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The Korteweg–de Vries equation modified by viscosity for waves in a two-layer fluid in a channel of arbitrary cross section

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Using a two-layer fluid model, Korteweg–de Vries (KdV) equations modified by viscosity are derived that describe weakly nonlinear long waves propagating along a channel of uniform but arbitrary cross section. Equations are deduced for both surface waves and internal waves. The case of high Reynolds number is considered, and the method of matched asymptotic expansion is employed. The coefficients of the KdV equation, which depend on the geometry of the channel cross section, are determined exactly for a rectangular cross section. Some particular cases including the Boussinesq limit are considered.

I. INTRODUCTION

Modification of the Korteweg–de Vries (KdV) equation by viscosity has been considered by some authors. Miles[1] derived a KdV equation modified by viscosity for weakly nonlinear long waves on the free surface of a two-dimensional layer of fluid having a finite depth. He started with equations derived by Chester[2] for oscillation of a liquid in a tank near resonant frequency. Kakutani and Matsuuchi[3] used the method of matched asymptotic expansion to derive the same equation obtained by Miles.[1] Koop and Butler[4] have derived a KdV equation modified by viscosity for internal wave propagation in a density-stratified fluid confined between two rigid horizontal boundaries. They have also derived an expression for the decay caused by viscosity of the amplitude of the internal solitary wave in a two-layer fluid confined between two rigid horizontal planes. This decay of amplitude for the case of a free upper boundary has been considered by Leone, Segur, and Hammack.[5] Recently one of the present authors, Das,[6] derived a KdV equation modified by viscosity for weakly nonlinear long waves propagating along a channel of uniform but arbitrary cross section by using the method of matched asymptotic expansion.[6]

In the present paper we derive KdV equations modified by viscosity for weakly nonlinear long surface, and internal waves in a two-layer fluid propagating along a channel of uniform but arbitrary cross section. An equation describing weakly nonlinear long waves propagating along a channel of uniform but arbitrary cross section (without considering the effect of viscosity) was derived by Pregrine.[7] Using a two-layer model, Grimshaw[8] derived a KdV equation that describes propagation of weakly nonlinear long waves in a channel of arbitrary cross section without considering the effect of viscosity. Peters[9] investigated rotational and irrotational solitary waves in a stream of constant density when the stream is confined to a channel of infinite length with arbitrary cross section. Here also the effect of viscosity was not considered.

We consider the case $R^{-1}, R'^{-1} \ll kh \ll 1$, which is equivalent to the condition $\delta, \delta' \ll h \ll k^{-1}$. Here R, R' are Reynolds numbers for the upper and lower layers, respectively, k is the wavenumber, h is the mean depth, and δ, δ' are thicknesses of the boundary layers in the upper and lower fluids. Obviously this condition corresponds to the case of high Reynolds number and long wavelength. This implies that the viscosity is dominant in the three boundary layers: (i) the boundary layer adjacent to the free surface, (ii) the boundary layer adjacent to the channel surface, and (iii) the boundary layer adjacent to the interface of the two layers. The flow field is divided into two regions. One is the region consisting of the three thin boundary layers mentioned above, which we call the inner region. The other is the area outside the boundary layers, which we call the outer region. The solutions in the two regions are matched by the well-known matched asymptotic method.[10,11]

The KdV equations derived here have the same form as those obtained by Miles,[1] Kakutani and Matsuuchi,[3] and Das.[6] To determine the coefficients of the dispersive term in the KdV equation, it is necessary to solve two two-dimensional Neumann-type boundary value problems. These problems are solved for a rectangular cross section of the channel.

Reduced forms of the KdV equations are obtained in the following cases: (i) Boussinesq limit, (ii) no stratification, (iii) channel with a rectangular cross section, and (iv) equal layer depths in a channel with a rectangular cross section.

II. DERIVATION OF KdV EQUATIONS

The undisturbed free surface and the interface are taken, respectively, as the $z = 0$ and $z = -d^*$ planes; the x axis is directed along the channel, while the z axis is vertical and intersects the lowest portion of the channel. Figure 1 shows the vertical section of the channel.

Let S^* be the cross-sectional area of the channel filled with upper water of lighter density ρ^* at equilibrium; S'^* is the cross-sectional area of the channel at equilibrium filled with lower water of heavier density ρ'^*. Here l^* and l'^* are, respectively, the width of the equilibrium free surface and the interface. Let $h^* = (S^* + S'^*)/l^*$ be the mean depth and $c^* = \sqrt{gh^*}$ be the characteristic velocity, where g is the acceleration caused by gravity.

