Physics Letters B 710 (2012) 223–229

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Two-potential theory of electric and magnetic charges via duality transformation

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ARTICLE INFO

Article history:
Received 13 January 2012
Received in revised form 20 February 2012
Accepted 23 February 2012
Available online 28 February 2012
Editor: M. Cvetič

Keywords:
Duality transformation
Two-potential theory

ABSTRACT

Dirac, Schwinger and Zwanziger theories of electric and magnetic charges are obtained via duality transformation. Analogous construction for three Euclidean dimensions, with magnetic charges interacting with electric currents, is also done. The role of Dirac strings as dislocations in the configurations of gauge potential is emphasized.

1. Introduction

In this Letter we obtain Dirac, Schwinger and Zwanziger theories [1,2] of electric and magnetic charges via duality transformation. Our aim is to build unified techniques for handling a field and its dual on an equal footing. The reason is that the dual field plays an important role in many contexts. It behaves as a disorder parameter and drives the properties of the theory in some phases. The dual field couples locally to certain topological configurations of the original field. Therefore by keeping them we will be able to naturally handle some non-perturbative aspects of the theory. A plasma or condensate of such topological configurations may be qualitatively and quantitatively relevant in some phases. By having a formalism which has both fields, we can handle such effects on the original field. Also combinations of the field and its dual close together often have exotic properties and play crucial role for the properties of the theory.

We illustrated our techniques for a scalar field in two Euclidean dimensions in [3]. The end point of a line discontinuity is the source for the dual field, much like the end point of a Dirac string behaving as a magnetic monopole in electrodynamics [1]. The two-dimensional local Lagrangian involving both the scalar field and its dual [4] is the analogue of Zwanziger's [2] two-potential local theory of electric and magnetic charges [5,6]. The dislocation line becomes invisible for a quantization of dual charges as in Dirac theory of magnetic monopoles. The correlation of the field with its dual has unusual properties.

All these show that the two-potential formalism is not restricted to electrodynamics. Any field theory can be recast as a local theory of the fields and their duals present together. This is important for quantum chromodynamics because confinement property is expected to be driven by topological configurations such as monopoles and vortices. We need to be able to see their effects on gluons and quarks. Therefore a formalism with both fields together is very useful.

The scheme of this Letter is as follows. We use Euclidean formalism throughout this Letter to highlight the role played by $\sqrt{-1}$. In Section 2, we start with a real massless scalar field in three Euclidean dimensions. We relate it to Abelian gauge theory by a duality transformation. Point sources for the scalar field are mapped to Dirac strings acting as dislocation lines in the configurations of the gauge potential. We further relate this to a local theory with both the scalar and gauge potential present simultaneously and coupling locally to the magnetic charges and electric currents. We demonstrate how the interactions amongst magnetic charges and electric currents is recovered. In Section 3, we begin with the Abelian gauge theory in three Euclidean dimensions and recover the same formalism. Here the electric currents act as sources of surface dislocations in the configuration of magnetic scalar potential. In Section 4, we obtain the Dirac, Schwinger and Zwanziger formulations of electric and magnetic charges via duality transformation in four Euclidean dimensions.

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0370-2693/$ – see front matter
doi:10.1016/j.physletb.2012.02.069

We discuss the relevance of our techniques in Section 5. In Appendix A, we give some techniques useful for handling the Dirac potential of a magnetic monopole.

2. Line dislocation and a local action with both the 'photon' and the 'dual photon' in three Euclidean dimensions

We begin with a free massless real scalar field $\chi(x)$ in three Euclidean dimensions. Its correlation functions can be obtained from the functional integral

$$\mathcal{Z}[\rho] = N_1 \int \mathscr{D}\chi(x)\, e^{\int d^3x [-\frac{1}{2}(\vec{\nabla}\chi(x))^2 + i\rho(x)\chi(x)]}. \tag{1}$$

Here N_1 is a normalization factor such that $\mathcal{Z}[\rho = 0] = 1$, and $\rho(x)$ is an external source coupling locally to $\chi(x)$. We have deliberately included $\sqrt{-1}$ in this source term for convenience in performing the duality transformation below. We linearize the dependence on $\chi(x)$ in the exponent in Eq. (1) by introducing an auxiliary field $\vec{B}(x)$:

$$\mathcal{Z}[\rho] = N_2 \int \mathscr{D}\vec{B}(x)\, \mathscr{D}\chi(x)\, e^{\int d^3x [-\frac{1}{2}\vec{B}(x)^2 - i\vec{B}(x)\cdot\vec{\nabla}\chi(x) + i\rho(x)\chi(x)]}. \tag{2}$$

A formal integration over $\chi(x)$ gives

$$\mathcal{Z}[\rho] = N_3 \int \mathscr{D}\vec{B} \prod_{\vec{x}} \delta\big(\vec{\nabla}\cdot\vec{B}(x) + \rho(x)\big) e^{\int d^3x [-\frac{1}{2}\vec{B}(x)^2]}. \tag{3}$$

