM + E(M) = eM + E(M)$.

Conversely we assume that $aM + E(M) = eM + E(M)$ for some weak idempotent e of R. Since e is weak idempotent, we have $eM = e^2M$. Clearly,

$a^2M + E(M) \subseteq aM + E(M)$. For the reverse inclusion, let $am + m_1 \in aM + E(M)$ for some $m \in M$ and $m_1 \in E(M)$. Then $am + m_1 \in aM + E(M) = eM + E(M)$ and thus $am + m_1 = em_2 + m_3$ for some $m_2 \in M$ and $m_3 \in E(M)$. Again, $em_2 = em_2 + em_2' + em_2 \in eM + E(M) = aM + E(M)$ implies $em_2 = am_4 + m_5$ for some $m_4 \in M$ and $m_5 \in E(M)$. Therefore, $am + m_1 = em_2 + m_3 = e^2m_2 + m_3 = e(am_4 + m_5) + m_3 = a(em_4) + em_5 + m_3 = a(em_4) + m_6$, where $m_6 = em_5 + m_3 \in E(M)$. Again, $em_4 = em_4 + em_4' + em_4 \in eM + E(M) = aM + E(M)$ implies $em_4 = am_7 + e_8$ for some $m_7 \in M$ and $m_8 \in E(M)$. Thus, $am + m_1 = a(am_7 + e_8) + m_6 = a^2m_7 + ae_8 + m_6 \in a^2M + E(M)$, as $ae_8 + m_6 \in E(M)$. Hence $aM + E(M) \subseteq a^2M + E(M)$ and thus $aM + E(M) = a^2M + E(M)$. Consequently, a is M-vn-regular.

□

Corollary 3.14 *Suppose M is a finitely generated R-semimodule such that $E(M)$ is a k-set. Then an element a of R is M-vn-regular if and only if there exists a weak idempotent e of R such that $aM + E(M) = (1 + 1' + e)M$.*

Lemma 3.15 *Let M be an R-semimodule and e, f be two weak idempotents of R. Then there exists a weak idempotent $g \in R$ such that $eM + fM = gM$.*

Proof. Now by Lemma 3.5, it follows that $g = e + f(1 + e')$ is a weak idempotent of R. We now show that for this g, we have $eM + fM = gM$.

Let $m \in M$. Then

$$\begin{aligned} egm = e(e + f(1 + e'))m &= (e^2 + f(e + (e^2)'))m \\ &= e^2m + f(em + (e^2m)') \\ &= em + f(em + e'm) \\ &= (e + f(e + e'))m \\ &= em \text{ [by condition \textit{(iii)} in Corollary 2.2].} \end{aligned}$$

This implies $eM \subseteq gM$.

Again,

$$\begin{aligned} fgm = f(e + f(1 + e'))m &= fem + f^2m + (f^2em)' \\ &= fem + fm + fe'm \\ &= (f + f(e + e'))m \\ &= fm \text{ [by condition \textit{(iii)} in Corollary 2.2].} \end{aligned}$$

Therefore, $fM \subseteq gM$ and hence $eM + fM \subseteq gM$. The reverse inclusion $gM = (e + f(1 + e'))M \subseteq eM + fM$ follows trivially. Consequently, we have $eM + fM = gM$.

□

Definition 3.16 *Let I be an ideal of a semiring R. Then I defines an equivalence relation ρ_I on R, called the Bourne relation [1], given by $a\, \rho_I\, b$ if and only if there exist elements $i_1, i_2 \in I$ satisfying $a + i_1 = b + i_2$. One can easily check that ρ_I is a congruence relation on R, called Bourne congruence [1]. We denote the semiring of all equivalence classes of elements of R under this* *relation ρ_I by R/I and the equivalence class containing an element $r \in R$ by r/I.*

Lemma 3.17 *Suppose M is a finitely generated vn-regular R-semimodule such that $E(M)$ is a k-set. Then $R/(E(M):M)$ is a vn-regular ring.*

