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Uniform and Nonuniform Estimates in the CLT for Banach Valued Dependent Random Variables

A. K. BASU

University of Calcutta, Calcutta, India

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A uniform estimate of the rate of convergence in the central limit theorem (CLT) in certain Banach spaces for dependent random variables is established when the Gaussian measure of the ε-neighbourhood of the boundary of a set is proportional to ε and the third order moment is finite in the strong sense. A uniform estimate in the CLT for Banach valued dependent random variables is carried out when the B-space is well behaved for a martingale transform.

1. INTRODUCTION

Throughout the paper B will denote a real separable Banach space with norm $\|\cdot\|$ and B^* the conjugate space. We shall assume that B satisfies the following condition. The norm $\|\cdot\|$ of B as a function from $B-\{0\}$ to R is three times continuously Fréchet-differentiable, and its differentials satisfy

$$\max\{\|D_x^1\|,\ \|D_x^2\|,\ \|D_x^3\|: \|x\|=1\}=C<\infty, \tag{1.1}$$

where D_x^i denotes the differential of order i of $\|\cdot\|$.

Let $(\Omega, \mathscr{F}, P\}$ be a fixed probability space. A B-valued random variable X is a Bochner measurable map from Ω to B. We denote by L_B^p the set of B-valued random variable X such that $\|X\|^p$ $(1 \leqslant p < \infty)$ is integrable. Throughout the paper we consider random variables (r.v.'s) having third order moment in the strong sense (i.e., $E\|X\|^3 < \infty$); this condition ensures the existence of the expectation and covariance operators, which are defined in the following way:

$$a = EX \text{ if } f(a) = Ef(X) \qquad \text{for all} \quad f \in B^*.$$

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The covariance operator T, being a bounded linear symmetric and non-negative operator mapping B^* into the second dual space B^{**}, is defined by the formula (assuming that $EX=0$)

$$(Tf)(g)=E(f(X)\,g(X)), \qquad f, g \in B^*.$$

We denote by $\mu(0, T)$ the Gaussian distribution in B with zero mean and covariance operator T. A B-valued r.v. Y is said to be Gaussian if for each $f \in B^*$, $f \circ Y$ is a real-valued Gaussian random variable. We denote by C absolute constants, and by $C(\cdot)$ constants depending on the arguments in parentheses.

Let

$$S_r(a)=\{x \in B: \|x-a\|<r\}$$

and

$$S_{r,\varepsilon}(a)=\{x \in B: r \leqslant \|x-a\|<r+\varepsilon\}.$$

It is known that if B is a Hilbert space, then the Gaussian measure $\mu=\mu(0, T)$ satisfies the following condition:

For all $\varepsilon>0$, $r \geqslant 0$ and $a \in B$, there is a constant $C_0(\mu, B)$ such that

$$\mu(S_{r,\varepsilon}(a)) \leqslant C_0(\mu, B)(1+\|a\|^2)\,\varepsilon \tag{1.2}$$

and

$$C_0(\mu_\alpha, B) \leqslant \alpha^{-C_{11}(B)} C_0(\mu, B), \qquad \text{where } \mu_\alpha=\mu(0, T\alpha),\ 0<\alpha \leqslant 1, \tag{1.3}$$

and $C_{11}(B)$ is a positive constant depending only on B. Inequalities (1.2) are not true in general for an arbitrary Banach space but they are true for some of the L^p spaces $(1<p<\infty)$. The condition (1.1) implies that B is of type 2 and that there exists a Gaussian r.v. Y which has the same covariance as $S=\sum_{i \leqslant n} X_i$ if $\{X_i\}$ is in L_B^2. Paulauskas [15] obtained bounds of $\Delta_n(a)=\sup_{r>0}|P(\|S-a\| \leqslant r)-P(\|Y\| \leqslant r)|$ under the hypotheses (1.1) and (1.2), where $\{X_i\}$ are zero mean independent r.v.'s in L_B^3. This result was generalised for m-dependent sequences $\{X_i\}$ by Rhee and Talagrand [16] in the special case $a=0$. We generalise Paulaskas's result for martingale sequences. It has been pointed out by the referee that Götze [7] has recently obtained much sharper results (namely $O(n^{-1/2})$) for i.i.d. cases through Edgeworth expansions.

