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To cite this article: Shyamal Biswas *et al* 2010 *New J. Phys.* **12** 063039

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The critical Casimir force in the superfluid phase: effect of fluctuations

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New Journal of Physics **12** (2010) 063039 (10pp)
Received 15 March 2010
Published 30 June 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/6/063039

Abstract. We have considered the critical Casimir force on a ^{4}He film below and above the bulk λ point. We have explored the role of fluctuations around the mean field theory in a perturbative manner, and have substantially improved the mean field result of Zandi *et al* (2007 *Phys. Rev.* E **76** 030601(R)). The Casimir scaling function obtained by us approaches a universal constant $(-\zeta(3)/8\pi)$ for $T \lesssim 2.13\,\text{K}$.

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Contents

1. Introduction

Recently, Garcia and Chan [1] and Ganshin *et al* [2] measured the Casimir force-induced thinning of the liquid ^{4}He film near the bulk λ point ($T_\lambda = 2.1768$ K). They obtained a universal scaling function (ϑ) for the critical Casimir force below and above the λ point, and observed a dip minimum and a non-vanishing constant tail in the ϑ below the λ point. This experiment challenges our understanding of the finite size effects of the films near their bulk critical points. On this issue, the Casimir effects on different critical films have been the subject of a number of experimental [1]–[7] and theoretical [8]–[16] works within the last few years.

Although the scaling function was appreciably obtained by the Monte Carlo simulations of Hucht [10] and Vasilyev *et al* [11], yet this problem is still unsolved analytically. That the confinement of the critical fluctuations may give rise to a (classical) Casimir force was first proposed by Nightingale and Indekeu [17]. Thereafter, a renormalization group calculation for the ϑ above the T_λ was presented by Krech and Dietrich [18]. For $T < T_\lambda$, a mean field theory with the Ginzburg–Landau (G–L) model was recently presented by Zandi *et al* [9]. They obtained an analytic expression for the ϑ in terms of the maximum of the superfluid order parameter. By proposing that their mean field calculation could be improved by the confinement of the critical fluctuations (at the Gaussian level), they nicely improved their result only at the λ point.

We analytically improve the mean field result of Zandi *et al* [9] as proposed by them for $T < T_\lambda$. The improvement for $T > T_\lambda$ was already done by Krech and Dietrich [18] even beyond the Gaussian level. However, we present a physically motivated regularization technique for obtaining the critical Casimir force above the λ point. Thus we build a unified picture for the theory of critical Casimir force acting on a ^{4}He film below and above the λ point. Our theory interestingly predicts the non-vanishing constant tail of ϑ as -0.0478, which agrees well with the numerical result of Hucht [10] but differs by a factor of 5 from the experimental value (-0.24) [2]. Nonetheless, it is a considerable improvement over the mean field calculation, which predicts it to be zero [9].

We start from the G–L model. For $T > T_\lambda$, we obtain the free energy in terms of the discrete Fourier modes. The Casimir force is then obtained in the Fisher–de Gennes form [19] by applying the Poisson summation formula [20]. The use of this summation formula distinguishes our approach from that of Krech and Dietrich [18]. For $T < T_\lambda$, we transform the critical fields by introducing the superfluid order parameter, and express the G–L free energy in a decoupled form of the mean field and fluctuating parts. The fluctuating part is treated as we do for $T > T_\lambda$, and the mean field part is treated in the manner of Zandi *et al* [9]. It is necessary to know the maximum of the order parameter for plotting the mean field part of the Casimir force. Although the graphical solutions of the maximum of the order parameter are exact, yet the solutions do not appear in a closed form. We predict a closed form of the maximum of the order parameter from asymptotic analyses, and obtain an approximate mean field Casimir force that matches very well with the exact mean field result [9]. Finally, we improve the mean field result by adding the contribution of the fluctuating part.

