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Unsteady multiple boundary layers on a porous plate in a rotating system

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An initial value investigation is made of the motion of an incompressible, homogeneous, viscous fluid bounded by a porous plate with uniform suction or blowing. Both the plate and the fluid are in a state of solid body rotation with constant angular velocity about the z axis normal to the plate, and additionally a nontorsional oscillation of a given frequency is superimposed on the plate for the generation of an unsteady flow in the rotating system. By using the Laplace transform technique, an exact solution of the three dimensional Navier–Stokes equations for unsteady flow is obtained. The structure of the associated multiple boundary layers is determined. Effects of uniform suction (or blowing) and rotation on the flow phenomena are analyzed. Several known results of interest are found to follow as particular cases of the solution of the problem considered.

I. INTRODUCTION

In a recent paper, Gupta[1] obtained an exact solution of the steady three-dimensional Navier–Stokes equations for the flow past a plate with uniform suction in a rotating coordinate system. He has investigated the structure of the steady velocity field and the associated boundary layer on the porous plate. It has been shown that the thickness of the boundary layer becomes thinner due to the presence of suction. Although in an inertial coordinate frame an asymptotic solution does not exist for steady flow past a porous plate with uniform blowing, a steady solution has been found for the same configuration in the rotating coordinate system. It has been argued that rotation is entirely responsible for the existence of a solution for the case of blowing.

With the view of making an extension of Gupta's work, this paper is concerned with an unsteady boundary layer flow generated in an incompressible homogeneous viscous fluid bounded by an infinite porous plate with uniform suction or blowing. Both the fluid and the plate are in a state of solid body rotation with constant angular velocity Ω about the z axis normal to the plate. The unsteady flow is induced in the fluid by nontorsional oscillations of the plate in its own plane with a fixed frequency ω. It is of interest to determine the structure of the unsteady velocity field and the associated multiple boundary layers adjacent to the plate. The initial value problem is solved in order to determine the significant effects of rotation, suction, or blowing. Several known results of interest are found to follow as particular cases of this analysis.

II. MATHEMATICAL FORMULATION

We consider the unsteady flow generated in a semi-infinite expanse of fluid bounded by an infinite porous plate at $z=0$ subjected to uniform suction or blowing. Both the fluid and the plate are in a state of solid body rotation with constant angular velocity Ω about the z axis normal to the plate and additionally, a nontorsional oscillation of frequency ω is imposed on the plate in its own plane at time $t>0$.

The unsteady motion of the viscous fluid in this rotating coordinate system is governed by the Navier–Stokes equations and the continuity equation in the usual notation

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}+2\Omega\mathbf{k}\times\mathbf{u}=-\rho^{-1}\nabla p+\nu\nabla^2\mathbf{u}, \quad (1)$$

$$\operatorname{div}\mathbf{u}=0, \quad (2)$$

where $\mathbf{u}=(u, v, w)$ is the velocity vector, $\mathbf{k}$ is the unit vector along the z axis, p is the pressure including the centrifugal term, ρ is the density, and ν is the kinematic viscosity of the fluid.

We assume the velocity field depends on z and t alone so that

$$\mathbf{u}(z,t)=[u(z,t), v(z,t), w(z,t)]. \quad (3)$$

It follows from (2), together with uniform suction or blowing, that $w=-w_0$ is a constant. Obviously, $w_0>0$ for suction and $w_0<0$ for blowing.

In the absence of the pressure gradient, the equations of motion (1) take the form

$$\frac{\partial u}{\partial t}-w_0\frac{\partial u}{\partial z}-2\Omega v=\nu\frac{\partial^2 u}{\partial z^2}, \quad (4)$$

$$\frac{\partial v}{\partial t}-w_0\frac{\partial v}{\partial z}+2\Omega u=\nu\frac{\partial^2 v}{\partial z^2}. \quad (5)$$

Introducting the notation $q=u+iv$, Eqs. (4) and (5) can be combined into a single equation

$$\frac{\partial q}{\partial t}-w_0\frac{\partial q}{\partial z}+2i\Omega q=\nu\frac{\partial^2 q}{\partial z^2}. \quad (6)$$

In view of the imposed oscillations on the plate, Eq. (6) has to be solved subject to a no-slip condition at the plate and no disturbance at infinity as

$$q(z,t)=-U+U^*(ae^{i\omega t}+be^{-i\omega t}) \qquad \text{on } z=0,\ t>0, \quad (7)$$

$$q(z,t)\to 0 \qquad \text{as } z\to\infty,\ t>0, \quad (8)$$

where U and U^* are constants with the dimension of velocity, and a, b, are complex constants so that the real and the imaginary parts of $q(z,t)$ become real on the plate.

