

THE GENERALIZED APPROXIMATE PERRON INTEGRAL

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ABSTRACT. In this paper we introduce the concept of the generalized AP-integral and discuss the properties of this integral.

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1. Introduction

The AP-integral or more precisely the approximately continuous Perron integral was first defined by Burkil [1] and its Riemann-type definition was given by Bullen [2]. Schwabik [5] presented a generalized version of the Perron integral leading to the new approach to a Generalized Ordinary Differential Equation. Kurzweil (1957) and Henstock (1961) independently found that Perron integration could be given by a generalized Riemann integral. Perron integration, being equivalent to special Denjoy integration, includes Lebesgue integration so that it was clear that the latter is included in generalized Riemann integration ([4]). The approximate Perron integral of Burkil (1931) has a generalized Riemann integral form that is easily obtained ([3]). A distinguishing feature of modern Mathematics is the establishment of patterns of behaviour in mathematical systems, leading to a unified set of proofs of results instead of a heterogeneous collection of proofs, one set for each system. In this paper, our aim is to unify the variety of generalized Riemann-type, Riemann-Stieltjes type and approximate Perron integrals by using functions of two variables from $[a, b] \times [a, b]$ to \mathbb{R} .

We now recall the following definitions presented in the book [4].

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DEFINITION 1.1. A collection Δ of closed subintervals of $[a, b]$ is called an *approximate full cover (AFC)* of $[a, b]$ if for every $x \in [a, b]$ there exists a measurable set $D_x \subset [a, b]$ such that $x \in D_x$ and D_x has density 1 at x , with $[u, v] \in \Delta$ whenever $u, v \in D_x$ and $u \leq x \leq v$.

A division of $[a, b]$ obtained by $a = x_0 < x_1 < \dots < x_n = b$ and $\{\xi_1, \xi_2, \dots, \xi_n\}$ is called a Δ -*division* if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, $x_{i-1} \leq \xi_i \leq x_i$ and $x_{i-1}, x_i \in D_{\xi_i}$ for all i . We call ξ_i the *associated point* of $[x_{i-1}, x_i]$ and x_i ($i = 0, 1, \dots, n$) the *division points*.

A division of $[a, b]$ given by $a \leq y_1 \leq \zeta_1 \leq z_1 \leq y_2 \leq \zeta_2 \leq z_2 \leq \dots \leq y_m \leq \zeta_m \leq z_m \leq b$ is called a Δ -*partial division* if Δ is an approximate full cover with $([y_i, z_i], \zeta_i) \in \Delta$, for $i = 1, 2, \dots, m$.

The next Cousin-type lemma from [4] makes it possible to give a Riemann-type definition of the AP-integral.

LEMMA 1.2. *If Δ is an approximate full cover of $[a, b]$, then there exists a Δ -division of $[a, b]$.*

DEFINITION 1.3. A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be *AP-integrable to a real number A* if for every $\epsilon > 0$ there is an AFC Δ of $[a, b]$ such that for every Δ -division $D = ([u, v], \xi)$ of $[a, b]$ we have

$$\left| (D) \sum f(\xi)(v - u) - A \right| < \epsilon$$

and we write $A = (AP) \int_a^b f$.

Now the generalized approximate Perron integral is defined as follows:

DEFINITION 1.4. A function $U: [a, b] \times [a, b] \rightarrow \mathbb{R}$ is said to be *generalized AP (GAP)-integrable to a real number A* if for every $\epsilon > 0$ there is an AFC Δ of $[a, b]$ such that for every Δ -division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$\left| (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - A \right| < \epsilon$$

and we write $A = (GAP) \int_a^b U$.

The set of all functions U which are generalized approximate Perron integrable on $[a, b]$ is denoted by $GAP[a, b]$. We use the notation

$$S(U, D) = (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\}$$

for the Riemann-type sum corresponding to the function U and the Δ -division $D = ([\alpha, \beta], \tau)$ of $[a, b]$.

Note that the integral is uniquely determined.

Remark 1.5. If the AFC Δ in Definition 1.4 is replaced by an ordinary full cover, that is, the family of all interval-point pairs $([u, v], \xi)$ which are δ -fine for some $\delta(\xi) > 0$, then we have the definition of Henstock integral ([4]).