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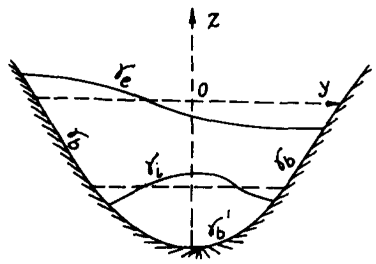

FIG. 1. Channel cross section.

The equations of the perturbed free surface and interface are, respectively, $z^* = \zeta^*(x^*, y^*, t^*)$ and $z^* = -d^* + \zeta'^*(x^*, y^*, t^*)$, where x^*, y^*, z^*, t^* are dimensional space coordinates and time. Let $A^*(x^*, t^*)$ and $A'^*(x^*, t^*)$ be the values of S^* and S'^*, respectively, in the perturbed state.

We introduce dimensionless variables as follows:

$$
\begin{aligned}
&(x, y, z) = (x^*, y^*, z^*)/h^*, \quad t = t^*(g/h^*)^{1/2}, \\
&\mathbf{u} = (u, v, w) = (u^*, v^*, w^*)/c^*, \\
&\mathbf{u}' = (u', v', w') = (u'^*, v'^*, w'^*)/c^*, \\
&p = p^*/(h^*\rho^* g), \quad p' = p'^*/(h^*\rho^* g), \\
&A = A^*/h^{*2}, \quad A' = A'^*/h^{*2}, \\
&\zeta = \zeta^*/h^*, \quad \zeta' = \zeta'^*/h^*,
\end{aligned}
\tag{1}
$$

where $\mathbf{u}$ is the velocity and p is the pressure in the upper, lighter water, and the corresponding quantities in the lower, heavier water are, respectively, $\mathbf{u}', p'$.

The Navier–Stokes equation and the equation of continity in the two layers are

$$\frac{\partial \mathbf{u}^j}{\partial t} + (\mathbf{u}^j \cdot \nabla)\mathbf{u}^j + \beta^j \nabla p^j + (0, 0, 1) = \frac{\beta^j}{R^j} \nabla^2 \mathbf{u}^j, \tag{2}$$

$$\nabla \cdot \mathbf{u}^j = 0, \tag{3}$$

where $(\mathbf{u}^j, p^j, \beta^j) = (\mathbf{u}, p, 1)$ and $(\mathbf{u}', p', \beta)$, respectively, for the upper and lower layer and

$$\beta = \rho/\rho', \quad R = c^* h^*/\nu, \quad R' = c^* h^*/\nu'. \tag{4}$$

Here ν and ν' are the kinematic coefficients of viscosity in the upper and lower layers, and R, R' are the corresponding Reynolds numbers.

The equations of continuity can also be written as

$$\frac{\partial A^j}{\partial t} + \frac{\partial Q^j}{\partial x} = 0, \tag{5}$$

where

$$Q^j = \int_{A^j}\int u^j \, dS \tag{6}$$

and the superscript j has the same meaning as before.

The equation of the free surface and the interface in the dimensionless variables are, respectively, $z = \zeta(x, y, t)$ and $z = -d + \zeta'(x, y, t)$.

The boundary conditions at the free surface are the following:

$$w - \frac{\partial \zeta'}{\partial t} = u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y}, \quad \text{at } z = \zeta, \tag{7}$$

$$
\begin{aligned}
p - p_0 = \frac{2}{R} \Bigg[&\left(\frac{\partial \zeta}{\partial x}\right)^2 \frac{\partial u}{\partial x} + \left(\frac{\partial \zeta}{\partial y}\right)^2 \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
&- \frac{\partial \zeta}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \frac{\partial \zeta}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\
&+ \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\Bigg] \\
&\times \left[1 + \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]^{-1/2}, \\
&\text{at } z = \zeta,
\end{aligned}
\tag{8}
$$

where p_0 is the constant atmospheric pressure.