Proof. Since $E(M)$ is a k-set, we can easily prove that $(E(M):M)$ is a k-ideal of R. First we show that $R/(E(M):M)$ is a ring. For this it is enough to show that $R/(E(M):M)$ contains a unique additive idempotent element. Now, $(e/(E(M):M))+(e/(E(M):M)) = e/(E(M):M)$, for any $e/(E(M):M) \in E^+(R/(E(M):M))$. Therefore $(e+e)/(E(M):M) = e/(E(M):M)$, i.e., $e+e+i = e+j$ for some $i,j \in (E(M):M)$. This implies $e+i = e'+e+j$ and thus $e \in (E(M):M)$, as $(E(M):M)$ is a k-ideal of R. Now for any two elements $a/(E(M):M), b/(E(M):M) \in E^+(R/(E(M):M))$, we have $a,b \in (E(M):M)$. Also $a+b = b+a$. This implies $a/(E(M):M) = b/(E(M):M)$ and hence $R/(E(M):M)$ is ring.

We now show that $R/(E(M):M)$ is a vn-regular ring. Let $a \in R$. Since M is finitely generated, it follows that aM is also finitely generated, and hence $aM = \sum_{i=1}^{n} Rx_i$ for some $x_1, x_2, \ldots, x_n \in aM \subseteq M$. Since M is a vn-regular semimodule, for each i, there exists a M-vn-regular element a_i of R such that $Rx_i + E(M) = a_iM + E(M)$. As M is finitely generated semimodule, by Theorem 3.13, for each i, there exists a weak idempotent e_i of R such that $a_iM + E(M) = e_iM + E(M)$. Again by Lemma 3.15, $\sum_{i=1}^{n} Rx_i + E(M) = eM + E(M)$ for some weak idempotent e of R. Thus $aM + E(M) = eM + E(M)$ and hence by Theorem 3.13, it follows that a is M-vn-regular and thus by Lemma 3.12, we have $\langle a \rangle + (E(M):M) = \langle a^2 \rangle + (E(M):M)$. Now, $a = a + a' + a \in \langle a \rangle + (E(M):M) = \langle a^2 \rangle + (E(M):M)$ implies $a = ra^2 + r_1$, for some $r \in R$ and $r_1 \in (E(M):M)$. This implies $a + (a' + a) = a = ra^2 + r_1$ with $a' + a, r_1 \in (E(M):M)$. Therefore, $a/(E(M):M) = a^2r/(E(M):M)$ for some element $r/(E(M):M) \in R/(E(M):M)$. Hence $R/(E(M):M)$ is a vn-regular ring.

□

Definition 3.18 *An R-semimodule M is said to be a multiplication semimodule if for each subsemimodule N of M there exists an ideal I of R such that $N = IM$.*

Theorem 3.19 *An R-semimodule M is a multiplication semimodule if and only if for each $m \in M$ there exists an ideal I of R such that $Rm = IM$.*

Proof. First suppose that an R-semimodule M is a multiplication semimodule. Then for each $m \in M$, we have Rm is a subsemimodule of M and hence there exists an ideal I of R such that $Rm = IM$.

Conversely, suppose that the given condition holds. To show M is a multiplication semimodule, let N be a subsemimodule of M. Then for each

$m \in N$, there exists an ideal I_m of R such that $Rm = I_m M$. Let $I = \sum_{m \in N} I_m$. Then I is an ideal of R such that $N = IM$. Consequently, M is a multiplication semimodule.

□

Lemma 3.20 *Let M be a multiplication R-semimodule such that $E(M)$ is a k-set. If $R/(E(M) : M)$ is a vn-regular ring, then $M$is a vn-regular semimodule.*