Paulaskas [15] obtained a nonuniform bound for independent Hilbert space valued r.v.'s. We obtain a nonuniform bound in B-space when the random variables are dependent. Let $B_n > 0$ be a normalizing constant; we may take $B_n^2 = \sum_{i \leqslant n} E\,\|X_i\|^2$ and denote $\bar{T}_n = B_n^{-2} \sum_{i \leqslant n} T_i$. Let $\mu_i = \mu(0, T_i)$ and F_i be the distribution of a B-valued r.v. X_i.

2. Some Preliminary Lemmas

The results of this section are either well known or easy and hence are stated without proof.

Lemma 1. *For $x \in B$, $x \neq 0$, $\lambda \neq 0$. We have $D_{\lambda x} = D_x$, $D^2_{\lambda x} = \lambda^{-1} D^2_x$, $D^3_{\lambda x} = \lambda^{-2} D^3_x$. Hence $\|D_x\| \leqslant C$, $\|D^2_x\| \leqslant C\,\|x\|^{-1}$, $\|D^3_x\| \leqslant C\,\|x\|^{-2}$.*

Lemma 2. *If B is of type 2 with constant C then for all independent B-valued zero mean r.v.'s $X_1, X_2, \ldots, X_n$ which are in L^2_B, $E\,\|\sum X_i\|^2 \leqslant C \sum E\,\|X_i\|^2$.*

Lemma 3. *There exists a constant C_1 such that for $\varepsilon > 0$, $r > 0$, there exists $f\colon R \to [0, 1]$, $f(x) = 0$ if $x \leqslant r$, $f(x) = 1$ if $x \geqslant r + \varepsilon$, f is three times continuously differentiable, and $\|f^{(3)}\|_\infty \leqslant C_1 \varepsilon^{-3}$.*

Lemma 4. *Suppose $f\colon R \to R$ is three times continuously differentiable and $f(x) = 0$ if $x \leqslant 0$.*

Let x, $y \in B$, $h(\lambda) = f(\|x + \lambda y\|)$. Then h is three times continuously differentiable. If $v = x + \lambda y = 0$,

$$h(\lambda) = h'(\lambda) = h''(\lambda) = h^{(3)}(\lambda) = 0.$$

If $\|v\| = \|x + \lambda y\| \neq 0$,

$$\begin{aligned} h'(\lambda) &= D_v(y)\, f'(\|x + \lambda y)\|) \\ h''(\lambda) &= (D_v(y))^2 f''(\|v\|) + D_v^2(y)^2 f'(\|v\|) \\ h^{(3)}(\lambda) &= (D_v(y))^3 f^{(3)}(\|v\|) + 3 D_v(y)\, D_v^2(y)^2 f''(\|v\|) \\ &\quad + D_v^3(y)^3 f'(\|v\|). \end{aligned}$$

Lemma 5 (Fernique [6]). *There exists a constant C_2 such that for all B-valued Gaussian r.v.'s Y one has*

$$P(\|Y\| \geqslant t) \leqslant \exp(-t^2/C_2 E\,\|Y\|^2)$$

and

$$E\,\|Y\|^p \leqslant C_p (E\,\|Y\|^2)^{p/2} \quad \text{for} \quad p \geqslant 2. \tag{2.0}$$

Let $P(B)$ be the set of all probability measures on $(B, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-field of B. Let us denote the set of Gaussian measures on B by $G(B)$. We provide $G(B)$ with the weakest topology that makes evaluation or the projection map in the space of Gaussian measures continuous bounded on B. A map $w \to u_w$ from Ω into $G(B)$ is said to be measurable if for f continuous and bounded on B the map $w \to u_w(f) = \int f \, d\mu_w$ is measurable (see, e.g., [13, p. 40]).