The following are the boundary conditions at the interface:

$$w - \frac{\partial \zeta'}{\partial t} = u \frac{\partial \zeta'}{\partial x} + v \frac{\partial \zeta'}{\partial y}, \quad \text{at } z = -d + \zeta', \tag{9}$$

$$w' - \frac{\partial \zeta'}{\partial t} = u' \frac{\partial \zeta'}{\partial x} + v' \frac{\partial \zeta'}{\partial y}, \quad \text{at } z = -d + \zeta', \tag{10}$$

$$
\begin{aligned}
&p - \frac{2}{R}\left[\left(\frac{\partial \zeta'}{\partial x}\right)^2 \frac{\partial u}{\partial x} + \left(\frac{\partial \zeta'}{\partial y}\right)^2 \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{\partial \zeta'}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \frac{\partial \zeta'}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{\partial \zeta'}{\partial x}\frac{\partial \zeta'}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
&\quad \times \left[1 + \left(\frac{\partial \zeta'}{\partial x}\right)^2 + \left(\frac{\partial \zeta'}{\partial y}\right)^2\right]^{-1/2} = p' - \frac{2\beta}{R'}\left[\left(\frac{\partial \zeta'}{\partial x}\right)^2 \frac{\partial u'}{\partial x} + \left(\frac{\partial \zeta'}{\partial y}\right)^2 \frac{\partial v'}{\partial y}\right. \\
&\quad \left. + \frac{\partial w'}{\partial z} - \frac{\partial \zeta'}{\partial y}\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right) - \frac{\partial \zeta'}{\partial x}\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right) + \frac{\partial \zeta'}{\partial x}\frac{\partial \zeta'}{\partial y}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\right] \\
&\quad \times \left[1 + \left(\frac{\partial \zeta'}{\partial x}\right)^2 + \left(\frac{\partial \zeta'}{\partial y}\right)^2\right]^{-1/2}, \quad \text{at } z = -d + \zeta'.
\end{aligned}
\tag{11}
$$

Since the velocity should satisfy the no-slip condition at the channel surface, we have

$$\mathbf{u}^j = 0 \tag{12}$$

at the channel surface.

Integrating the x component of Eq. (2) over the cross-sectional area $A^j(x, t)$, we obtain the following equations:

$$\int_A\int\left(\frac{\partial u}{\partial t}+2u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}-\frac{1}{R}\frac{\partial^2 u}{\partial x^2}\right)dS+\int_{\gamma_e}\frac{u[\partial\zeta/\partial t+u(\partial\zeta/\partial x)]}{(1+\zeta_y^2)^{1/2}}dl-\int_{\gamma_i}\frac{u[\partial\zeta'/\partial t+u(\partial\zeta'/\partial x)]}{(1+\zeta'^2_y)^{1/2}}dl$$
$$=\frac{1}{R}\int_{\gamma_e}\frac{\partial u}{\partial n}dl+\frac{1}{R}\int_{\gamma_i}\frac{\partial u}{\partial n}dl+\frac{1}{R}\int_{\gamma_b}\frac{\partial u}{\partial n}dl, \tag{13}$$
$$\int_{A'}\int\left(\frac{\partial u'}{\partial t}+2u'\frac{\partial u'}{\partial x}+\beta\frac{\partial p'}{\partial x}-\frac{\beta}{R'}\frac{\partial^2 u'}{\partial x^2}\right)dS+\int_{\gamma_i}\frac{u'[\partial\zeta'/\partial t+u'(\partial\zeta'/\partial x)]}{(1+\zeta'^2_y)^{1/2}}dl$$
$$=\frac{\beta}{R'}\int_{\gamma_i}\frac{\partial u'}{\partial n}dl+\frac{\beta}{R'}\int_{\gamma'_b}\frac{\partial u'}{\partial n}dl, \tag{14}$$
where γ_e and γ_b are boundary curves for the area A, which are adjacent, respectively, to the free surface and the channel surface; γ'_b is a boundary curve of the area A', which is adjacent to the channel surface; and γ_i is the common boundary curve of the two areas A, A'. In deriving Eqs. (13) and (14), use has been made of Eqs. (7), (9), (10), and (12).