Proof. Let $a \in R$. Then $a/(E(M) : M) \in R/(E(M) : M)$. Since $R/(E(M) : M)$ is a vn-regular ring, so there exists an element $x \in R$ such that $a/(E(M) : M) = (a^2/(E(M) : M))(x/(E(M) : M))$. So there exist $p, q \in (E(M) : M)$ such that $a + p = a^2x + q$. Then $a + (a^2x)' + p = (a^2 + (a^2)')x + q \in (E(M) : M)$. As $(E(M) : M)$ is a k-ideal of R, it follows that $a + (a^2x)' \in (E(M) : M)$. So $a + (a^2x)' = r_1$ for some $r_1 \in (E(M) : M)$. Therefore, $a + (a^2x)' + a^2x = r_1 + a^2x$. Then $a = r_1 + a^2x$. Now if we take $e = ax$, then clearly $aM + E(M) = eM + E(M)$. Also e is weak idempotent of R. Then a is a M-vn-regular element of R. Let $m \in M$. Since M is a multiplication semimodule, there exists an ideal I of R such that $Rm = IM$. Let $r \in I$. Then for any $m_1 \in M$, we have $rm_1 \in IM = Rm$ and thus $r \in (Rm : M)$. This implies $I \subseteq (Rm : M)$. Hence $Rm = IM \subseteq (Rm : M)M \subseteq Rm$ and therefore $Rm = (Rm : M)M$. Now $m \in Rm = (Rm : M)M$ implies $m = \sum_{i=1}^{k} a_i m_i$, where $a_i \in (Rm : M)$ and $m_i \in M$ for all $i = 1, 2, \ldots, k$. Let $J = \langle a_1, a_2, \ldots, a_k \rangle$. Then J is a finitely generated ideal of R such that $J \subseteq (Rm : M)$. Moreover, it is easy to verify that $Rm = JM$. Since J is finitely generated and for each $a \in R$, $aM + E(M) = eM + E(M)$ for some weak idempotent e of R, therefore by Lemma 3.15, we have $Rm + E(M) = JM + E(M) = fM + E(M)$ for some weak idempotent f of R. Consequently, M is a vn-regular semimodule.

□

Definition 3.21 *A subsemimodule N of a semimodule M is called full if $E(M) \subseteq N$. A subsemimodule N is said to be M-cyclic if $N = aM$ for some $a \in R$ and in this case we say that the subsemimodule N is an M-cyclic subsemimodule of M generated by the element a of R. An M-cyclic subsemimodule of an R-semimodule M of the form $E(M) + aM$ is called a full M-cyclic subsemimodule of M and in this case we say that $E(M) + aM$ is a full M-cyclic subsemimodule of M generated by the element a of R.*

Lemma 3.22 *$E(M)$ is a full M-cyclic subsemimodule of M.*

Proof. Since M is an inverse R-semimodule, we have $E(M) + (1 + 1')M \subseteq E(M)$. Let $m \in E(M)$. Now $m = 1 \cdot m = (1 + 1' + 1)m = m + (1' + 1)m \in E(M) + (1 + 1')M$. Hence $E(M) = E(M) + (1 + 1')M$ and consequently $E(M)$ is a full M-cyclic subsemimodule.

□

Remark 3.2 We note that for any $m \in E(M)$, we have $m = 1 \cdot m = (1 + g + g')m = m + gm = (1 + g)m \in (1 + g)M$, where $g \in R$. Thus any M-cyclic subsemimodule of the form $(1 + g)M$ is a full subsemimodule of M. Hence, in particular, for any weak idempotent e of R, $(1 + e)M$ is also a full subsemimodule of M. In this connection we have the following result.

Theorem 3.23 *Let e be a weak idempotent of R. Then the M-cyclic subsemimodule eM is full M-cyclic if and only if $eM = (1 + f')M$ for some weak idempotent f of R.*

Proof. First suppose that eM is a full M-cyclic subsemimodule. Since the element e is a weak idempotent of R, by Lemma 3.5, we have $(1+e')$ is also a weak idempotent of R. Let $f = 1 + e'$. Now, $1 + f' = 1 + 1' + e$ implies $(1 + f')M = (1 + 1' + e)M \subseteq E(M) + eM \subseteq eM$. Also, for any $m \in M$, we have $em = (e+e'+e)m = (e+e')m+e^2m = (1+1'+e)em = (1+f')em \in (1 + f')M$. Thus $eM \subseteq (1 + f')M$. Consequently, $eM = (1 + f')M$.

Converse part follows from Remark 3.2.