2. Free energy of the critical fluctuations for $T > T_\lambda$

According to the experimental setup, ^{4}He vapor comes in contact with a plate, and upon liquefaction it forms a film of 238–340 Å thickness [1, 2]. We consider the plate to be along the x–y plane of the co-ordinate system, the area of the film to be A and the thickness of the film to be L along the z-direction. Near the λ point ^{4}He behaves critically, and its local free energy can be written in the G–L framework as

$$F_l = \int \mathrm{d}^3\mathbf{r} \left[\frac{1}{2} |\nabla \phi(\mathbf{r})|^2 + \frac{a}{2} |\phi(\mathbf{r})|^2 + \frac{b}{4} |\phi(\mathbf{r})|^4 \right], \tag{1}$$

where $\phi(\mathbf{r}) = \phi_1(\mathbf{r}) + \mathrm{i}\phi_2(\mathbf{r})$ is a complex scalar critical field at the position vector $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, a is the inverse square of the correlation length ($\xi = \xi_0 t^{-\nu}$), $t = T/T_\lambda - 1$ is the reduced temperature, ν is the correlation length exponent and b is a positive coupling constant [21, 22]. The quartic term in equation (1) is neglected in the Gaussian approximation.

Let us first calculate the Casimir force for $T > T_\lambda$. In conformity with the Dirichlet boundary conditions, the Fourier expansion of the critical fields is given by $\phi_{1,2}(\mathbf{r}) = \sqrt{2/L} \sum_{n=1}^{\infty} \int \phi_{1,2,n}(\mathbf{k}) \sin(n\pi z/L) \mathrm{e}^{\mathrm{i}\mathbf{k}(x\hat{i}+y\hat{j})} \mathrm{d}^2\mathbf{k}/(2\pi)^2$. In the basis of the Fourier modes, we obtain the partition function ($Z = \int \mathrm{D}[\phi_1]\mathrm{D}[\phi_2]\mathrm{e}^{-F_l/k_\mathrm{B}T}$) within the Gaussian approximation, and get the standard form of the free energy ($-k_\mathrm{B}T \ln Z$) of the critical fluctuations of the film as [18]

$$F = 2 \times \frac{k_\mathrm{B} T A}{2} \sum_{n=1}^{\infty} \int_0^{\infty} \ln\left(k^2 + a + \frac{n^2\pi^2}{L^2}\right) \frac{k\,\mathrm{d}k}{2\pi}. \tag{2}$$

The factor 2 of the above equation comes from the fact that ϕ has two components.

3. Critical Casimir force for $T > T_\lambda$

From equation (2) we get the force acting on the film as

$$f_L = -\frac{\partial F}{\partial L} = 2\frac{\pi^2 k_\mathrm{B} T A}{L^3} S, \tag{3}$$

where $S=\sum_{n=1}^{\infty}\int_0^{\infty}[n^2/(k^2+a+(n^2\pi^2/L^2))]k\,\mathrm{d}k/2\pi$. This expression can be recast as

$$S=\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}n^2\mathrm{e}^{-(k^2+a+(n^2\pi^2/L^2))t_1}\mathrm{d}t_1\frac{k\,\mathrm{d}k}{2\pi}$$

$$=-\frac{1}{4\pi}\int_0^{\infty}t_1^{-1}\mathrm{e}^{-at_1}\frac{\partial}{\partial\tau_1}\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\tau_1}\,\mathrm{d}t_1, \tag{4}$$

where $\tau_1=t_1\pi^2/L^2$. Using the Poisson summation formula in equation (4), we recast equation (3) as

$$f_L=2\times\frac{Ak_{\mathrm{B}}T\pi}{4L^3}\int_0^{\infty}\mathrm{d}t_1\frac{\mathrm{e}^{-at_1}}{t_1}\left(\frac{\sqrt{\pi}}{4\tau_1^{3/2}}+\frac{\sqrt{\pi}}{2\tau_1^{3/2}}\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi^2/\tau_1}-\sqrt{\frac{\pi}{\tau_1}}\sum_{n=1}^{\infty}\frac{n^2\pi^2}{\tau_1^2}\mathrm{e}^{-n^2\pi^2/\tau_1}\right). \tag{5}$$