For a point source

$$\rho(x) = g\delta^3(\vec{x} - \vec{y}), \tag{4}$$

the δ-functional constraint in (3) corresponds to a magnetic monopole of strength g at the point $\vec{y}$. For solving it, we choose the particular integral in the form of a Dirac string from $\vec{y}$ along the negative z-direction. This singular solution gives a net flux g through any surface enclosing $\vec{y}$. Thus the solution for a general $\rho(x)$ is

$$\vec{B}(x) = \vec{\nabla}\times\vec{A}(x) - \hat{n}_3\partial_3^{-1}\rho(x) \tag{5}$$

where

$$\partial_3^{-1}\rho(x) = -\int_{x_3}^{\infty} dx_3'\, \rho\big(x_1, x_2, x_3'\big) \tag{6}$$

and $\hat{n}_3$ is the unit vector in the 3-direction. The use of Eq. (5) in Eq. (3) rewrites the massless scalar theory of (1) as an Abelian gauge theory. The gauge field has one transverse degree of freedom in three Euclidean dimensions, matching that of the scalar theory.

We shall refer to $\vec{A}(x)$ as the 'photon' and $\chi(x)$ as the 'dual photon'. We are interested in their mutual correlations. We therefore include a source $\vec{j}(x)$ for $\vec{A}(x)$:

$$\mathcal{Z}[\rho, \vec{j}] = N_4 \int \mathscr{D}\vec{A}\, e^{\int d^3x [-\frac{1}{2}(\vec{\nabla}\times\vec{A}(x) - \hat{n}_3\partial_3^{-1}\rho(x))^2 + i\vec{j}(x)\cdot\vec{A}(x)]}. \tag{7}$$

The dual photon $\chi(x)$ couples locally to the magnetic monopole density $\rho(x)$. Thus Eq. (7) gives a (gauge) theory with both electric current and magnetic charges. Eq. (7) shows that a point magnetic charge at $\vec{y}$ has the effect of a line dislocation (the Dirac string) starting at $\vec{y}$ in the configuration space of the gauge potential.

As a consequence of these singular dislocation lines, the configurations $\vec{A}(x)$ which matter in the functional integral (7) are not the usual plane waves. For the action to be finite, $\vec{\nabla}\times\vec{A}$ should also be singular and cancel the Dirac string singularities. Thus the configurations which matter are precisely the Dirac potential $\vec{A}^D(\vec{x} - \vec{y})$ due to a magnetic monopole at $\vec{y}$ and its distortions. This is explicitly seen as follows. Let us shift $\vec{A}(x)$ to $\vec{a}(x)$ as

$$\vec{A}(x) = \vec{a}(x) + \int d^3y\, \vec{A}^D(\vec{x} - \vec{y})\rho(y). \tag{8}$$

Now use the representation of $\vec{A}^D(\vec{x} - \vec{y})$ in the form of Green function for $\partial_3\nabla^2$, as given in (54). This leads to

$$\vec{\nabla}\times\vec{A}(x) = \vec{\nabla}\times\vec{a}(x) + \frac{1}{4\pi}\int d^3y\, \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3}\rho(y) + \hat{n}_3\partial_3^{-1}\rho(x). \tag{9}$$

So this shift cancels the Dirac string in (7). The second term on the r.h.s. of (9) is simply the magnetic field at $\vec{x}$ due to a magnetic monopole density $\rho(y)$. It is the gradient $-\vec{\nabla}\chi(x)$ of a scalar potential

$$\chi(x) = \int d^3y\, \Delta(\vec{x} - \vec{y})\rho(y), \tag{10}$$

$$\Delta(\vec{x} - \vec{y}) = \frac{1}{4\pi}\frac{1}{|\vec{x} - \vec{y}|}. \tag{11}$$

Therefore

$$\mathcal{Z}[\rho,\vec{j}] = N_4 \int \mathcal{D}\vec{a}\, e^{\int d^3x[-\frac{1}{2}(\vec{\nabla}\times\vec{a}(x)-\vec{\nabla}\chi(x))^2 + i\vec{j}(x)\cdot\vec{a}(x)] + i\int d^3x d^3y\, \vec{j}(x)\cdot\vec{A}^D(\vec{x}-\vec{y})\rho(y)}. \tag{12}$$