Introducing slow space and time scales as
$$\xi=\epsilon^{1/2}(x-C_0t),\quad \tau=\epsilon^{3/2}t, \tag{15}$$
where C_0 is the linear phase velocity of long waves propagating along the channel, expanding field quantities for the outer region as
$$\begin{bmatrix}\zeta^j_e\\ u^j_e\\ p^j_e\\ Q^j_e\\ A^j_e\end{bmatrix}=\begin{bmatrix}0\\0\\p_0+1-\dfrac{z+1}{\beta^j}\\0\\S^j\end{bmatrix}+\epsilon\begin{bmatrix}\zeta^{j(1)}_e\\ u^{j(1)}_e\\ p^{j(1)}_e\\ Q^{j(1)}_e\\ A^{j(1)}_e\end{bmatrix}+\epsilon^2\begin{bmatrix}\zeta^{j(2)}_e\\ u^{j(2)}_e\\ p^{j(2)}_e\\ Q^{j(2)}_e\\ A^{j(2)}_e\end{bmatrix}+\cdots, \tag{16}$$
$$\begin{bmatrix}v^j_e\\ w^j_e\end{bmatrix}=\epsilon^{3/2}\begin{bmatrix}v^{j(1)}_e\\ w^{j(1)}_e\end{bmatrix}+\epsilon^{5/2}\begin{bmatrix}v^{j(2)}_e\\ w^{j(2)}_e\end{bmatrix}+\cdots, \tag{17}$$
where the subscript e stands for the quantities in the inner region, and solving Eqs. (2)–(14) recursively as in Das,[6] we find the following for C_0^2 from the lowest-order equations,
$$(C_0^2-H)[C_0^2-(1-\beta)H']-\beta\gamma H'C_0^2=0, \tag{18}$$
and the following equation from the next higher-order equations,
$$\frac{\partial\zeta^{(1)}_e}{\partial\tau}+\alpha_1\zeta^{(1)}_e\frac{\partial\zeta^{(1)}_e}{\partial\xi}+\alpha_2\frac{\partial^3\zeta^{(1)}_e}{\partial\xi^3}=\frac{C_0\lambda^{-1}}{l}\{C_0^2F'-[H'(1-\beta)-C_0^2]F\}, \tag{19}$$
where
$$\alpha_1=C_0\bigg[3(H+\gamma\beta^2H')-\frac{C_0^2}{l}[C_0^2-(1-\beta)H']\left(\frac{1}{f'(l_2)}-\frac{1}{f'(-l_1)}\right)$$
$$-\frac{H'}{C_0^2\gamma^2 l}(1-\beta)(C_0^2-H)^2\left(\frac{1}{f'(l'_2)}-\frac{1}{f'(-l'_1)}\right)-\frac{3HH'}{C_0^2}(1-\beta)+\frac{6\beta H'}{C_0^2}(1-\beta)(C_0^2-H)$$
$$+\frac{3H'}{C_0^4\gamma}(1-\beta)^2(C_0^2-H)^2\bigg]\lambda^{-1}, \tag{20}$$
$$\alpha_2=\frac{C_0^3}{l}\bigg[C_0^2\int_{-l_1}^{l_2}\psi(y,0)dy-\int_S\int\psi\,dS-\beta\int_{S'}\int\psi'\,dS-H'(1-\beta)\int_{-l_1}^{l_2}\psi(y,0)dy$$
$$+\frac{H'}{\gamma C_0^2}(C_0^2-H)(1-\beta)\int_{-l'_1}^{l'_2}\psi'(y,-d)dy-\beta H\left(\int_{-l'_1}^{l'_2}\psi(y,-d)dy-\int_{-l'_1}^{l'_2}\psi'(y,-d)\,dy\right)$$
$$+\frac{H'}{C_0^2}(1-\beta)\int_S\int\psi\,dS-\frac{1}{\gamma C_0^2}(C_0^2-H)(1-\beta)\int_{S'}\int\psi'\,dS\bigg]\lambda^{-1}, \tag{21}$$

$$F=\frac{1}{R}\int_{\gamma_e}\frac{\partial u_i^{(1)}}{\partial n}dl+\frac{1}{R}\int_{\gamma_i}\frac{\partial u_i^{(1)}}{\partial n}dl+\frac{1}{R}\int_{\gamma_b}\frac{\partial u_i^{(1)}}{\partial n}dl, \tag{22}$$

$$F'=\frac{\beta}{R'}\int_{\gamma_i}\frac{\partial u_i'^{(1)}}{\partial n}dl+\frac{\beta}{R'}\int_{\gamma_b}\frac{\partial u_i'^{(1)}}{\partial n}dl, \tag{23}$$

$$\lambda=2C_0^2(H+\beta\gamma H')+2H'(1-\beta)(C_0^2-2H). \tag{24}$$

In the above expressions, l, l' are the equilibrium breadths of the free surface and the interface; $H=S/l$, $H'=S'/l'$, $\gamma=l'/l$; $z=f(y)$ is the equation for the intersection curve of the channel surface with the $x=0$ plane; and $-l_1$, l_2 are the y coordinates of the point where the curve intersects with the line $x=0$, $z=0$ and $(-l_1', l_2')$ are the same with the line $z=-d$, $x=0$. Here $\psi(y,z)$, $\psi'(y,z)$ satisfy the following equations and boundary conditions:

$$\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=1,\quad \text{in } S,$$
$$\left(\frac{\partial\psi}{\partial z}\right)_{z=0}=C_0^2,\quad \left(\frac{\partial\psi}{\partial z}\right)_{z=-d}=\frac{C_0^2-H}{\gamma}, \tag{25}$$
$$\frac{\partial\psi}{\partial n}=0,\quad \text{on the channel surface},$$

$$\frac{\partial^2\psi'}{\partial y^2}+\frac{\partial^2\psi'}{\partial z^2}=1,\quad \text{in } S',\quad \left(\frac{\partial\psi'}{\partial z}\right)_{z=-d}=H', \tag{26}$$
$$\frac{\partial\psi'}{\partial n}=0,\quad \text{on the channel surface.}$$

In order to determine the quantities $(1/R^j)\int(\partial u_i^j/\partial n)dl$ appearing in Eq. (19), in the lowest order we proceed in the same way as in Das'[6] derivation of equations in the inner layer adjacent to the free surface, and obtain the following equations for the inner region adjacent to the interface:

$$-\epsilon^{1/2}C_0\frac{\partial u_i^j}{\partial\xi}+\epsilon^{3/2}\frac{\partial u_i^j}{\partial\tau}+\epsilon^{1/2}u_i^j\frac{\partial u_i^j}{\partial\xi}+w_i^j\frac{\partial u_i^j}{\partial Z}+v_i^j\frac{\partial u_i^j}{\partial Y}+\epsilon^{1/2}\beta^j\frac{\partial p_i^j}{\partial\xi}$$
$$=\frac{\epsilon^{7/2}}{R^j}\beta^j\frac{\partial^2u_i^j}{\partial\xi^2}+\frac{\epsilon^{5/2}\beta^j}{R^j}\left(\frac{\partial^2u_i^j}{\partial Y^2}+\frac{1}{\epsilon^2}\frac{\partial^2u_i^j}{\partial Z^2}\right), \tag{27}$$

$$-\epsilon^{1/2}C_0\frac{\partial v_i^j}{\partial\xi}+\epsilon^{3/2}\frac{\partial v_i^j}{\partial\tau}+\epsilon^{1/2}u_i^j\frac{\partial v_i^j}{\partial\xi}+v_i^j\frac{\partial v_i^j}{\partial Y}+w_i^j\frac{\partial v_i^j}{\partial Z}+\beta^j\frac{\partial p_i^j}{\partial Y}$$
$$=\frac{\epsilon^{7/2}\beta^j}{R^j}\frac{\partial^2v_i^j}{\partial\xi^2}+\frac{\epsilon^{5/2}\beta^j}{R^j}\left(\frac{\partial^2v_i^j}{\partial Y^2}+\frac{1}{\epsilon^2}\frac{\partial^2v_i^j}{\partial Z^2}\right), \tag{28}$$

$$-\epsilon^{3/2}C_0\frac{\partial w_i^j}{\partial\xi}+\epsilon^{5/2}\frac{\partial w_i^j}{\partial\tau}+\epsilon^{3/2}u_i^j\frac{\partial w_i^j}{\partial\xi}+\epsilon v_i^j\frac{\partial w_i^j}{\partial Y}+\epsilon w_i^j\frac{\partial w_i^j}{\partial Z}+\frac{\beta^j}{\epsilon}\frac{\partial p_i^j}{\partial Z}+1$$
$$=\frac{\epsilon^{9/2}\beta^j}{R^j}\frac{\partial^2w_i^j}{\partial\xi^2}+\frac{\epsilon^{5/2}\beta^j}{R^j}\left(\epsilon\frac{\partial^2w_i^j}{\partial Y^2}+\frac{1}{\epsilon}\frac{\partial^2w_i^j}{\partial Z^2}\right). \tag{29}$$

The continuity of velocity and the tangential stress across the interface give the following equations [for the quantities in the inner region, $(\ )_1$ will designate the value of the quantity inside the bracket at $z=0$]:

$$(u_i)_1-(u_i')_1=\text{nonlinear terms}, \tag{30}$$

$$\nu\left[-\epsilon^{3/2}C_0\left(\frac{\partial w_i}{\partial\xi}\right)_1+\frac{1}{\epsilon}\left(\frac{\partial u_i}{\partial Z}\right)_1\right]-\beta\nu'\left[-\epsilon^{3/2}C_0\left(\frac{\partial w_i'}{\partial\xi}\right)_1+\frac{1}{\epsilon}\left(\frac{\partial u_i'}{\partial Z}\right)_1\right]=\text{nonlinear terms}. \tag{31}$$