Setting $U(\tau, t) = f(\tau)t$ and $U(\tau, t) = f(\tau)g(t)$ where $f, g: [a, b] \rightarrow \mathbb{R}$ and $\tau, t \in [a, b]$, we obtain Riemann-type and Riemann-Stieltjes type integrals for the functions f, g and a given Δ -division D of $[a, b]$.

Considering $U(\tau, t) = f(\tau)t$ in Definition 1.4 we obtain the classical approximately continuous Perron integral.

This definition is given in a more general form because of the general form of the function U . Setting $U(\tau, t) = f(\tau)t$ and replacing AFC Δ by an ordinary full cover it becomes Perron's major and minor functions in the known form.

2. Fundamental properties

THEOREM 2.1. *If $U, V \in GAP[a, b]$ and $c_1, c_2 \in \mathbb{R}$, then $(c_1U + c_2V) \in GAP[a, b]$ and*

$$(GAP) \int_a^b (c_1U + c_2V) = c_1(GAP) \int_a^b U + c_2(GAP) \int_a^b V.$$

Proof. Let $A = (GAP) \int_a^b U$ and $B = (GAP) \int_a^b V$. Then given $\epsilon > 0$ there is an approximate full cover Δ_1 of $[a, b]$ such that for every Δ_1 -division $D_1 = ([\alpha', \beta'], \tau')$ of $[a, b]$ we have

$$\left| (D_1) \sum \{U(\tau', \beta') - U(\tau', \alpha')\} - A \right| < \epsilon/2. \quad (2.1)$$

Similarly, there is an approximate full cover Δ_2 of $[a, b]$ such that for every Δ_2 -division $D_2 = ([\alpha'', \beta''], \tau'')$ of $[a, b]$ we have

$$\left| (D_2) \sum \{V(\tau'', \beta'') - V(\tau'', \alpha'')\} - B \right| < \epsilon/2. \quad (2.2)$$

Putting $\Delta = \Delta_1 \cap \Delta_2$, we get an approximate full cover of $[a, b]$.

Therefore, for any Δ -division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$\begin{aligned} & \left| (D) \sum \{ (c_1 U + c_2 V)(\tau, \beta) - (c_1 U + c_2 V)(\tau, \alpha) \} - (c_1 A + c_2 B) \right| \\ & \leq \left| (D) \sum c_1 \{ U(\tau, \beta) - U(\tau, \alpha) \} - c_1 A \right| \\ & \quad + \left| (D) \sum c_2 \{ V(\tau, \beta) - V(\tau, \alpha) \} - c_2 B \right| \\ & < |c_1| \epsilon / 2 + |c_2| \epsilon / 2, \quad \text{by (2.1) and (2.2).} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this gives

$$(GAP) \int_a^b (c_1 U + c_2 V) = c_1 (GAP) \int_a^b U + c_2 (GAP) \int_a^b V. \quad \square$$

THEOREM 2.2 (Cauchy Criterion). *$U \in GAP[a, b]$ if and only if for every $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that*

$$|S(U, D_1) - S(U, D_2)| < \epsilon$$

for any two Δ -divisions D_1, D_2 of $[a, b]$.

PROOF. Let $U \in GAP[a, b]$. Then to each $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that for any Δ -division D of $[a, b]$ we have

$$\left| S(U, D) - (GAP) \int_a^b U \right| < \epsilon / 2.$$

Hence for any two Δ -divisions D_1, D_2 of $[a, b]$ we have

$$|S(U, D_1) - S(U, D_2)| < \epsilon.$$

Conversely, suppose that the Cauchy Criterion is satisfied. That is, for every $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that for any two Δ -divisions D_1, D_2 of $[a, b]$ we have

$$|S(U, D_1) - S(U, D_2)| < \epsilon.$$

Let $\epsilon_1 > \epsilon_2 > \dots > \epsilon_n > \dots > 0$ with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

For each positive integer n , there is an approximate full cover Δ'_n of $[a, b]$ such that for any two Δ'_n -divisions D_n, D'_n of $[a, b]$ we have

$$|S(U, D_n) - S(U, D'_n)| < \epsilon_n.$$

Let $\Delta_1 = \Delta'_1$, $\Delta_n = \Delta'_1 \cap \Delta'_2 \cap \dots \cap \Delta'_n$ ($n = 2, 3, \dots$) and fix a Δ_n -division D_n of $[a, b]$.