Now

$$\left(\vec{\nabla}\times\vec{a}(x)-\vec{\nabla}\chi(x)\right)^2 = \left(\vec{\nabla}\times\vec{a}(x)\right)^2 - 2\vec{\nabla}\cdot\left(\chi(x)\vec{\nabla}\times\vec{a}(x)\right) + \vec{\nabla}\cdot(\chi\vec{\nabla}\chi) + \chi(x)\rho(x) \tag{13}$$

as $\nabla^2\chi(x) = -\rho(x)$. There is no boundary contribution from the total divergence terms. So we get

$$\mathcal{Z}[\rho,\vec{j}] = N_4 \int \mathcal{D}\vec{a}\, e^{[\int d^3x(-\frac{1}{2}(\vec{\nabla}\times\vec{a}(x))^2 + i\vec{j}(x)\cdot\vec{a}(x)) - \int d^3x d^3y (\frac{1}{2}\rho(x)\Delta(\vec{x}-\vec{y})\rho(y) - i\vec{j}(x)\cdot\vec{A}^D(\vec{x}-\vec{y})\rho(y))]} \tag{14}$$

which has a conventional action for the new fluctuations $\vec{a}(x)$. This completes our contention that the configurations that contribute to (7) are the Dirac potential $\int d^3y\, \vec{A}^D(\vec{x}-\vec{y})\rho(y)$ and its fluctuations. We may fix a gauge for $\vec{a}(x)$ as usual and integrate over $\vec{a}(x)$ in (14). This gives

$$\mathcal{Z}[\rho,\vec{j}] = N_5\, e^{\int d^3x d^3y[-\frac{1}{2}j^i(x)\Delta_{ij}(\vec{x}-\vec{y})j^j(y) - \frac{1}{2}\rho(x)\Delta(\vec{x}-\vec{y})\rho(y) + i\vec{j}(x)\cdot\vec{A}^D(\vec{x}-\vec{y})\rho(y)]}, \tag{15}$$

where $\Delta(\vec{x}-\vec{y})$ and $\Delta_{ij}(\vec{x}-\vec{y})$ are respectively the propagator for a massless real scalar and the Abelian gauge potential in three Euclidean dimensions [7]. It shows the electric currents $\vec{j}(x)$ interacting via the Biot-Savart law. (Gauge fixing permits us to extend the law to currents that need not be conserved.) It also has the magnetic monopoles interacting via the Coulomb potential. In addition it shows that the magnetic monopoles $\rho(x)$ interact with electric current $\vec{j}(x)$ through the Dirac potential $\vec{A}^D(\vec{x}-\vec{y})$. (Note the $\sqrt{-1}$ in this term in Eq. (15), even in our Euclidean theory. It is not strange as the interaction of the current $\vec{j}(x)$ with a gauge potential $\vec{A}(x)$ is given by $i\vec{j}(x)\cdot\vec{A}(x)$ even in the Euclidean theory.) This represents the net effect of the line discontinuity (the Dirac string) in the configurations of the gauge potential. Only the end point of the string matters and behaves like a magnetic monopole.

The magnetic charge density $\rho(x)$ has a non-local coupling to the 'photon' field $\vec{A}(x)$ in (7), though it couples locally to the dual photon $\chi(x)$ in (1). We now present a local action that has both the photon and the dual photon fields present together. For this we rewrite (7) introducing an auxiliary field $\vec{b}(x)$:

$$\mathcal{Z}[\rho,\vec{j}] = N_6 \int \mathcal{D}\vec{b}\,\mathcal{D}\vec{A}\, e^{\int d^3x[-\frac{1}{2}\vec{b}(x)^2 + i\vec{b}(x)\cdot(\vec{\nabla}\times\vec{A}(x) - \hat{n}_3\partial_3^{-1}\rho(x)) + i\vec{j}(x)\cdot\vec{A}(x)]}. \tag{16}$$

Thus $\rho(x)$ couples locally to $\partial_3^{-1}b_3(x)$, which is to be identified with the dual photon $\chi(x)$:

$$\chi(x) = \partial_3^{-1}b_3(x). \tag{17}$$

We may integrate back over $b_1(x)$ and $b_2(x)$, to get

$$\mathcal{Z}[\rho,\vec{j}] = N_7 \int \mathcal{D}\chi\,\mathcal{D}\vec{A}\, e^{\int d^3x[-\frac{1}{2}(\partial_3\chi(x))^2 - \frac{1}{2}(\hat{n}_3\times(\vec{\nabla}\times\vec{A}(x)))^2 + i\partial_3\chi(x)\hat{n}_3\cdot\vec{\nabla}\times\vec{A}(x) + i\vec{j}(x)\cdot\vec{A}(x) + i\rho(x)\chi(x)]}. \tag{18}$$

This gives the local field theory of electric currents and magnetic charges in three Euclidean dimensions. It is the analogue of the two-potential formalism in four dimensions [2] and of the local field theory involving the scalar field and its dual in two dimensions [3,4]. Note the following unusual features:

- The action is not manifestly rotation invariant. Nevertheless, the rotation covariance is restored for physical observables when the Dirac quantization condition for electric and magnetic charges is met. (See below.)
- The 'kinetic energy' terms for $\chi(x)$ and $\vec{A}(x)$ have derivatives only in some directions. However, if we integrate over $\chi(x)$ (correspondingly $\vec{A}(x)$), we recover the conventional action for $\vec{A}(x)$ (correspondingly $\chi(x)$).
- The action in (18) is not real (even with the imaginary sources switched off). The term bilinear in χ and $\vec{A}(x)$ is pure imaginary.