LEMMA 6. *Let $\mathscr{F}^*$ be a countably generated σ-field of $\mathscr{F}$ and denote by E^* the expectation with respect to $\mathscr{F}^*$. Let $X \in L_B^2$. Then there exists an $\mathscr{F}^*$ measurable map $w \to \mu_w \in G(B)$ such that for each $f, g \in B^*$*

$$E^*(f(X)\, g(X))(w) = \int f(X)\, g(X)\, d\mu_w(x) \tag{2.1}$$

and for each $p \geqslant 2$, if $X \in L_B^p$,

$$\int \left(\int \|x\|^p \, d\mu_w(x) \right) dP(w) \leqslant C_p E \|X\|^p. \tag{2.2}$$

Proof. The proof depends on existence of regular random measure in B-space by choosing a regular version of the conditional distribution, i.e., $\mu: B \times \Omega \to [0, 1]$ is a function such that for every $w \in \Omega$, $\mu_w(\cdot)$ is a probability measure on $\mathscr{B}$ and for every $A \in \mathscr{B}$, $\mu_{(\cdot)}(A)$ is a version of $P(X \in A \mid \mathscr{F}^*)$ and hence is $\mathscr{F}^*$-measurable (see [9]). The rest follows from the properties of conditional expectation.

3. MAIN RESULTS

Let $\{\mathscr{F}_i\}$ be an increasing sequence of σ-algebras of $\mathscr{F}$. For a B-valued r.v. or a real r.v. X, we denote conditional expectation with respect to $\mathscr{F}_i$ by E^i. Let Y be Gaussian with the same covariance as $S = \sum_{i \leqslant n} X_i$. Let X_i be $\mathscr{F}_i$-measurable.

Suppose also that

$$E^{i-1}(X_i) = 0 \qquad \text{for } i \geqslant 2. \tag{3.1}$$

For all $f, g \in B^*$,

$$\sum_{i \leqslant n} E^{i-1}(f(X_i)\, g(X_i)) \text{ is a constant a.s.} \tag{3.2}$$

LEMMA 7. *Under conditions* (3.1) *and* (3.2) *there exists a probability space $(\Omega^*, \mathscr{F}_*, P^*)$, sub-$\sigma$-algebras $\mathscr{F}_1^*, ..., \mathscr{F}_n^*$ of $\mathscr{F}_*$, B-valued r.v.'s X_1^*,*

$X_2^*, \ldots, X_n^*$, Y_1^*, $Y_2^*, \ldots, Y_n^*$ on Ω^* such that the following conditions are satisfied if we denote by F^i the conditional expectation with respect to $\mathscr{F}_i^*$.

$$\text{The law of } (X_1^*, X_2^*, \ldots, X_n^*) \text{ is the law of } (X_1, X_2, \ldots, X_n). \tag{3.3}$$

$$X_i^* \text{ is } \mathscr{F}_l^* \text{ measurable for } i \leqslant l-1, \tag{3.4}$$

$$F^i(X_i^*) = F^i(Y_i^*) = 0. \tag{3.5}$$

For each $k \leqslant n$,

$$F^k(f(X_k^*)\, g(X_k^*)) = F^k(f(Y_k^*)\, g(Y_k^*)). \tag{3.6}$$

Let $X^* = \sum_{k \leqslant n} X_k^*$ and $Y^* = \sum_{k \leqslant n} Y_k^*$; then Y^* is Gaussian with the same covariance as X^*.

For $l \geqslant 2$ and $k \leqslant n$,

$$E\, \|Y_k^*\|^l \leqslant C_l E\, \|X_k^*\|^l, \qquad \text{where } C_l \text{ is a constant depending on } l \text{ only.} \tag{3.7}$$