As $L\to\infty$, only the first term of the parentheses of equation (5) survives. This is the bulk force acting on the film. By the standard analytic continuation technique we get the expression of this bulk force as $f_\infty=2\times(Ak_{\mathrm{B}}T/16)(a/\pi)^{3/2}\Gamma(-3/2)$. Subtracting this bulk part from f_L we get the Casimir force in the Fisher–de Gennes form [19] $f_{\mathrm{C}}[L,t]=(Ak_{\mathrm{B}}T_\lambda/L^3)\vartheta(t)$, where $\vartheta(t)$ is the Casimir scaling function, which can be expressed in terms of a scaled temperature $\tau=L^{1/\nu}t$ as

$$\vartheta(\tau)=-2\times\frac{1}{8\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{2\tau^{\nu}}{\xi_0 n^2}+\frac{2\tau^{2\nu}}{n\xi_0^2}\right)\mathrm{e}^{-2n\tau^{\nu}/\xi_0}. \tag{6}$$

We have $\nu=1/2$ in the Gaussian as well as in the mean field approximations [22]. However, if we want to include the effect of the $|\phi|^4$ term within the above prescription, we must put $\nu=0.670\,16$ ($\approx 2/3$) in equation (6) [1, 2, 22]. From equation (6) we get the value of $\vartheta(\tau)$ at the λ point as $-\zeta(3)/4\pi=-0.0956$, which matches well with the experimental data obtained by Garcia and Chan [1]. The same number at the λ point was also obtained in [9, 18] with different regularization techniques. We need to know the value of ξ_0 for plotting the $\vartheta(\tau)$ against τ. The experimental value of ξ_0 for $T>T_\lambda$ varies from 1.2 to 1.43 Å [23]–[25]. With no *a priori* reason, we take $\xi_0=1.3$ Å for $T>T_\lambda$ [24].

4. Free energy of the critical fluctuations for $T<T_\lambda$

In addressing the situation below the λ point we note that the Casimir scaling function looks qualitatively similar to the ultrasonic attenuation and finite size specific heat (i.e. both have a peak below T_λ) [26, 27]. We anticipate that the Casimir effect for $T<T_\lambda$ can be thought of as coming from mean field and fluctuating parts. Splitting the ultrasonic attenuation into a sum of mean field and fluctuating parts was the original contribution of Landau and Khalatnikov, and it gives a good account of the ultrasonic attenuation below the λ point [28]. Here we show how a similar approach for the critical Casimir effect can be adopted below the λ point.

We return to equation (1) and note that for $T<T_\lambda$, a becomes negative, and accordingly we write $a=-|a|$. This leads to a broken symmetry, and we handle it by transforming the fields ϕ_2 and ϕ_1 to $\psi_2=\phi_2$ and $\psi_1=\phi_1-m(z)$ where $m(z)$ is the superfluid order parameter. The expectation value $\langle\phi_1\rangle$ is now z dependent because we are considering a finite size system in the z-direction and, consequently, we expect an inhomogeneity in the superfluid density ($\sim m^2(z)$).

The fields ψ_1, ψ_2 are such that $\langle\psi_i\rangle = 0$ $(i = 1, 2)$, and the local free energy in equation (1) in terms of ψ_1, ψ_2 becomes

$$F_l = \int d^3(\mathbf{r})\Big[\Big[\frac{1}{2}\left(\frac{dm}{dz}\right)^2 - \frac{|a|m^2}{2} + \frac{bm^4}{4}\Big] + \Big[-\frac{d^2m}{dz^2} - |a|m + bm^3\Big]\psi_1$$
$$+\frac{1}{2}[(3bm^2 - |a|)\psi_1^2 + (\nabla\psi_1)^2] + \frac{1}{2}[(bm^2 - |a|)\psi_2^2 + (\nabla\psi_2)^2]$$
$$+\Big[bm\psi_1(\psi_1^2 + \psi_2^2) + \frac{b}{4}(\psi_1^2 + \psi_2^2)^2\Big]\Big]. \quad (7)$$