The 'propagators' can be calculated using Fourier modes, as follows. In the action of Eq. (18), we use the identity $(\hat{n}_3\times(\vec{\nabla}\times\vec{A}))^2 = (\vec{\nabla}\times\vec{A})^2 - (\hat{n}_3\cdot(\vec{\nabla}\times\vec{A}))^2$. The resulting $-\frac{1}{2}(\vec{\nabla}\times\vec{A})^2$ term in the action becomes $\frac{1}{2}\vec{A}\cdot\nabla^2\vec{A}$, upon adding the gauge-fixing term corresponding to the Feynman gauge. Also $\hat{n}_3\cdot(\vec{\nabla}\times\vec{A}) = (\hat{n}_3\times\vec{\nabla})\cdot\vec{A}$, which is $i\vec{k}_\perp\cdot\vec{A}(k)$ in momentum space, where $\vec{k}_\perp \equiv \hat{n}_3\times\vec{k}$. Thus we obtain

$$\begin{aligned}\mathcal{Z}[\rho,\vec{j}] = N_7 \int \mathcal{D}\chi(k)\,\mathcal{D}\vec{A}(k)\exp\bigg[\int d^3k\bigg(&-\frac{1}{2}\chi(-k){k_3}^2\chi(k) - \frac{1}{2}A_i(-k)(k^2\delta_{ij} - k_{\perp i}k_{\perp j})A_j(k)\\ &+ iA_i(-k)k_{\perp i}k_3\chi(k) + ij_i(-k)A_i(k) + i\rho(-k)\chi(k)\bigg)\bigg].\end{aligned} \tag{19}$$

The propagators are then obtained by the inversion of a matrix:

$$\begin{bmatrix} k_3^2 & -ik_3k_{\perp i} \\ -ik_3k_{\perp j} & k^2\delta_{ij} - k_{\perp i}k_{\perp j}\end{bmatrix}^{-1} = \frac{1}{k^2}\begin{bmatrix} 1 & i\frac{k_{\perp i}}{k_3} \\ i\frac{k_{\perp j}}{k_3} & \delta_{ij}\end{bmatrix}. \tag{20}$$

In position space, the propagators are

$$\langle \chi(x)\chi(y)\rangle = \Delta(\vec{x}-\vec{y}), \tag{21}$$

$$\langle A_i(x)A_j(y)\rangle = \Delta_{ij}(\vec{x}-\vec{y}) = \delta_{ij}\Delta(\vec{x}-\vec{y}), \tag{22}$$

$$\langle A_i(x)\chi(y)\rangle = iA_i^D(\vec{x}-\vec{y}), \tag{23}$$

where $\Delta(\vec{x}-\vec{y})$ is given by (11), and we have used (57). Alternatively, we can read off these position space propagators from (15).

Even though $\chi(x)$ and $\vec{A}(x)$ are real fields, the propagator $\langle A_i(x)\chi(y)\rangle$ is pure imaginary. This is possible because the action is not real. We see that $iA_i^D(\vec{x}-\vec{y})$ serves as the 'propagator' connecting the electric currents and magnetic charges. The correlation of $\chi(x)$ with the 'magnetic field' $\vec{B}(x) = \vec{\nabla}\times\vec{A}(x)$ has the Dirac string singularity:

$$\langle \vec{B}(x)\chi(0)\rangle = i\left(\frac{1}{4\pi}\frac{\vec{x}}{|\vec{x}|^3} + \hat{n}_3\delta(x_1)\delta(x_2)\theta(-x_3)\right). \tag{24}$$

Because of the explicit presence of the Dirac string, rotation invariance in (23) and (15) is not manifest. Dirac [1] argued that with a 'quantization' of electric (e) and magnetic (g) charges, the Dirac string becomes invisible and rotation covariance is restored. Consider a point magnetic charge given by (4), and a loop C carrying a current

$$j_i(x) = e\oint_C d\tau \frac{dX_i(\tau)}{d\tau}\delta^3\big(\vec{x}-\vec{X}(\tau)\big) \tag{25}$$

where τ is an arbitrary parametrization of the loop C. The contribution to the cross-correlation of $\exp[ie\oint dx^i A_i(x)]$ with $\exp[ig\chi(y)]$ comes from the last term in the exponent of (15) for the sources (4) and (25). (This is the analogue of the cross-correlation between the vertex operators for the scalar field and the dual field in two dimensions, which was considered in [3].) Thus this cross-correlation equals $\exp[ieg\oint dx^i A_i^D(\vec{x}-\vec{y})]$. Using Stokes' theorem, this is

$$\exp\left(-i\frac{eg}{4\pi}\Omega(C)\right) \tag{26}$$

where $\Omega(C)$ is the solid angle subtended by C at the site $\vec{y}$ of the magnetic charge, and the solid angle is to be computed by using a surface bounding C which does not intersect the Dirac string. Therefore when an infinitesimal loop C does not enclose the string, we get the contribution 1, but when it encloses the string, we get the contribution e^{-ieg}. Only with the quantization condition

$$eg = 2\pi n \tag{27}$$

the latter contribution is also 1, and the Dirac string is then invisible to any current loop.