Proof. The proof is based on an idea of Dvoretzky [4, 5] and its corrected version provided by Klopotowski [11, 12] (who uses increasing sequences of σ-fields $\mathscr{F}_{n1} \supset \mathscr{F}_{n2} \cdots \mathscr{F}_*$ and adapts $X_{n1}, X_{n2},$ to them). Let

$$\Omega^* = \Omega \times \overbrace{B \times B \times \cdots \times B}^{n\text{-times}}$$

and $\mathscr{B}$ be the σ-field of the subsets of B. $\mathscr{F}_*$ is generated by the sets $A \times B_1 \times B_2 \times \cdots \times B_n$, where $A \in \Omega$, $B_i \in \mathscr{B}$ for $i \leqslant n$, and $\mathscr{F}_i^*$ is generated by the sets $A \times B_1 \times \cdots \times B_n$, where $A \in \mathscr{F}_{i-1}$, $B_i \in \mathscr{B}$ for $i \leqslant n$. By Lemma 6 construct the random measure $v_w = \mu_w^1 \otimes \mu_w^2 \otimes \cdots \otimes \mu_w^n$. Denote by P^* the unique probability on $(\Omega^*, \mathscr{F}_*, P^*)$ such that for $A \in \Omega$, $B \in \mathscr{B}^n$

$$P^*(A \times B) = \int_A v_w(B)\, dp(w).$$

Now (3.5) follows from conditions (3.1), and (3.6) follows from (3.2). Since $\{Y_i^*\}$ are Gaussian, (3.7) follows from (2.2) of Lemma 6 and (2.2) of Lemma 5.

Let $F_i^*(\cdot)$ be the conditional (conditioning with respect to $\mathscr{F}_i^*$) distribution of the B-valued r.v. X_i^* and similarly let $\mu_i^*(\cdot)$ be the conditional distribution of the B-valued Gaussian random variable $Y_i^* \cdot \{F_i^* - \mu_i^*\}$ is a finite random signed measure on $\mathscr{B}$. The conditional absolute pseudomoment of third order is given by $v_3(i) = \int \|y\|^3 \; |F_i^* - \mu_i^*|\,(dy)$, where $|F_i^* - \mu_i^*|\,(y)$ denotes the total variation of the signed measure for a fixed $y \in B$.

THEOREM 1. *Under the preceding conditions* (1.1), (1.2), (3.1), *and* (3.2) *there is a constant* $c(B, \bar{T}_n)$ *such that for all* $n \geqslant 1$,

$$\Delta_n(a) \leqslant C(B, \bar{T}_n)(1 + \|a\|^2)\left(B_n^{-3} \sum_{i \leqslant n} Ev_3(i)\right)^{1/4} \tag{3.8}$$

$$\leqslant C_1(B, \bar{T}_n)(1 + \|a\|^2)\left(B_n^{-3} \sum_{i \leqslant n} E\,\|X_i\|^3\right)^{1/4}. \tag{3.9}$$

Proof. By Lemma 3 there exists a sequence of functions f_n: $R \to [0, 1]$ such that $f_n(x) = 1$ if $x \geqslant r + \varepsilon_n$, $f_n(x) = 0$ if $x \leqslant r$, $f_n^{(3)}$ is continuous in R and $|f_n^{(3)}(x)| \leqslant C\varepsilon_n^{-3} I[r, r + \varepsilon_n]$. To avoid notational complexity, let us dispense with the norming constant B_n for the time being; we shall introduce it at an appropriate time.

It is well known (see, e.g., [15]) that

$$\Delta_n = \Delta_n(0) \leqslant \sup_{r > 0} |Ef_n(S) - Ef_n(Y)| + \bar{\mu}_n(S_{r, \varepsilon_n}),$$

where $S_{r, \varepsilon_n} = S_{r, \varepsilon_n}(0)$. Therefore using condition (1.2)

$$\Delta_n \leqslant \sup_{r > 0} |Ef_n(S) - Ef_n(Y)| + C(B, \bar{T}_n)\, \varepsilon_n. \tag{3.10}$$

As in [1]

$$\Delta f_n = \sup_{r > 0} |Ef_n(S) - Ef_n(Y)|$$

$$\leqslant \sum_{i \leqslant n} |Ef_n(\|U_i + X_i\|) - Ef_n(\|U_i + Y_i\|)|,$$

where $U_i = \sum_{j < i} X_j + \sum_{i < j \leqslant n} Y_j$. Therefore $\Delta f_n \leqslant \sum_{i \leqslant n} V_i$, where

$$V_i = |E\{f_n(\|U_i + X_i\|) - f_n(\|U_i + Y_i\|)\}|.$$

Let us dispense with f_n and work with f only. Let

$$h_1(\lambda) = f(\|U_i + \lambda X_i\|) \qquad \text{and} \qquad h_2(\lambda) = f(\|U_i + \lambda Y_i\|).$$