The inner and outer solutions should satisfy the following matching conditions:

$$\lim_{Z\to\infty}u_i=(u_e)_1,\quad \lim_{Z\to\infty}p_i=(p_e)_1,$$
$$\lim_{Z\to-\infty}u_i'=(u_e')_1,\quad \lim_{Z\to-\infty}p_i'=(p_e')_1. \tag{32}$$

Substituting the perturbation expansion

$$\begin{bmatrix}u_i^j\\p_i^j\end{bmatrix}=\begin{bmatrix}0\\p_0+1-1/\beta j\end{bmatrix}+\epsilon\begin{bmatrix}u_i^{j(1)}\\-z/\beta j+p_i^{j(1)}\end{bmatrix}+\epsilon^2\begin{bmatrix}u_i^{j(2)}\\p_i^{j(2)}\end{bmatrix}+\cdots, \tag{33}$$

$$\begin{bmatrix}v_i^j\\w_i^j\end{bmatrix}=\epsilon^{3/2}\begin{bmatrix}v_i^{j(1)}\\w_i^{j(1)}\end{bmatrix}+\epsilon^{5/2}\begin{bmatrix}v_i^{j(2)}\\w_i^{j(2)}\end{bmatrix}+\cdots \tag{34}$$

for the quantities in the inner region in Eqs. (27)–(32) and solving the equations in the lowest order, we obtain solutions for $u_i^{j(1)}$. Substituting these solutions of $u_i^{j(1)}$, we obtain expressions for the integrals $\int_{\gamma_i}(\partial u_i^j/\partial n)dl$ in the lowest order.

Proceeding in the same way as in Das,[6] we obtain expressions for $\int_{\gamma_b^j}(\partial u_i^j/\partial n)dl$ in the lowest order, and find that $\int_{\gamma_e}(\partial u_i/\partial n)dl$ vanishes. Substituting these expressions for the integrals, we obtain the following KdV equation modified by viscosity for the surface wave mode:

$$\frac{\partial\zeta_e^{(1)}}{\partial\tau}+\alpha_1\zeta_e^{(1)}\frac{\partial\zeta_e^{(1)}}{\partial\xi}+\alpha_2\frac{\partial^3\zeta_e^{(1)}}{\partial\xi^3}=\alpha_3\int_{-\infty}^{\infty}\frac{\partial\zeta_e^{(1)}}{\partial\xi'}\frac{1-\text{sgn}(\xi-\xi')}{|\xi-\xi'|^{1/2}}d\xi', \tag{35}$$

where

$$\alpha_3=\Bigg[\frac{C_0^2L}{2l\sqrt{\pi R}}+\frac{C_0^2\beta^2L'}{2l\sqrt{\pi R'}}-\frac{C_0^{1/2}\beta^{1/2}(1-\beta)[H-C_0^2(1-\gamma)]}{2\sqrt{\pi RR'}(\sqrt{R}\,\nu+\sqrt{\beta R'}\,\nu')}\times\left(R\nu+R'\nu'\frac{\beta\gamma H'}{C_0^2-H}\right)-(1-\beta)\left(\frac{LH'}{2l\sqrt{\pi R}}-\frac{\beta L'(C_0^2-H)}{2l\gamma\sqrt{\pi R'}}\right)\Bigg]\lambda^{-1}, \tag{36}$$

and $C_0^2=C_s^2$, the greater of the two roots of Eq. (18), which corresponds to the surface wave mode.

The KdV equation modified by viscosity for the internal wave, obtained from (35) by setting $\zeta_e^{(1)} = C_0^2\gamma(C_0^2 - H)^{-1}\zeta_e'^{(1)}$, is as follows:

$$\frac{\partial \zeta_e'^{(1)}}{\partial \tau} + \alpha_1'\zeta_e'^{(1)}\frac{\partial \zeta_e'^{(1)}}{\partial \xi} + \alpha_2'\frac{\partial^3 \zeta_e'^{(1)}}{\partial \xi^3}$$

$$= \alpha_3'\int_{-\infty}^{\infty}\frac{\partial \zeta_e'^{(1)}}{\partial \xi'}\frac{1-\mathrm{sgn}(\xi-\xi')}{|\xi-\xi'|^{1/2}}d\xi', \quad (37)$$

where

$$\alpha_1' = \alpha_1 C_0^2\gamma(C_0^2 - H)^{-1}, \quad \alpha_2' = \alpha_2, \quad \alpha_3' = \alpha_3, \quad (38)$$

and $C_0^2 = C_I^2$, the smaller of the two roots of Eq. (18), which corresponds to the internal wave mode.