3. Surface dislocations and scalar potential theory of electric currents and magnetic charges in three Euclidean dimensions

In Section 2, we started with a massless real scalar field and obtained the two-potential theory of magnetic charges interacting with electric currents in three Euclidean dimensions. In this section, we begin with Abelian gauge theory and obtain the same two-potential formalism. This exercise is instructive for the case of four Euclidean dimensions.

We begin with

$$\mathcal{Z}[\vec{j}] = N_8\int \mathscr{D}\vec{A}(x)\, e^{\int d^3x[-\frac{1}{2}(\vec{\nabla}\times\vec{A}(x))^2 + i\vec{j}(x)\cdot\vec{A}(x)]} \tag{28}$$

describing the current $\vec{j}(x)$ interacting via the gauge potential $\vec{A}(x)$ in three Euclidean dimensions. Rewriting

$$\mathcal{Z}[\vec{j}] = N_9\int \mathscr{D}\vec{b}\,\mathscr{D}\vec{A}\, e^{\int d^3x[-\frac{1}{2}\vec{b}(x)^2 + i\vec{b}(x)\cdot\vec{\nabla}\times\vec{A}(x) + i\vec{j}(x)\cdot\vec{A}(x)]} \tag{29}$$

$$= N_{10}\int \mathscr{D}\vec{b}\prod_{\vec{x}}\delta\big(\vec{\nabla}\times\vec{b}(x) + \vec{j}(x)\big)e^{\int d^3x[-\frac{1}{2}\vec{b}(x)^2]}. \tag{30}$$

The consistency of the constraint requires

$$\vec{\nabla}\cdot\vec{j}(x) = 0. \tag{31}$$

Choosing the solution

$$b_i(x) = \partial_i\chi(x) - \epsilon_{3il}\partial_3^{-1}j_l(x) \tag{32}$$

for the δ-functional constraint, we get

$$\mathcal{Z}[\rho,\vec{j}] = N_{11}\int \mathscr{D}\chi\, e^{\int d^3x[-\frac{1}{2}(\partial_1\chi(x) - \partial_3^{-1}j_2(x))^2 - \frac{1}{2}(\partial_2\chi(x) + \partial_3^{-1}j_1(x))^2 - \frac{1}{2}(\partial_3\chi(x))^2 + i\rho(x)\chi(x)]} \tag{33}$$

where we now introduced the source for χ. Note that the component $j_3(x)$ is not explicitly present. However, by the conservation law (31), we can write

$$j_3(x) = -\partial_3^{-1}\big(\partial_1 j_1(x) + \partial_2 j_2(x)\big) \tag{34}$$

and therefore it is implicitly present.

Eq. (33) is giving the interaction of magnetic charges and electric currents using the scalar potential $\chi(x)$ encountered in magnetostatics. Consider a current loop C in the 1–2 plane with a charge e flowing in it. Eq. (33) shows that for the action to be finite in this case, there should be a discontinuity in the scalar potential χ in the form of a surface dislocation. This dislocation is along a cylindrical domain wall with C as the mouth and extending all the way to infinity in the 3-direction. The gradient of the potential jumps by e across the domain wall. This is the conventional description of using a multivalued magnetic scalar potential in the presence of electric currents [8].

We linearize (33) in a specific way:

$$\begin{aligned}\mathcal{Z}[\rho,\vec{j}] = N_{12}\int \mathscr{D}\chi\,\mathscr{D}\vec{A}(x)\prod_x \delta\big(A_3(x)-\alpha_3(x)\big)\exp\bigg\{\int_x\bigg[-\frac{1}{2}\big(\partial_3\chi(x)\big)^2-\frac{1}{2}\big(\partial_2A_3(x)-\partial_3A_2(x)\big)^2\\ -\frac{1}{2}\big(\partial_3A_1(x)-\partial_1A_3(x)\big)^2 - i\big(\partial_2A_3(x)-\partial_3A_2(x)\big)\big(\partial_1\chi(x)-\partial_3^{-1}j_2(x)\big)\\ - i\big(\partial_3A_1(x)-\partial_1A_3(x)\big)\big(\partial_2\chi(x)+\partial_3^{-1}j_1(x)\big)+i\rho(x)\chi(x)\bigg]\bigg\}.\end{aligned} \tag{35}$$

Eq. (33) is reproduced from Eq. (35) by shifting A_I, $I=1,2$ to $A'_I = A_I - \partial_3^{-1}\partial_I A_3$ and integrating over A'_I. The potential A_3 has been introduced in this step to have a gauge-invariant Lagrangian. The gauge-fixing condition $A_3(x)=\alpha_3(x)$ corresponding to the axial gauge is here the simplest choice as it is not affected by the shift from A_I to A'_I.