The initial condition of the problem is

$$q(z,t)=0, \quad (9)$$

at $t\leq 0$ for all z.

III. THE SOLUTION OF THE PROBLEM

It is convenient to introduce the nondimensional variables

$$z'=zU^*/\nu, \qquad t'=\Omega t, \qquad q'=q/U^*, \quad (10)$$

and the nondimensional parameters

$$S=w_0/U^*, \qquad E=2\Omega\nu/U^{*2}, \qquad \sigma=\omega/\Omega. \quad (11)$$

In terms of these nondimensional quantities, Eq. (6) and the boundary and initial conditions (7)–(9) become, on dropping the primes,

$$\frac{\partial^2 q}{\partial z^2}+S\frac{\partial q}{\partial z}-iEq=\tfrac{1}{2}E\frac{\partial q}{\partial t}, \qquad z>0, \quad (12)$$

$$q=(ae^{i\sigma t}+be^{-i\sigma t})-U/U^* \qquad \text{on } z=0,\ t>0, \quad (13)$$

$$q\to 0 \qquad \text{as } z\to\infty,\ t>0, \quad (14)$$

$$q=0 \text{ at } t\leq 0 \qquad \text{for all } z>0. \quad (15)$$

In order to solve the initial value problem, we introduce the Laplace transform $\bar{q}(z,p)$ of $q(z,t)$ defined by the integral

$$\bar{q}(z,p)=\int_0^\infty e^{-pt}q(z,t)\,dt. \quad (16)$$

In view of this transformation, Eq. (3) reduces to a second-order ordinary differential equation which can readily be solved subject to the transformed boundary conditions. The solution for the transformed velocity field is obtained as

$$\bar{q}(z,p)=\left(\frac{a}{p-i\sigma}+\frac{b}{p+i\sigma}-\frac{U}{U^*}p^{-1}\right)\times\exp\{-\tfrac{1}{2}z[S+(S^2+4iE+2pE)^{1/2}]\}. \quad (17)$$

Using the table of the inverse Laplace transform due to Campbell and Foster,[2] the exact solution for $q(z,t)$ is given by

$$\begin{aligned}
q(z,t)=\frac{a}{2}&\exp[i\sigma t-\tfrac{1}{2}(Sz)]\\
&\times\left\{\exp\left[z(\tfrac{1}{2}E)^{1/2}\left(i\sigma+\frac{S^2+4iE}{2E}\right)^{1/2}\right]\right.\\
&\times\operatorname{erfc}\left[\frac{z}{2}\left(\frac{E}{2t}\right)^{1/2}+\left(i\sigma+\frac{S^2+4iE}{2E}\right)^{1/2}t^{1/2}\right]\\
&+\exp\left[-z(\tfrac{1}{2}E)^{1/2}\left(i\sigma+\frac{S^2+4iE}{2E}\right)^{1/2}\right]\\
&\left.\times\operatorname{erfc}\left[\tfrac{1}{2}z\left(\frac{E}{2t}\right)^{1/2}-\left(i\sigma+\frac{S^2+4iE}{2E}\right)^{1/2}t^{1/2}\right]\right\}\\
&+\tfrac{1}{2}b\exp[-i\sigma-\tfrac{1}{2}(Sz)]\\
&\times\left\{\exp\left[z(\tfrac{1}{2}E)^{1/2}\left(\frac{S^2+4iE}{2E}-i\sigma\right)^{1/2}\right]\right.\\
&\times\operatorname{erfc}\left[\tfrac{1}{2}z\left(\frac{E}{2t}\right)^{1/2}+\left(\frac{S^2+4iE}{2E}-i\sigma\right)^{1/2}t^{1/2}\right]\\
&+\exp\left[-z(\tfrac{1}{2}E)^{1/2}\left(\frac{S^2+4iE}{2E}-i\sigma\right)^{1/2}\right]\\
&\left.\times\operatorname{erfc}\left[\tfrac{1}{2}z\left(\frac{E}{2t}\right)^{1/2}-\left(\frac{S^2+4iE}{2E}-i\sigma\right)^{1/2}t^{1/2}\right]\right\}\\
&-\frac{U}{2U^*}e^{-Sz/2}\left\{\exp\left[-z(\tfrac{1}{2}E)^{1/2}\left(\frac{S^2+4iE}{2E}\right)^{1/2}\right]\right.\\
&\times\operatorname{erfc}\left[\tfrac{1}{2}z\left(\frac{E}{2t}\right)^{1/2}-\left(\frac{S^2+4iE}{2E}\right)^{1/2}t^{1/2}\right]\\
&+\exp\left[z(\tfrac{1}{2}E)^{1/2}\left(\frac{S^2+4iE}{2E}\right)^{1/2}\right]\\
&\left.\times\operatorname{erfc}\left[\tfrac{1}{2}z\left(\frac{E}{2t}\right)^{1/2}+\left(\frac{S^2+4iE}{2E}\right)^{1/2}t^{1/2}\right]\right\}, \quad (18)
\end{aligned}$$