By Lemmas 3 and 7 we get

$$F^i(h_1'(0)) = F^i(h_2'(0)) \qquad \text{and} \qquad F^i(h_1''(0)) = F^i(h_2''(0)).$$

Hence $E(h_1'(0)) = E(h_2'(0))$ and $E(h_1''(0)) = E(h_2''(0))$. Therefore by Taylor expansion

$$V_i \leqslant \tfrac{1}{6} E\,|F^i(h_1^{(3)}(\tau) - h_2^{(3)}(\tau))|,$$

where $|\tau_1| < 1$ and $|\tau_2| < 1$.

We know that for $1 \leqslant j \leqslant 3$, $f^{(j)}(u) = C_1 \varepsilon^{-3} u^{3-j} I_{[r, r+\varepsilon]}(u)$ and $\|D_x^j\| \leqslant C \|x\|^{-j+1}$. Therefore

$$V_i \leqslant \tfrac{1}{6} C(B)\, \varepsilon_n^{-3} B_n^{-3} E \int_B \|y\|^3 \, |F_i^* - \mu_i^*| \, (dy). \tag{3.11}$$

Therefore $\Delta f_n \leqslant C_1(B)\, \varepsilon_n^{-3} B_n^{-3} \sum_{i \leqslant n} E v_3(i)$.

Choosing $\varepsilon_n = (C_1(B)\, B_n^{-3} \sum_{i \leqslant n} E v_3(i))^{1/4}\, (C(B, \bar{T}_n))^{-1/4}$ we get (3.8) in case $a = 0$.

In case $a \neq 0$ we change the definition of f_n to $g_n(t) = f_n(\|t - a\|)$. It is clear that (3.11) is unaltered and the second term of (3.10) will be $C(B, \bar{T}_n)$ $(1 + \|a\|^2)\, \varepsilon_n$. Now $E v_3(i) \leqslant E \|X_i\|^3 + E \|y_i\|^3$ and by Lemma 5 and a result of Hoffman-Jorgensen and Pisier [8],

$$E v_3(i) \leqslant C E \|X_i\|^3.$$

This proves Theorem 1 completely.

4. Nonuniform Bound

To get a nonuniform bound for dependent B-valued r.v.'s we need some results and ideas due to Burkholder [2, 3].

LEMMA 8. *If $\{X_i\}$ is a B-valued r.v. in L_B^p satisfying condition (2.1) and the B-space is well behaved for martingale transform ($B \in MT$) or has the unconditionality property for martingale differences ($B \in VMD$) (see [3]) then the following inequality holds:*

$$E\left(\left\| \sum_{i \leqslant n} X_i \right\|^p\right) \leqslant C_p E\left[\sum_{i \leqslant n} \|X_i\|^2\right]^{p/2} \qquad \text{for} \quad p \geqslant 1.$$

Proof. Now $\{S_n = \sum_{i \leqslant n} X_i\}$ is a B-valued martingale with respect to nondecreasing σ-algebras $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}$ and $g_n = \sum_{k \leqslant n} \varepsilon_k X_k$ is its transform by the independent Rademacher sequence $\{\varepsilon_n\}$ and conversely. Let $\|S\|_p = \sup_n \|S_n\|_p$, $g^*(w) = \sup_n \|g_n(w)\|$, and $\|g\|_p = \sup_n \|g_n\|_p$. Then by [3] $\lambda P(g^* > \lambda) \leqslant C \|S\|_1$, $\lambda > 0$, and $\|g\|_p \leqslant C_p \|S\|_p$.