Finally it can be checked, using (34), that the action in (35) is equal to the action in (18). The Lagrangian being gauge-invariant, we can now pass from the axial gauge to any other gauge we find convenient.

Instead of choosing an infinite domain wall, we can simply choose a finite surface S enclosing the current loop C to be the dislocation for the scalar potential. For that case, in the place of (32), we have

$$b_i(x) = \partial_i\chi(x)+\partial_i\chi_s(x), \tag{36}$$

where χ_s in discontinuous across the surface S:

$$\vec{\nabla}\chi_s(x) = \int_S d^2X(s)\,\hat{n}(s)e\delta^3\big(\vec{x}-\vec{X}(s)\big), \tag{37}$$

$\vec{X}(s)$ being a point on the surface S, and $\hat{n}(s)$ the normal to S at this point. Because of this discontinuity, $(\partial_i\partial_j-\partial_j\partial_i)\chi_s(x)\neq 0$, and (36) gives $\vec{\nabla}\times\vec{b}(x)+\vec{j}(x)=0$ with $\vec{j}(x)$ given by (25). (This can be checked by using the identity $\int d\vec{S}\times\vec{\nabla}\psi = \oint d\vec{l}\,\psi$ where ψ is a scalar function.) When (36) is used in (30), the finiteness of the action requires χ to have a discontinuity across S so as to cancel the discontinuity in χ_s. Therefore for a loop C' linked to C, we get

$$e^{ig\oint_{C'}dx^i\,\partial_i\chi(x)} = e^{iegN_{CC'}} \tag{38}$$

where $N_{CC'}$ is the linking number: the number of times loop C' winds around loop C in the clockwise sense. (This is because the integral of the current $\vec{j}$ over any open surface bounding the loop C' equals $eN_{CC'}$.) Thus, (38) equals 1 when the quantization condition (27) is satisfied.

4. Two-potential theory of electric and magnetic charges in four Euclidean dimensions

Consider the quantized Abelian gauge field in four Euclidean dimensions:

$$\mathcal{Z}[j_\mu] = N_{13}\int \mathscr{D}A_\mu\, e^{\int d^4x[-\frac{1}{4}(\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x))^2+ij_\mu A_\mu(x)]}. \tag{39}$$

Here j_μ ($\mu=1,2,3,4$) is the external current. We have

$$\begin{aligned}\mathcal{Z}[j_\mu] = N_{14}\int \mathscr{D}b_i\,\mathscr{D}e_i\,\mathscr{D}A_\mu\exp\bigg\{\int d^4x\bigg[-\frac{1}{2}e_i(x)^2-\frac{1}{2}b_i(x)^2+ie_i(x)\big(\partial_4A_i(x)-\partial_iA_4(x)\big)\\ +i\epsilon_{ijk}b_i(x)\partial_jA_k(x)+ij_i(x)A_i(x)+ij_4(x)A_4(x)\bigg]\bigg\}\end{aligned} \tag{40}$$

$$= N_{15}\int \mathscr{D}b_i\mathscr{D}e_i\prod_x\delta\big(\partial_ie_i(x)+j_4(x)\big)\prod_x\delta\big(\epsilon_{ijk}\partial_jb_k(x)-\partial_4e_i(x)+j_i(x)\big)\exp\bigg\{\int d^4x\bigg[-\frac{1}{2}e_i(x)^2-\frac{1}{2}b_i(x)^2\bigg]\bigg\}. \tag{41}$$

From the δ-functional constraints we have the consistency condition

$$\partial_ij_i(x)+\partial_4j_4(x)=0. \tag{42}$$

We solve the first constraint in (41) as

$$e_i(x)=\epsilon_{ijk}\partial_jC_k(x)-\delta_{i3}\partial_3^{-1}j_4(x), \tag{43}$$

corresponding to choosing the Dirac string along the 3-direction (similar to (5)). Using this, the second constraint becomes

$$\epsilon_{ijk}\partial_j(b_k - \partial_4 C_k) = -j_i - \delta_{i3}\partial_3^{-1}\partial_4 j_4. \tag{44}$$

We solve this in the form (similar to (32))

$$b_k(x) - \partial_4 C_k(x) = -\partial_k C_4(x) - \epsilon_{3kl}\partial_3^{-1} j_l(x). \tag{45}$$