III. SPECIFIC CASES

In this section we consider some specific cases. The reduced forms of the coefficients of the KdV equations (35) and (37) are given in the following four cases.

A. Boussinesq limit

The Boussinesq limit corresponds to very small $1-\beta$. In this limit C_S^2 and C_I^2 become

$$C_S^2 = H + \gamma H' + O(1-\beta),$$
$$C_I^2 = \frac{HH'(1-\beta)}{H+\gamma H'} + O(1-\beta)^2. \quad (39)$$

The coefficients of Eqs. (35) and (37) become

$$\alpha_1 = \frac{3}{2}(H+\gamma H')^{-1/2}\left[1 + \frac{(H+\gamma H')}{3l}\left(\frac{1}{f'(l_2)} - \frac{1}{f'(-l_1)}\right)\right] + O(1-\beta), \quad (40)$$

$$\alpha_2 = \frac{(H+\gamma H')^{-1/2}}{2l}\left((H+\gamma H')\int_{-l_1}^{l_2}\psi_s(y,0)dy - H'\right.$$
$$\left.\times\int_{-l_1'}^{l_2'}\psi_3(y,-d)dy + H'\int_{-l_1'}^{l_2'}\psi_s'(y,-d)dy - \int_S\int\psi_S\,dS - \int_{S'}\int\psi_S'\,dS\right) + O(1-\beta), \quad (41)$$

$$\alpha_3 = \frac{1}{4l\sqrt{\pi}}(H+\gamma H')^{-1}\left(\frac{L}{\sqrt{R}} + \frac{L'}{\sqrt{R'}}\right) + O(1-\beta), \quad (42)$$

$$\alpha_1' = -\frac{3\gamma(1-\beta)^{1/2}H^{-1/2}H'^{1/2}}{2(H+\gamma H')}\left[1 - \frac{H}{\gamma H'} + \frac{H}{3\gamma^2 l}\left(\frac{1}{f'(l_2')} - \frac{1}{f'(-l_1')}\right)\right], \quad (43)$$

$$\alpha_2' = -\frac{(1-\beta)^{1/2}(HH')^{1/2}}{2l(H+\gamma H')^{3/2}}\left(\frac{\gamma H'}{H}\int_S\int\psi_I\,dS + \frac{H}{\gamma H'}\int_{S'}\int\psi_I'\,dS - \frac{H}{\gamma}\int_{-l_1'}^{l_2'}\psi_I'(y,-d)dy - H'\int_{-l_1'}^{l_2'}\psi_I(y,-d)dy\right), \quad (44)$$

$$\alpha_3' = \frac{(HH')^{-1}(H+\gamma H')^{-1}}{2l\sqrt{\pi}}\left(\frac{\gamma L H'^2}{\sqrt{R}} + \frac{L'H^2}{\gamma\sqrt{R'}}\right). \quad (45)$$

Here ψ_S, ψ_S', ψ_I, ψ_I' satisfy the following equations and boundary conditions:

$$\frac{\partial^2\psi_{S,I}}{\partial y^2} + \frac{\partial^2\psi_{S,I}}{\partial z^2} = 1, \quad \frac{\partial^2\psi_{S,I}'}{\partial y^2} + \frac{\partial^2\psi_{S,I}'}{\partial z^2} = 1,$$
$$\left(\frac{\partial\psi_S}{\partial z}\right)_{z=0} = H + \gamma H', \quad \left(\frac{\partial\psi_I}{\partial z}\right)_{z=0} = 0,$$
$$\left(\frac{\partial\psi_s}{\partial z}\right)_{z=-d} = H',$$
$$\left(\frac{\partial\psi_I}{\partial z}\right)_{z=-d} = -\frac{H}{\gamma}, \quad \left(\frac{\partial\psi_{S,I}'}{\partial z}\right)_{z=-d} = H'. \quad (46)$$

B. No stratification

In this case there is only a surface wave, and the KdV equation modified by viscosity for this wave is Eq. (35), where the coefficients are now given by (40)–(42) with the $O(1-\beta)$ terms omitted. The KdV equation obtained in this case is the same as (70) in Das' paper.[6]

C. Channel of rectangular cross section

Let the cross section of the channel be rectangular and d, d' be the depths of the upper and lower layers.