Now following Theorem 10 of [2] and Marcinkiewicz' interpolation formula we get

$$E \|g_n\|^p \leqslant M_p\, E \|S_n\|^p.$$

A result of Kahane [10] asserts that for each element $x_1, \ldots, x_n$ of any Banach space,

$$\int \left\| \sum_{i \leqslant n} \varepsilon_i(w)\, x_i \right\|^p \leqslant N_p \left(\int \left\| \sum_{i \leqslant n} \varepsilon_i(w)\, x_i \right\|^2 \right)^{p/2}$$

where N_p is a constant.

So we get

$$\int \left\| \sum_{i \leqslant n} \varepsilon_i(w)\, X_i(w) \right\|^p dw \leqslant \int \left\| \sum_{i \leqslant n} \varepsilon_i(w_i)\, X_i(w_2) \right\|^p dw_1\, dw_2.$$

Therefore

$$\begin{aligned} & A_p M_p^{-1} E\left(\sum_{i \leqslant n} \|X_i\|^2 \right)^{p/2} \\ & \leqslant M_p^{-1} E\left(\int_0^1 \left\| \sum_{i \leqslant n} \varepsilon_i(w_1)\, X_i(w_2) \right\|^p dw_1 \right) \\ & = \int_0^1 M_p^{-1} E \left\| \sum_{i \leqslant n} \varepsilon_i(w_1)\, X_i(w_2) \right\|^p dw_1 \\ & \leqslant E\, \|S_n\|^p \\ & \leqslant B_p M_p E\left(\sum_{i \leqslant n} \|X_i\|^2 \right)^{p/2} \\ & = C_p E\left[\sum_{i \leqslant n} \|X_i\|^2 \right]^{p/2}. \end{aligned}$$

Hence the lemma is proved.

THEOREM 2. *Let* $V_i^2 = \sum_{j \leqslant i} E^{j-i}(\|X_j\|^2)$, $U_i^2 = \sum_{j \leqslant i} \|X_i\|^2$ and $L_n = \sum_{i \leqslant n} E\, \|X_i\|^{2+2\delta} + E\,(\|V_n^2 - 1\|^{1+\delta})$ for $0 < \delta \leqslant 1$. Suppose $\{S_i = \sum_{j \leqslant i} X_i, \mathscr{F}_i, 1 \leqslant i \leqslant n\}$ is a zero mean B-valued martingale and $X_i \in L_B^{2+2\delta}$. Assume that the condition of Lemma 8 holds. There exists constant A_δ, depending on δ, such that whenever $L_n \leqslant 1$

$$\begin{aligned} \Delta(x) &= |P(\|S_n\| \leqslant x) - P(\|Y\| \leqslant x)| \\ &\leqslant A_\delta L_n^{1/(3+2\delta)} [1 + x^{4(1+\delta)^2/(3+2\delta)}]^{-1} \qquad \text{for } x \geqslant 0. \end{aligned}$$

Proof.

$$\begin{aligned} \Delta(x) &\leqslant P(\|S_n\| > x) + P(\|Y\| > x) \\ &\leqslant x^{-2+2\delta}(E\, \|S_n\|^{2+2\delta} + E\, \|Y\|^{2+\delta}). \end{aligned} \tag{4.1}$$

$$
\begin{aligned}
E\,\|S_n\|^{2+2\delta} &\leqslant c_\delta E\Big(\sum_{i\leqslant n}\|X_i\|^2\Big)^{1+\delta} \qquad \text{(by Lemma 8)}\\
&= c_\delta E(U_n^{2+2\delta})\\
&\leqslant c'_\delta[E\,\|U_n^2 - V_n^2\|^{1+\delta} + E\,\|V_n^2 - 1\|^{1+\delta} + 1]]\\
&\leqslant c''_\delta(L_n + 1). \qquad (4.2)
\end{aligned}
$$

Applying Burkholder's [2] inequality again to the real martingale with differences $\|X_i\|^2 - E^{i-1}(\|X_i\|^2)$ and using the fact that $\delta \leqslant 1$, we get

$$
\begin{aligned}
E\,\|U_n^2 - V_n^2\|^{1+\delta} &= E\Big|\sum_{i\leqslant n}(\|X_i\|^2 - E^{i-1}(\|X_i\|^2)\Big|^{1+\delta}\\
&\leqslant c'_\delta E\Big|\sum_{i\leqslant n}\{\|X_i\|^2 - E^{i-1}(\|X_i\|^2)\}^2\Big|^{(1+\delta)/2}\\
&\leqslant c''_\delta \sum_{i\leqslant n} E\,|\,\|X_i\|^2 - E^{i-1}(\|X_i\|^2)|^{1+\delta}
\end{aligned}
$$