Therefore we get

$$\begin{aligned}\mathcal{Z}[j_\mu, k_\mu] = N_{16}\int \mathscr{D}C_\mu \exp\bigg\{\int d^4x\bigg[&-\frac{1}{2}\big(\partial_1 C_2(x) - \partial_2 C_1(x) - \partial_3^{-1} j_4(x)\big)^2 - \frac{1}{2}\big(\partial_I C_3(x) - \partial_3 C_I(x)\big)^2 \\ &- \frac{1}{2}\big(\partial_I C_4(x) - \partial_4 C_I(x) + \epsilon_{IJ}\partial_3^{-1} j_J(x)\big)^2 - \frac{1}{2}\big(\partial_3 C_4(x) - \partial_4 C_3(x)\big)^2 + ik_\mu(x)C_\mu(x)\bigg]\bigg\}\end{aligned} \tag{46}$$

where we have introduced a source k_μ for C_μ. Here the indices I, J run over only 1 and 2. This gives the Dirac and Schwinger formulations. As in Section 4, we linearize (46) in a particular way. The first term in the Lagrangian is linearized to

$$-\frac{1}{2}\big(\partial_3 A_4(x) - \partial_4 A_3(x)\big)^2 + i\big(\partial_3 A_4(x) - \partial_4 A_3(x)\big)\big(\partial_1 C_2(x) - \partial_2 C_1(x) - \partial_3^{-1} j_4(x)\big)$$

while the third term is linearized to

$$-\frac{1}{2}\big(\partial_3 A_I(x) - \partial_I A_3(x)\big)^2 + i\epsilon_{IJ}\big(\partial_3 A_I(x) - \partial_I A_3(x)\big)\big(\partial_J C_4(x) - \partial_4 C_J(x) + \epsilon_{JK}\partial_3^{-1} j_K(x)\big).$$

The axial gauge condition is imposed on A_3. (We get back (46) by shifting A_4 and A_I, and then integrating.) This linearized form of (46), on using (42), is equal to

$$\begin{aligned}\mathcal{Z}[j_\mu, k_\mu] = N_{17}\int \mathscr{D}C_\mu\,\mathscr{D}A_\mu \prod_x \delta\big(A_3(x) - \alpha_3(x)\big)\exp\bigg\{\int d^4x\bigg[&-\frac{1}{2}\big(\partial_I A_3(x) - \partial_3 A_I(x)\big)^2 - \frac{1}{2}\big(\partial_3 A_4(x) - \partial_4 A_3(x)\big)^2 \\ &- \frac{1}{2}\big(\partial_I C_3(x) - \partial_3 C_I(x)\big)^2 - \frac{1}{2}\big(\partial_3 C_4(x) - \partial_4 C_3(x)\big)^2 + i\big(\partial_3 A_4(x) - \partial_4 A_3(x)\big)\big(\partial_1 C_2(x) - \partial_2 C_1(x)\big) \\ &+ i\epsilon_{IJ}\big(\partial_3 A_I(x) - \partial_I A_3(x)\big)\big(\partial_J C_4(x) - \partial_4 C_J(x)\big) + ij_\mu(x)A_\mu(x) + ik_\mu(x)C_\mu(x)\bigg]\bigg\}.\end{aligned} \tag{47}$$

Thus we have recovered Zwanziger's two-potential theory of electric and magnetic changes via a duality transformation. If we had chosen $\hat{n}$ as the direction of the Dirac string instead of the negative z-axis, we would have got for the exponent on the r.h.s. of (47)

$$\begin{aligned}\int d^4x\bigg[&-\frac{1}{2}\big(\hat{n}\cdot\vec{E}(x)\big)^2 - \frac{1}{2}\big(\hat{n}\times\vec{B}(x)\big)^2 - \frac{1}{2}\big(\hat{n}\cdot\vec{\mathcal{E}}(x)\big)^2 - \frac{1}{2}\big(\hat{n}\times\vec{\mathcal{B}}(x)\big)^2 + i\big(\hat{n}\cdot\vec{\mathcal{B}}(x)\big)\big(\hat{n}\cdot\vec{E}(x)\big) \\ &+ i\big(\hat{n}\times\vec{\mathcal{E}}(x)\big)\cdot\big(\hat{n}\times\vec{B}(x)\big) + ij_\mu(x)A_\mu(x) + ik_\mu(x)C_\mu(x)\bigg],\end{aligned} \tag{48}$$

where $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}A_4 + \partial_4\vec{A}$, $\vec{\mathcal{B}} = \vec{\nabla}\times\vec{C}$ and $\vec{\mathcal{E}} = -\vec{\nabla}C_4 + \partial_4\vec{C}$.

5. Discussion

For a variety of reasons, it is useful to have a formulation with both a field and its dual field simultaneously present in a local theory. In [3] this was done for a scalar theory in two Euclidean dimensions and the advantages were highlighted. In this Letter, we have carried this out for Abelian gauge theory in three and four Euclidean dimensions.