The free energy can be minimized from the condition that $\langle\delta F_l/\delta\psi_1\rangle = 0$, and can be recast from equation (7) as $[-d^2m/dz^2 - |a|m + bm^3] + bm[\langle\psi_2^2\rangle + 3\langle\psi_1^2\rangle] = 0$, which can only be solved analytically if we disregard the second square bracketed term by considering the necessary condition that the mean field part dominates over the fluctuating part (i.e. $m^2 \gg \langle\psi_i^2\rangle$). With this consideration we can write an approximate equation for the profile of $m(z)$ as

$$-\frac{d^2m}{dz^2} - |a|m + bm^3 = 0. \quad (8)$$

It is to be noted that equation (8) does not minimize the local free energy in equation (7). Hence, the quadratic terms in ψ_1 and ψ_2 may not be positive. However, equation (8) would minimize the local free energy if we replace $m(z)$ in the quadratic and higher order terms by its bulk value ($\sqrt{|a|/b}$). With all the above considerations (and with equation (8)) the fluctuating and mean field parts of the local free energy become decoupled, and consequently, we recast equation (7) as

$$F_l = F_{mf} + F_o + F_{int}, \quad (9)$$

where $F_{mf} = \int d^3(\mathbf{r})\,[\frac{1}{2}(dm/dz)^2 - (|a|m^2/2) + (bm^4/4)]$ is the mean field part, $F_o = \int d^3(\mathbf{r}) \times [\frac{1}{2}2|a|\psi_1^2 + \frac{1}{2}(\nabla\psi_1)^2 + \frac{1}{2}(\nabla\psi_2)^2]$ is the (Gaussian) fluctuating part and $F_{int} = \int d^3(\mathbf{r})[\sqrt{|a|b}\psi_1(\psi_1^2 + \psi_2^2) + (b/4)(\psi_1^2 + \psi_2^2)^2]$ is the interaction part. We now check that all the quadratic terms in F_l are positive. Hence, equation (8) minimizes the local free energy, and consequently, the quadratic terms in equation (9) on average dominate over the higher order terms because $|a|/b \approx m^2 \gg \langle\psi_i^2\rangle$. Evaluation of the partition function from the extremized local free energy in equation (9) leads to

$$Z = e^{-[F_{mf}/k_BT]}Z_o\left[1 - \left\langle\frac{F_{int}}{k_BT}\right\rangle_o + \frac{1}{2}\left\langle\left(\frac{F_{int}}{k_BT}\right)^2\right\rangle_o - \cdots\right]$$
$$\approx e^{-[F_{mf}/k_BT]}Z_o e^{-\left[\langle F_{int}/k_BT\rangle_o - (1/2)\langle(F_{int}/k_BT)^2\rangle_o\right]}, \quad (10)$$

where $Z_o = \int D[\psi_1]D[\psi_2]e^{-F_o/k_BT}$ is the partition function for the (Gaussian) fluctuating part, and the expectation value $\langle\cdots\rangle_o$ is taken with respect to F_o. The free energy obtained from equation (10) is given by

$$F = F_{mf} + F_{cf} + k_BT\left[\left\langle\frac{F_{int}}{k_BT}\right\rangle_o - \frac{1}{2}\left\langle\left(\frac{F_{int}}{k_BT}\right)^2\right\rangle_o + \cdots\right], \quad (11)$$

where $F_{cf} = -k_BT \ln Z_o$ is the (Gaussian) fluctuating part of the free energy for $T < T_\lambda$.

5. Critical Casimir force for $T < T_\lambda$

5.1. Fluctuating contribution

The fluctuating part of the free energy in equation (11) can be recast in a special form of equation (2) as $F_{\rm cf} = \frac{1}{2}k_B T \sum_{n=1}^{\infty}\int_0^{\infty}[\ln[k^2+2|a|+n^2\pi^2/L^2]+\ln[k^2+n^2\pi^2/L^2]]k\mathrm{d}k/2\pi$, which gives the Casimir scaling function

$$\vartheta_{\rm cf}(\tau) = -\frac{1}{8\pi}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{2^{3/2}|\tau|^{\nu}}{\xi_0 n^2}+\frac{4|\tau|^{2\nu}}{n\xi_0^2}\right)\mathrm{e}^{-(2^{3/2}n|\tau|^{\nu})/\xi_0}\right]-\frac{\zeta(3)}{8\pi} \tag{12}$$

by following the steps from equation (2) to (6).