In this case, Eq. (18), which determines C_S^2 and C_I^2, becomes

$$(C_0^2 - d)[C_0^2 - (1-\beta)d'] = \beta d' C_0^2; \quad (47)$$

the solutions of Eqs. (25) for ψ and Eqs. (26) for ψ' are, respectively,

$$\psi = \tfrac{1}{2}z^2 + C_0^2 z,$$
$$\psi' = \tfrac{1}{2}(z+1)^2 + \tfrac{1}{2}(d-d') - C_0^2 d. \quad (48)$$

In this case the coefficients of Eqs. (35) and (37) become

$$\alpha_1 = \left(3C_0(d+\beta^2 d') - \frac{3dd'}{C_0}(1-\beta) + \frac{6\beta d'}{C_0}(1-\beta)\right.$$
$$\left.\times(C_0^2 - d) + \frac{3d'}{C_0^3}(1-\beta)^2(C_0^2-d)^2\right)\lambda_0^{-1}, \quad (49)$$

$$\alpha_2 = [(C_0^3 d^2/6)(3C_0^2 - d) + (\beta C_0^3 d'/6) \times (6C_0^2 d - 3d + 3d' - d'^2) + \tfrac{1}{3} C_0 d'(1-\beta) \times (C_0^2 - d)(d'^2 - 3) + \tfrac{1}{6} C_0 d^2 d' \times (1-\beta)(d - 3C_0^2)]\lambda_0^{-1}, \tag{50}$$

$$\alpha_3 = \left[\frac{C_0^2 d}{l\sqrt{\pi R}} + \frac{C_0^2 \beta^2 (l + 2d')}{2l\sqrt{\pi R'}} - \frac{C_0^{1/2}\beta^{1/2} d(1-\beta)}{2\sqrt{\pi R R'}(\sqrt{R}\,\nu + \sqrt{\beta R'}\,\nu')}\left(R\nu + \frac{R'\nu'\beta d'}{C_0^2 - d}\right) - (1-\beta)\left(\frac{H'd}{l\sqrt{\pi R}} - \frac{\beta(l+2d')(C_0^2 - H)}{2l\sqrt{\pi R'}}\right)\right]\lambda_0^{-1}, \tag{51}$$

$$\lambda_0 = 2C_0^2(d + \beta d') + 2d'(1-\beta)(C_0^2 - 2d), \tag{52}$$

$$\alpha_1' = \alpha_1 C_0^2 \gamma (C_0^2 - d)^{-1}, \quad \alpha_2' = \alpha_2, \quad \alpha_3' = \alpha_3. \tag{53}$$

In the Boussinesq approximation, C_S^2 and C_I^2 become

$$C_S^2 = 1 + O(1-\beta), \quad C_I^2 = dd'(1-\beta), \tag{54}$$

and the above coefficients of the KdV equation are

$$\alpha_1 = \frac{3}{2}, \quad \alpha_2 = \frac{1}{6}, \quad \alpha_3 = \frac{1}{4\sqrt{\pi} l}\left(\frac{2d}{\sqrt{R}} + \frac{2d' + l}{\sqrt{R'}}\right),$$

$$\alpha_1' = -\frac{3}{2}\left(\frac{d'}{d}\right)^{1/2}\left(1 - \frac{d}{d'}\right)(1-\beta)^{1/2}, \tag{55}$$

$$\alpha_2' = -\tfrac{1}{6}(dd')^{1/2}(1-\beta)^{1/2}(3d^2 + 3d'^2 + 5dd'),$$

$$\alpha_3' = \frac{1}{4\sqrt{\pi} l d'}\left(\frac{2d'^2}{\sqrt{R}} + \frac{(2d' + l)d}{\sqrt{R'}}\right).$$

D. Equal layer depths in a channel of rectangular cross section

In this case the coefficients of Eqs. (35) and (37) can be obtained from Eqs. (49)–(53) by setting $d = d' = \frac{1}{2}$. In the Boussinesq limit these coefficients become

$$\alpha_1 = \frac{3}{2}, \quad \alpha_2 = \frac{1}{6}, \quad \alpha_3 = \frac{1}{4l\sqrt{\pi}}\left(\frac{1}{\sqrt{R}} + \frac{l+1}{\sqrt{R'}}\right),$$

$$\alpha_1' = -\frac{3}{8}(1-\beta)^{3/2}, \quad \alpha_2' = -\frac{11}{48}(1-\beta)^{1/2}, \tag{56}$$

$$\alpha_3' = \frac{1}{4l\sqrt{\pi}}\left(\frac{1}{\sqrt{R}} + \frac{l+1}{\sqrt{R'}}\right).$$

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