THE GENERALIZED APPROXIMATE PERRON INTEGRAL

Let $t_n = S(U, D_n)$. It is clear that $\Delta_{n+1} \subseteq \Delta_n$, $n = 1, 2, 3, \dots$

Note that $\{t_n\}$ is a Cauchy sequence of real numbers and let $A = \lim_{n \rightarrow \infty} t_n$. Then given $\epsilon > 0$ there exists a positive integer N such that $|t_N - A| < \epsilon_N < \epsilon/2$. Then for any Δ_N -division D'_N of $[a, b]$ we have

$$|S(U, D'_N) - A| \leq |S(U, D'_N) - t_N| + |t_N - A| < \epsilon_N + \epsilon_N < \epsilon.$$

Therefore, $U \in \text{GAP}[a, b]$ and $(\text{GAP}) \int_a^b U = A$. □

THEOREM 2.3. *If $U \in \text{GAP}[a, b]$, then for every $[c, d] \subset [a, b]$ we have $U \in \text{GAP}[c, d]$.*

Proof. Since $U \in \text{GAP}[a, b]$, by Cauchy criterion, given $\epsilon > 0$ there exists an approximate full cover Δ of $[a, b]$ such that for any two Δ -divisions D, D' of $[a, b]$ we have

$$|S(U, D) - S(U, D')| < \epsilon.$$

We assume that $a < c < d < b$. Take any two Δ -divisions D_1, D_2 of $[c, d]$ and another Δ -division D_3 of $[a, c] \cup [d, b]$.

Then $D_1 \cup D_3$ forms a Δ -division of $[a, b]$. Then $S(U, D_1 \cup D_3) = S(U, D_1) + S(U, D_3)$.

Similarly, $S(U, D_2 \cup D_3) = S(U, D_2) + S(U, D_3)$ and $D_2 \cup D_3$ is a Δ -division of $[a, b]$. So, by Cauchy criterion, we have

$$\begin{aligned} |S(U, D_1) - S(U, D_2)| &= |\{S(U, D_1) + S(U, D_3)\} - \{S(U, D_2) + S(U, D_3)\}| \\ &= |S(U, D_1 \cup D_3) - S(U, D_2 \cup D_3)| < \epsilon. \end{aligned}$$

Hence $U \in \text{GAP}[c, d]$. □

THEOREM 2.4. *If $a < c < b$ and $U \in \text{GAP}[a, c]$, $U \in \text{GAP}[c, b]$, then $U \in \text{GAP}[a, b]$ and*

$$(\text{GAP}) \int_a^b U = (\text{GAP}) \int_a^c U + (\text{GAP}) \int_c^b U.$$

Proof. Let $\epsilon > 0$ be arbitrary. Let $A = (\text{GAP}) \int_a^c U$ and $B = (\text{GAP}) \int_c^b U$. By the assumption, there exists an approximate full cover Δ_1 of $[a, c]$ such that for any Δ_1 -division $D_1 = ([\alpha, \beta], \tau)$ of $[a, c]$ we have

$$|S(U, D_1) - A| < \epsilon/2.$$

We may assume that when $\tau \in [a, c)$, the interval $[\alpha, \beta]$ of D_1 is contained in $[a, c)$.

Similarly, there exists an approximate full cover Δ_2 of $[c, b]$ such that for any Δ_2 -division $D_2 = ([\alpha', \beta'], \tau')$ of $[c, b]$ we have

$$|S(U, D_2) - B| < \epsilon/2.$$

When $\tau' \in (c, b]$, the interval $[\alpha', \beta']$ of D_2 is contained in $(c, b]$. Let $D_{1,c}$ be an approximate neighbourhood of c contained in $[a, c]$. Furthermore, $([\alpha, c], c) \in \Delta_1$ whenever $\alpha \in D_{1,c}$ and $\alpha < c$.

Similarly, we can choose an approximate neighbourhood $D_{2,c}$ of c contained in $[c, b]$ and $([c, \beta], c) \in \Delta_2$ whenever $\beta \in D_{2,c}$ and $c < \beta$.