where $\operatorname{erfc}(x)$ is the complementary error function[3] defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-\tau^2}\, d\tau. \qquad (19)$$

This solution describes a general feature of the unsteady boundary layer flows in a rotating fluid including the effects of uniform suction or blowing according as $S>0$ or $S<0$.

When $U=0$ and $S=0$, the velocity distribution (18) is exactly identical to that of Thornley.[4] She pointed out certain inherent difficulties of the problem and its solution related to the resonant frequency, $\sigma \to 2$ ($\omega \to 2\Omega$). First, it has been found that one of the two boundary layers formed on the plate grows without any limit as $\omega \to 2\Omega$. Second, the solution of the unsteady problem depends on the order of the double limit operation, $t \to \infty\, \sigma \to 2$.

In a recent work on the Ekman boundary layers, Faller and Kaylor[5] have also reported the resonant behavior of the solution at $\omega = 2\Omega$. Their numerical calculation exhibits the steady amplification of the surface velocity due to this resonance. It can readily be verified that unless $S=0$,

$$\lim_{t\to\infty} \lim_{\sigma\to 2} q(z,t) = \lim_{\sigma\to 2} \lim_{t\to\infty} q(z,t), \qquad (20)$$

where $q(z,t)$ is given by (18).

The significant conclusion of this analysis is that the suction is entirely responsible for the resolution of the above difficulties. We next turn our attention to the steady-state solution and its characteristic features.

IV. THE STEADY-STATE SOLUTION AND THE STRUCTURE OF THE BOUNDARY LAYERS

Using the asymptotic representation of the complementary error function[3] for $t\to\infty$, solution (18) reduces to its steady-state form

$$\begin{aligned} q(z,t) &\sim a \exp\left\{ i\sigma t - z\left[\tfrac{1}{2}S + (\tfrac{1}{2}E)^{1/2} \times \left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right]\right\} \\ &+ b \exp\left\{ -i\sigma t - z\left[\tfrac{1}{2}S - (\tfrac{1}{2}E)^{1/2} \left(\frac{S^2+4iE}{2E} - i\sigma\right)^{1/2}\right]\right\} \\ &- \left(\frac{U}{U^*}\right) \exp\left\{ -z\left[\tfrac{1}{2}S + (\tfrac{1}{2}E)^{1/2} \left(\frac{S^2+4iE}{2E}\right)^{1/2}\right]\right\}, \qquad (21) \\ &= a \exp[i\sigma t - z(\alpha_1 + i\beta_1)] \\ &\quad + b \exp[-i\sigma t - z(\alpha_2 + i\beta_2)] \\ &\quad - (U/U^*) \exp[-z(\alpha_3 + i\beta_3)], \qquad (22) \end{aligned}$$

where

$$\alpha_1 = \tfrac{1}{2}S + (2\sqrt{2})^{-1}\{[S^4 + 4E^2(2+\sigma)^2]^{1/2} + S^2\}^{1/2}, \qquad (23)$$

$$\beta_1 = (2\sqrt{2})^{-1}\{[S^4 + 4E^2(2+\sigma)^2]^{1/2} - S^2\}^{1/2}, \qquad (24)$$

$$\alpha_2 = \tfrac{1}{2}S + (2\sqrt{2})^{-1}[S^4 + 4E^2(2-\sigma)^2]^{1/2} + S^2]^{1/2}, \qquad (25)$$

$$\beta_2 = (2\sqrt{2})^{-1}\{[S^4 + 4E^2(2-\sigma)^2]^{1/2} - S^2\}^{1/2}, \qquad (26)$$

$$\alpha_3 = \tfrac{1}{2}S + (2\sqrt{2})^{-1}[(S^4 + 16E^2)^{1/2} + S^2]^{1/2}, \qquad (27)$$