Since $|a|/b \approx m^2 \gg \langle\psi_i^2\rangle$ we can easily check from equation (11) that $F_{\rm mf} \gg F_{\rm cf} \gg \langle F_{\rm int}\rangle_{\rm o}$. Thus we can ignore the $\langle F_{\rm int}\rangle_{\rm o}$ terms in equation (11), and can expect that the Casimir force obtained from the fluctuating part would be much smaller than that obtained from the mean field part.

5.2. Mean field contribution

Let us now evaluate the Casimir scaling function from the mean field part ($F_{\rm mf}$) in equation (11). From the consideration that the order parameter $m(z)$ is smooth and obeys the Dirichlet boundary conditions $m(0) = m(L) = 0$, $m(z)$ must be symmetric about $z = L/2$ and there would be a single maximum of $m(z)$ at $z = L/2$ for the lowest possible value of the mean field free energy. An analytical expression of the Casimir force $(-[\partial F_{\rm mf}/\partial L - \partial F_{\rm mf}/\partial L|_{L\to\infty}])$ from the mean field part $F_{\rm mf}$ (and from equation (8)) was nicely obtained in [9] in terms of $\eta = (b/2|a|)(m(L/2))^2$ as

$$f_{\rm mf} = -\frac{A|a|^2}{b}\left[\frac{1}{4}-\eta(1-\eta)\right]. \tag{13}$$

In equation (13), η as well as the maximum of the order parameter is restricted by [9, 29]

$$L\sqrt{|a|} = \frac{2K(\eta/(1-\eta))}{\sqrt{1-\eta}}, \tag{14}$$

where $K(x) = (\pi/2)[1 + x/4 + (9/64)x^2 + (25/256)x^3 + \cdots]$ is the complete elliptic integral of the first kind. Equation (14) gives the allowed range $0 \leqslant \eta < \frac{1}{2}$ for the corresponding domain $\pi \leqslant L\sqrt{|a|} < \infty$, and it can be exactly solved by the graphical method [29]. Although the graphical method does not provide η in a closed form of $L\sqrt{|a|}$, yet we can do so by the asymptotic analyses near $\eta \to 0$ and $\eta \to \frac{1}{2}$. For $L\sqrt{|a|} \to \infty$, the asymptotic solution of η in equation (14) is $\eta \to \frac{1}{2}\tanh^2(L\sqrt{|a|/2}/2)$ [29][8]. On the other hand, for $L\sqrt{|a|} \to \pi$, the asymptotic solution (up to the third order in η in equation (14)) is $\eta \to (2/3)(L\sqrt{\bar{a}}/\pi)^2(1 - (25/24)(L\sqrt{\bar{a}}/\pi)^2 + 1.04514(L\sqrt{\bar{a}}/\pi)^4 + \cdots)$, where $\bar{a} = |a| - \pi^2/L^2$ [29]. Corresponding to

[8] For $L\sqrt{|a|} \to \infty$, the asymptotic solution in [29] was given as $\frac{1}{2}\coth^2$ instead of $\frac{1}{2}\tanh^2$. The asymptotic behavior of coth and tanh are the same.

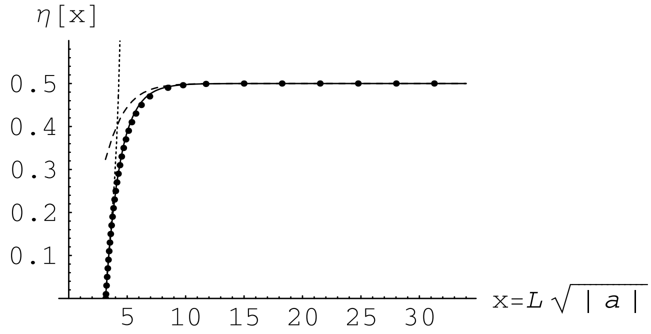

Figure 1. The dots are the graphical solutions of η in equation (14). The dotted and dashed lines are the asymptotic solutions near $L\sqrt{|a|} \to \pi$ and $L\sqrt{|a|} \to \infty$, respectively. The continuous line follows equation (15).