We define Δ as follows.

When $\tau \in [a, c)$ we choose the interval-point pair $([\alpha, \beta], \tau)$ from Δ_1 and we choose the interval-point pair $([\alpha, \beta], \tau)$ from Δ_2 if $\tau \in (c, b]$.

When $\tau = c$, we choose the interval-point pair $([\alpha, \beta], c)$ such that $([\alpha, c], c) \in \Delta_1$ and $([c, \beta], c) \in \Delta_2$.

This creates our required Δ .

Then for every Δ -division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we can express the sum $S(U, D)$ as the sum of $S(U, D_1)$ and $S(U, D_2)$ in which $\tau \in [a, c]$ and $\tau \in [c, b]$ respectively and D_1 is a Δ_1 -division of $[a, c]$ and D_2 is a Δ_2 -division of $[c, b]$.

Then

$$\begin{aligned} |S(U, D) - (A + B)| &= |S(U, D_1) + S(U, D_2) - (A + B)| \\ &\leq |S(U, D_1) - A| + |S(U, D_2) - B| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence $U \in \text{GAP}[a, b]$ and

$$(\text{GAP}) \int_a^b U = A + B = (\text{GAP}) \int_a^c U + (\text{GAP}) \int_c^b U.$$

□

The following theorem has an important use in the theory of generalized Perron integral.

THE GENERALIZED APPROXIMATE PERRON INTEGRAL

THEOREM 2.5 (Saks-Henstock Lemma). *Let $U : [a, b] \times [a, b] \rightarrow \mathbb{R}$ be GAP-integrable over $[a, b]$. Then given $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that for every Δ -division $D = \{([\alpha_{j-1}, \alpha_j], \tau_j) : j = 1, 2, \dots, q\}$ of $[a, b]$ we have*

$$\left| \sum_{j=1}^q \{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP) \int_a^b U \right| < \epsilon.$$

Then if $\{([\beta_j, \gamma_j], \zeta_j) : j = 1, 2, \dots, m\}$ represents a Δ -partial division of $[a, b]$ we have

$$\left| \sum_{j=1}^m \left[\{U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)\} - (GAP) \int_{\beta_j}^{\gamma_j} U \right] \right| < \epsilon.$$

Proof. Without any loss of generality it can be assumed that $\beta_j < \gamma_j$ for $j = 1, 2, \dots, m$. Denote $\gamma_0 = a$ and $\beta_{m+1} = b$. If $\gamma_j < \beta_{j+1}$ for some $j = 0, 1, \dots, m$ then the integral $(GAP) \int_{\gamma_j}^{\beta_{j+1}} U$ exists by Theorem 2.3 and therefore for every $\epsilon > 0$ there is an approximate full cover $\Delta_j \subseteq \Delta$ of $[\gamma_j, \beta_{j+1}]$ such that for every Δ_j -division D^j of $[\gamma_j, \beta_{j+1}]$ we have

$$\left| S(U, D^j) - (GAP) \int_{\gamma_j}^{\beta_{j+1}} U \right| < \epsilon/2(m+1).$$

If $\gamma_j = \beta_{j+1}$, then we take $S(U, D^j) = 0$. Then $\sum_{j=1}^m [U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)] + \sum_{j=1}^m S(U, D^j)$ represents an integral sum which corresponds to a certain Δ -division and hence

$$\left| \sum_{j=1}^m [U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)] + \sum_{j=1}^m S(U, D^j) - (GAP) \int_a^b U \right| < \epsilon/2.$$

Thus

$$\begin{aligned}
 & \left| \sum_{j=1}^m \left[\{U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)\} - (GAP) \int_{\beta_j}^{\gamma_j} U \right] \right| \\
 &= \left| \sum_{j=1}^m [U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)] + \sum_{j=1}^m S(U, D^j) - \sum_{j=1}^m S(U, D^j) \right. \\
 &\quad \left. - (GAP) \int_a^b U + \sum_{j=1}^m (GAP) \int_{\gamma_j}^{\beta_{j+1}} U \right| \\
 &\leq \left| \sum_{j=1}^m [U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)] + \sum_{j=1}^m S(U, D^j) - (GAP) \int_a^b U \right| \\
 &\quad + \sum_{j=1}^m \left| S(U, D^j) - (GAP) \int_{\gamma_j}^{\beta_{j+1}} U \right| \\
 &< \epsilon/2 + (m+1) \epsilon/2(m+1) < \epsilon/2 + \epsilon/2 = \epsilon.
 \end{aligned}$$

Hence the result is obtained. \square

Using the Saks-Henstock lemma presented in Theorem 2.5 we obtain the following corollary of it.