(by c_r-inequality, $(1+\delta)/2 \leqslant 1$)

$$
\begin{aligned}
&\leqslant c_\delta^3 \sum_{i\leqslant n}[E\,\|X_i\|^{2+2\delta} + E\,|E^{i-1}(\|X_i\|^2)|^{1+\delta}]\\
&\leqslant 2c_\delta^3 \sum_{i\leqslant n} E\|X_i\|^{2+2\delta}. \qquad (4.3)
\end{aligned}
$$

By [6] (see also Lemma 5) there exists constant M_p such that for $p \geqslant 2$

$$
E\,\|Y\|^p \leqslant M_p E(\|Y\|^2)^{p/2} \leqslant c_p(E\,\|S_n\|^2)^{P/2} \leqslant d_p E\,\|S_n\|^p.
$$

Hence

$$
E\,\|Y\|^{2+2\delta} \leqslant c_\delta^3(L_n + 1).
$$

Sustituting (4.2) in (4.1) we see that if $L_n \leqslant 1$,

$$
\Delta(x) \leqslant 2cx^{-2-2\delta}(L_n + 1) \leqslant 4cx^{-2-2\delta} \qquad \text{for } x > 0. \qquad (4.4)
$$

Suppose now that $L_n \leqslant 2^{-(3+2\delta)}$ and $L_n^{2/(3+2\delta)}(1 + x^\alpha)^{1/1+\delta} > \frac{1}{2}$. Then $(1 + x^\alpha)^{1/1+\delta} > 2$ and so $x^\alpha > 1$. Therefore $x^{-2-2\delta} \leqslant [1/2(1 + x^\alpha)]^{-2(1+\delta)/\alpha} = C(1 + x^\alpha)^{-2(1+\delta)/\alpha}$. Furthermore,

$$
1 \leqslant 2^{1/2} L_n^{1/(3+2\delta)}(1 + x^\alpha)^{1/2(1+\delta)}. \qquad (4.5)
$$

Substituting this inequality and (4.5) into (4.4) we deduce that

$$
\Delta(x) \leqslant cL_n^{1/(3+2\delta)}(1 + x^\alpha)^{1/2(1+\delta)}. \qquad (4.6)
$$

The best rate of convergence is obtained when α satisfies the equation

$$1/2(1+\delta)-2(1+\delta)/\alpha=-1$$

giving $\alpha=4(1+\delta)^2/(3+2\delta)$.

Suppose now $1>L_n>2^{-(3+2\delta)}$. In this case we have, for $0<x\leqslant 1$,

$$4L_n^{1/(3+2\delta)}(1+x^\alpha)^{-1}>1>1P(\|S_n\|<x)-P(\|Y\|<x),$$

and for $x>1$,

$$\begin{aligned}
&4L_n^{1/(3+2\delta)}(1+x^\alpha)^{-1}\\
&\quad\geqslant x^{-2-2\delta}=(c+E\|Y\|^{2+2\delta})^{-1}\\
&\quad\quad\times(C+E\|Y\|^{2+2\delta})\,x^{-2-2\delta})\,x^{-2-2\delta}\qquad\text{(for any } c>0)\\
&\quad\geqslant c'_\delta(E\|S_n\|^{2+2\delta}+E\|Y\|^{2+2\delta})\,x^{-2-2\delta}\\
&\quad\geqslant c'_\delta\,|P(\|S_n\|\leqslant x)-P(\|Y\|\leqslant x)|\qquad\text{(from (4.4))}.
\end{aligned}$$

Therefore

$$\Delta(x)\leqslant A_\delta L_n^{1/(3+2\delta)}[1+x^{4(1+2\delta)}]^{-1}\qquad\text{for } x>0.$$

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