□

Lemma 3.24 *In an R-semimodule M, the M-cyclic subsemimodules $(1 + e')M$ and $(1 + 1' + e)M$ are complement to each other, where e is a weak idempotent of R.*

Proof. Now for any $m \in M$, we have $m = 1 \cdot m = (1 + 1' + 1 + e + e')m = (1 + e')m + (1 + 1' + e)m \in (1 + e')M + (1 + 1' + e)M$ and thus $(1 + e')M + (1 + 1' + e)M = M$. Now, if $m_1 \in (1 + e')M \cap (1 + 1' + e)M$, then $m_1 = (1 + e')m_2 = (1 + 1' + e)m_3$ for some $m_2, m_3 \in M$. Then $m_1 = (1+e')m_1 = (1+1'+e)m_1$. Now $m_1 = (1+1'+e)m_1 = m_1+(1'+e)m_1 = m_1+m_1' \in E(M)$. Thus, $(1+e')M\cap(1+1'+e)M = E(M)$ and hence $(1+e')M$ and $(1 + 1' + e)M$ are complement to each other.

□

Combining Lemma 3.7 and Lemma 3.24, we at once have the following theorem.

Theorem 3.25 *Let M be a colon distributive R-semimodule. Then two subsemimodules N and K of M are complement to each other if and only if there exists a weak idempotent e of R such that $N = (1 + 1' + e)M$ and $K = (1 + e')M$.*

Combining Theorem 3.13 and Theorem 3.25, we have the following corollary.

Corollary 3.26 *Let M be a colon distributive R-semimodule. The following statements are equivalent :*

(1) M is vn-regular,

(2) For every $m \in M$, there exists a weak idempotent e of R such that $Rm + E(M) = (1 + 1' + e)M$,

(3) For every $m \in M$, the subsemimodule $Rm + E(M)$ has a complement.

Theorem 3.27 *Let M be an R-semimodule and a, b be two M-vn-regular elements of R. Then the sum of two full M-cyclic subsemimodules $E(M)+aM$ and $E(M)+bM$ is again a full M-cyclic subsemimodule of M.*

Proof. Now for M-vn-regular elements $a, b \in R$, there exist weak idempotent elements $e, f \in R$ such that $E(M)+aM = E(M)+eM$ and $E(M)+bM = E(M)+fM$. Then by Lemma 3.15, we have $(E(M)+aM)+(E(M)+bM) = (E(M)+eM)+(E(M)+fM) = E(M)+eM+fM = E(M)+(e+f(1+e'))M$, which is a full M-cyclic subsemimodule of M.

□

Lemma 3.28 *Let M be an R-semimodule and e be a weak idempotent element of R. Then $eM \cap aM = eaM$ for all $a \in R$.*

Proof. Clearly, $eaM \subseteq eM \cap aM$. For the reverse inclusion, let $x \in eM \cap aM$. Then $x = em = am_1$ for some $m, m_1 \in M$. Now, $ex = e^2m = em = x$ implies $x = ex = eam_1 \in eaM$. Therefore, $eM \cap aM \subseteq eaM$. Hence $eM \cap aM = eaM$.

□

Theorem 3.29 *Let M be an R-semimodule and a, b be two M-vn-regular elements of R. Then the intersection of two full M-cyclic subsemimodules $E(M)+aM$ and $E(M)+bM$ is again a full M-cyclic subsemimodule of M.*

Proof. Now for M-vn-regular elements a, b of R, there exist weak idempotent elements e, f of R such that $E(M)+aM = (1+1'+e)M$ and $E(M)+bM = (1+1'+f)M$. Then by Lemma 3.28, we have $(E(M)+aM)\cap(E(M)+bM) = (1+1'+e)M \cap (1+1'+f)M = (1+1'+ef)M = E(M)+efM$, which is a full M-cyclic subsemimodule of M.

□

Combining Lemma 3.6, Theorem 3.27 and Theorem 3.29, we have the following result.

Theorem 3.30 *The set $\widehat{\mathscr{L}}(M)$ of all full M-cyclic subsemimodules of M, generated by M-vn-regular elements of R, is a complemented modular lattice, partially ordered by the set inclusion relation, the meet being $\cap$ and the join is the sum of two full M-cyclic subsemimodules of M, its least element is $E(M)$ and the greatest element is M.*

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