The general features are:

- The sources for a field are certain types of singular dislocations in the configurations of the dual field and also vice versa.
- The role of these dislocations is to force discontinuous boundary conditions on the fields. Thereby new sectors of the field configurations are explored.
- A local theory with both the field and its dual present simultaneously has certain unusual features. Though there are more fields, it is equivalent to the original theory and the degrees of freedom are not changed. This happens because the dual fields are hidden in the auxiliary fields as specific non-local combinations. As a consequence correlations of fields with their duals have unusual properties. The theory is not manifestly rotation invariant. However, rotation covariance is recovered for the 'right' observables with a quantization of the charges of the field and the dual field.
- These features are already known in the context of Dirac's theory of magnetic monopoles. Our thrust is that they are general properties of dual fields and not restricted to electrodynamics. We have obtained Dirac, Schwinger and Zwanziger formulations of electric and magnetic charges via duality transformations. We have emphasized the role of Dirac string as singular dislocation in the configurations of the gauge potentials. These issues are relevant for non-Abelian gauge theory. Many non-perturbative aspects such as confinement are expected to be driven by topological configurations which couple locally to the dual field. This will be discussed elsewhere.

Acknowledgements

I.M. thanks UGC (DRS) for support.

Appendix A

In this appendix, we represent the Dirac vector potential of a monopole in the form of a Green function. The Dirac potential of a monopole located at the origin has the form

$$\vec{A}^D(x) = \frac{1}{4\pi}\frac{\sin\theta}{r(1+\cos\theta)}\hat{\phi} = \frac{1}{4\pi}\hat{n}_3 \times \frac{\hat{r}}{r+x_3} \tag{49}$$

with the Dirac string along the negative z-direction. Here $r = |\vec{x}|$ and $\hat{r} = \vec{x}/|\vec{x}|$. [For checking (49) and other results below, a useful formula is $\hat{n}_3 = \cos\theta\hat{r} - \sin\theta\hat{\theta}$.] Let us write

$$\vec{A}^D(x) = \hat{n}_3 \times \vec{c} \tag{50}$$

where the vector field $\vec{c}$ is undetermined up to addition of a vector in the 3-direction. We choose

$$\vec{c} = \frac{1}{4\pi}\frac{\hat{r}+\hat{n}_3}{r+x_3} \tag{51}$$

so that [9]

$$\vec{c} = \vec{\nabla} f, \quad f = \frac{1}{4\pi}\ln(r+x_3). \tag{52}$$

Note that $\partial_3 f = 1/4\pi r$, and so $\partial_3\nabla^2 f = -\delta(x)$. Thus the Dirac potential at x due to a monopole at x' can be expressed in terms of the Green function for the operator $\partial_3\nabla^2$:

$$\vec{A}^D(x-x') = \frac{1}{4\pi}\hat{n}_3 \times \vec{\nabla}\ln\left(\left|x-x'\right| + x_3 - x'_3\right) \tag{53}$$

$$= -\hat{n}_3 \times \vec{\nabla}\left[\left(\partial_3\nabla^2\right)^{-1}\left(x-x'\right)\right]. \tag{54}$$

[An alternative form of the Dirac potential is $\vec{A}^D(x) = -\hat{\phi}(1/4\pi r)\cot\theta = -\hat{n}_3 \times \hat{r}(x_3/4\pi\rho^2)$, with the Dirac strings along the $\pm z$ directions. Here $\rho^2 = {x_1}^2 + {x_2}^2$. This alternative form is half of the sum of the two forms of $\vec{A}^D(x)$, one having the Dirac string in the negative z-direction and the other having the Dirac string in the positiveve z-direction. In this case, we choose $\vec{c} = (r\hat{n}_3 - x_3\hat{r})/4\pi\rho^2$ in Eq. (50). Then $\vec{c} = \vec{\nabla} f$ and $\partial_3 f = 1/4\pi r$ continue to hold, but with $f = (1/8\pi)\ln((r+x_3)/(r-x_3))$. So Eq. (54) is still valid.]

The result given in Eq. (54) can also be seen by going over to the Fourier space. For the potential of Eq. (49),

$$\vec{\nabla} \times \vec{A}^D(x) = \frac{\hat{r}}{4\pi r^2} + \hat{n}_3\delta(x_1)\delta(x_2)\theta(-x_3). \tag{55}$$

Taking the Fourier transform, we get

$$\vec{k} \times \vec{A}^D(\vec{k}) = -\frac{\vec{k}}{k^2} + \frac{\hat{n}_3}{k_3}. \tag{56}$$

(The Fourier transform of the theta function can be obtained using $d\theta(x)/dx = \delta(x)$.) We now evaluate $\vec{k}\times$ both sides and use $\vec{k}\cdot\vec{A}^D(\vec{k}) = 0$ (since $\vec{\nabla}\cdot\vec{A}^D(x) = 0$) to obtain

$$\vec{A}^D(\vec{k}) = \frac{\hat{n}_3 \times \vec{k}}{k_3 k^2}. \tag{57}$$

This agrees with Eq. (54).

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