and

$$\beta_3 = (2\sqrt{2})^{-1}[(S^4 + 16E^2)^{1/2} - S^2]^{1/2}. \qquad (28)$$

Evidently, solution (21) or (22) represents the steady-state solution and indicates the existence of three distinct boundary layers of thicknesses of order $\nu/U^*\alpha_1$, $\nu/U^*\alpha_2$, and $\nu/U^*\alpha_3$ with $\alpha_1 > \alpha_3 > \alpha_2$. It is also clear that the steady-state solution and the associated boundary layers are modified due to the presence of uniform suction or blowing. Further, the thickness of the boundary layers decreases with an increase in the suction parameter S, and remains bounded for all values of the frequency of the imposed oscillations. When $\sigma=0$, $\alpha_1=\alpha_2=\alpha_3$ and the three boundary layers coalesce into a single layer of thickness $\nu/U^*\alpha_1$. In particular, when $a=b=0$, solution (22) reduces to the form

$$q(z,t) = -(U/U^*) \exp[-z(\alpha_3 + i\beta_3)]. \qquad (29)$$

In addition to uniform rotation of the system, if we now impose a constant velocity U in the positive x direction on both the plate and the fluid, then the plate comes to rest while the fluid moves with velocity U in the positive direction and the relative motion remains unchanged. Consequently, solution (29) assumes the dimensional form

$$q(z,t) = U\{1 - \exp[-z(\alpha_3 + i\beta_3)]\}. \qquad (30)$$

This corresponds to Gupta's result which he obtained independently from a steady analysis of the problem. He has also discussed some properties of the steady solution (30), so any further discussion here would be redundant.

Finally, it may be inferred from this analysis that in a rotating system if the fluid has a uniform stream in the positive x direction and the plate starts oscillating with a given frequency ω, then in addition to the boundary layer of thickness $\nu/U^*\alpha_3$ predicted by Gupta, two more oscillating boundary layers are found to occur with thicknesses of order $\nu/U^*\alpha_1$ and $\nu/U^*\alpha_2$ with $\nu/U^*\alpha_1 \le \nu/U^*\alpha_3 < \nu/U^*\alpha_2$.

V. THE STEADY SOLUTION FOR THE CASE OF BLOWING

We recall (22) and replace S by $-S_1$ so that $S_1>0$. The result (22) reduces to the form

$$q(z,t) = a \exp[i\sigma t + z(\alpha_1' - i\beta_1')] + b \exp[-i\sigma t + z(\alpha_2' - i\beta_2')] - (U/U^*) \exp[z(\alpha_3' - i\beta_3')], \quad (31)$$

where $\alpha_r'(r=1, 2, 3)$ are all negative and

$$\alpha_1' = \tfrac{1}{2}S_1 - (2\sqrt{2})^{-1}\{[S_1^4 + 4E^2(2+\sigma)^2]^{1/2} + S_1^2\}^{1/2}, \quad (32)$$

$$\alpha_2' = \tfrac{1}{2}S_1 - (2\sqrt{2})^{-1}\{[S_1^4 + 4E^2(2-\sigma)^2]^{1/2} + S_1^2\}^{1/2}, \quad (33)$$

$$\alpha_3' = \tfrac{1}{2}S_1 - (2\sqrt{2})^{-1}[(S_1^4 + 16E^2)^{1/2} + S_1^2]^{1/2}, \quad (34)$$

and $\beta_r'(r=1, 2, 3)$ are equal to β_r, respectively, with S replaced by $-S_1$.

The velocity field (31) describes the general features of the boundary layer flow in a rotating system with uniform blowing. An argument similar to that advanced earlier leads us to conclude that (31) also includes Gupta's solution for the case of blowing as a special case.

Although in an inertial system, a steady asymptotic solution does not exist for flow past a porous plate subjected to uniform blowing, but in a rotating frame, a solution exists and is given by (31). The present unsteady analysis confirms the existence of the steady asymptotic solution for the case of blowing in a rotating system. Gupta has explained physical mechanisms for the existence of a solution in a rotating system and the nonexistence of a solution in an inertial system. Any further discussion on this point seems to be superfluous.

Finally, in the absence of rotation ($E=0$), $\alpha_1' = \alpha_2' = \alpha_3' = 0$, so that the boundary layers associated with solution (31) become infinitely deep. These findings are in agreement with those of Faller and Kaylor.[5]

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