COROLLARY 2.6. *If $U \in GAP[a, b]$, then for every $\epsilon > 0$ there exists an approximate full cover Δ of $[a, b]$ such that for every Δ -division $D = \{([\alpha_{j-1}, \alpha_j], \tau_j) : j = 1, 2, \dots, q\}$ of $[a, b]$ we have*

$$\sum_{j=1}^q \left| [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] - (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U \right| < \epsilon.$$

Proof. Since $U \in GAP[a, b]$, given $\epsilon > 0$ there exists an approximate full cover Δ of $[a, b]$ such that for every Δ -division $D = \{([\alpha_{j-1}, \alpha_j], \tau_j) : j = 1, 2, \dots, q\}$ of $[a, b]$ we have

$$\left| \sum_{j=1}^q [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] - (GAP) \int_a^b U \right| < \epsilon/2$$

THE GENERALIZED APPROXIMATE PERRON INTEGRAL

or,

$$\left| \sum_{j=1}^q [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] - \sum_{j=1}^q (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U \right| < \epsilon/2$$

or,

$$\left| \sum_{j=1}^q \left[\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U \right] \right| < \epsilon/2.$$

We decompose the division D into two systems D^+ and D^- consisting of non-negative and negative contributions of $\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U$ respectively.

By Saks-Henstock lemma, we obtain

$$(D^+) \sum_{j=1}^q \left[\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U \right] < \epsilon/2$$

and

$$(D^-) \sum_{j=1}^q \left[(GAP) \int_{\alpha_{j-1}}^{\alpha_j} U - \{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} \right] < \epsilon/2.$$

Hence

$$\begin{aligned} & \sum_{j=1}^q \left| [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] - (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U \right| \\ &= (D^+) \sum_{j=1}^q \left[\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP) \int_{\alpha_{j-1}}^{\alpha_j} U \right] \\ &+ (D^-) \sum_{j=1}^q \left[(GAP) \int_{\alpha_{j-1}}^{\alpha_j} U - \{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} \right] \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence the result is obtained. □

3. The indefinite generalized Perron integral

DEFINITION 3.1. Let $U \in \text{GAP}[a, b]$. The function $\phi: [a, b] \rightarrow \mathbb{R}$ defined by $\phi(s) = (\text{GAP}) \int_a^s U$, $a < s \leq b$, $\phi(a) = 0$ is called the *indefinite GAP-integral* of U .

According to Theorem 2.3 this definition makes sense.

LEMMA 3.2. Let $U: [a, b] \times [a, b] \rightarrow \mathbb{R}$ be such that $U \in \text{GAP}[a, b]$ and $c \in [a, b]$. Then

$$\lim_{s \rightarrow c} \left[(\text{GAP}) \int_a^s U - U(c, s) + U(c, c) \right] = (\text{GAP}) \int_a^c U.$$

PROOF. Let $\epsilon > 0$ be given. Since $U \in \text{GAP}[a, b]$, given $\epsilon > 0$ there exists an approximate full cover Δ of $[a, b]$ such that for every Δ -division D of $[a, b]$ we have

$$\left| S(U, D) - (\text{GAP}) \int_a^b U \right| < \epsilon.$$

We may suppose that c is always an associated point in Δ -division D of $[a, b]$. Let D_c be an approximate neighbourhood of c contained in $[a, b]$. Furthermore, $([s, c], c) \in \Delta$ whenever $s \in D_c$ and $s < c$.

Similarly, we can choose an approximate neighbourhood D_c of c contained in $[a, b]$ and $([c, s], c) \in \Delta$ whenever $s \in D_c$ and $c < s$.

Then Saks-Henstock lemma gives

$$\left| U(c, s) - U(c, c) - (\text{GAP}) \int_c^s U \right| < \epsilon \quad \text{if } c < s.$$

That is,

$$\begin{aligned} & \left| (\text{GAP}) \int_a^s U - U(c, s) + U(c, c) - (\text{GAP}) \int_a^c U \right| \\ &= \left| (\text{GAP}) \int_c^s U - U(c, s) + U(c, c) \right| < \epsilon. \end{aligned}$$

This yields the relation.

The proof is similar if $s < c$. □

THE GENERALIZED APPROXIMATE PERRON INTEGRAL

THEOREM 3.3. *If $U \in GAP[a, b]$ and ϕ is the indefinite GAP-integral of U , then ϕ is approximately continuous at a point $s_0 \in [a, b]$ if and only if the function $U(s_0, \cdot): [a, b] \rightarrow \mathbb{R}$ is continuous at the point s_0 .*

P r o o f. Let $U \in GAP[a, b]$ and $s_0 \in [a, b]$. We assume that the function $U(s_0, \cdot)$ is continuous at the point s_0 . We shall show that ϕ is approximately continuous at s_0 . By the above lemma, we obtain

$$\lim_{s \rightarrow s_0} \left[(GAP) \int_a^s U - U(s_0, s) + U(s_0, s_0) \right] = (GAP) \int_a^{s_0} U.$$

i.e.

$$\lim_{s \rightarrow s_0} [\phi(s) - U(s_0, s) + U(s_0, s_0)] = \phi(s_0). \quad (3.1)$$

Again, since $U(s_0, \cdot)$ is continuous at the point s_0 , for each $\epsilon > 0$ there exists a $\eta > 0$ such that

$$|s - s_0| < \eta \implies |U(s_0, s) - U(s_0, s_0)| < \epsilon/2.$$

Or, in other words, $\lim_{s \rightarrow s_0} U(s_0, s) = U(s_0, s_0)$. Then (3.1) gives, $\lim_{s \rightarrow s_0} \phi(s) = \phi(s_0)$. That is, ϕ is continuous at the point s_0 . Thus ϕ is approximately continuous at the point s_0 .

Conversely, suppose that ϕ is approximately continuous at a point $s_0 \in [a, b]$. That is, for every $\epsilon > 0$ there exists a set D_{s_0} of density 1 at s_0 such that

$$|\phi(s) - \phi(s_0)| < \epsilon/2 \quad \text{for every } s \in D_{s_0}. \quad (3.2)$$

Since $U \in GAP[a, b]$, given $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that for every Δ -division D of $[a, b]$ we have

$$\left| S(U, D) - (GAP) \int_a^b U \right| < \epsilon/2.$$

We may suppose that s_0 is always an associated point in Δ -division D of $[a, b]$ and $|s - s_0| < \eta$ ($\eta > 0$). Let D_{s_0} be an approximate neighbourhood of s_0 contained in $[a, b]$. Furthermore, $([s, s_0], s_0) \in \Delta$ whenever $s \in D_{s_0}$ and $s < s_0$. Similarly, we can choose an approximate neighbourhood D_{s_0} of s_0 contained in $[a, b]$ and $([s_0, s], s_0) \in \Delta$ whenever $s \in D_{s_0}$ and $s_0 < s$.

Then by Saks-Henstock lemma applied to $([s, s_0], s_0)$ or $([s_0, s], s_0)$ we have

$$\left| U(s_0, s_0) - U(s_0, s) - (GAP) \int_s^{s_0} U \right| < \epsilon/2 \quad \text{if } s < s_0.$$

$$\left| U(s_0, s) - U(s_0, s_0) - (GAP) \int_{s_0}^s U \right| < \epsilon/2 \quad \text{if } s_0 < s.$$

If $s < s_0$, then

$$\begin{aligned} |U(s_0, s) - U(s_0, s_0)| &= |U(s_0, s_0) - U(s_0, s)| \\ &\leq \left| U(s_0, s_0) - U(s_0, s) - (GAP) \int_s^{s_0} U \right| + \left| (GAP) \int_s^{s_0} U \right| \\ &< \epsilon/2 + |\phi(s_0) - \phi(s)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{by (3.2)}. \end{aligned}$$

Similar, if $s_0 < s$.

Hence $U(s_0, \cdot)$ is continuous at the point s_0 . □

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