the above asymptotic solutions we can take a fitting function for the domain $\pi \leqslant L\sqrt{|a|} < \infty$ as [16, 29]

$$\eta(L\sqrt{|a|}) = \frac{1}{2}\tanh^2\left(\frac{\sqrt{(L^2|a|-\pi^2)/2}}{2}\right). \tag{15}$$

We see in figure 1 that all the asymptotic and graphical solutions match very well with equation (15). Hence, we consider equation (15) as an approximate solution for the rest of this paper.

With the consideration of equation (15), and that $\eta = 0$ [9] for $0 \leqslant L\sqrt{|a|} \leqslant \pi$, one can recast equation (13) in terms of the reduced temperature and the mean field correlation length as

$$f_{\mathrm{mf}} = \begin{cases} -\dfrac{A|t|^2}{4b\xi_0^4} & \text{for } \pi \geqslant L/\xi \geqslant 0, \\ -\dfrac{A|t|^2}{4b\xi_0^4}\,\mathrm{sech}^4\left(\sqrt{\dfrac{(L/\xi)^2-\pi^2}{8}}\right) & \text{for } L/\xi \geqslant \pi. \end{cases} \tag{16}$$

From equation (16), a dip minimum with discontinuous slope is expected to occur at $L\sqrt{|a|} = \pi$. This point is fitted to the experimental dip at $\tau = -9.7$ [2]. The modifications to equation (16) would come from the higher-order fluctuating terms, and the primary correction would be to keep the form of the f_{mf} unaltered with the mean field ξ replaced by $\xi = \xi_0|t|^{-\nu}$, where ν to the lowest order in b is $\frac{1}{2} + \frac{1}{2}b(n+2)$ [22]. Using the fixed point value of b we can get the usual ν at one loop order. We can safely assume that the effect of the different loops will be to make $\xi = \xi_0 t^{-\nu}$ with ν acquiring the value 0.670 16 ($\approx 2/3$) correct to all orders [2, 22]. From equation (16) we get the modified mean field Casimir scaling function in terms of $\tau = L^{1/\nu}t$ as

$$\vartheta_{\mathrm{mf}}(\tau) = \begin{cases} -\dfrac{|\tau|^2}{4b\xi_0^4 k_{\mathrm{B}}T_\lambda} & \text{for } -9.7 \leqslant \tau \leqslant 0, \\ -\dfrac{|\tau|^2}{4b\xi_0^4 k_{\mathrm{B}}T_\lambda}\,\mathrm{sech}^4\left(\sqrt{\dfrac{|\tau|^{2\nu}/\xi_0^2-\pi^2}{8}}\right) & \text{for } \tau \leqslant -9.7. \end{cases} \tag{17}$$

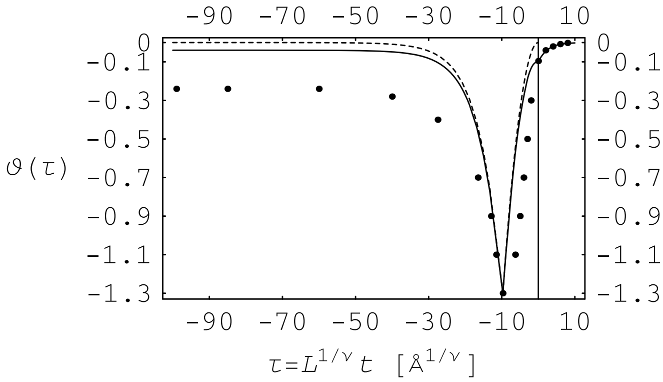

Figure 2. The continuous line for $\tau > 0$ follows equation (6) with $\xi_0 = 1.3$ Å [24] and that for $\tau < 0$ follows equation (18) with $\xi_0 = 1.4593$ Å, $\frac{1}{4} b \xi_0^4 k_B T_\lambda = 0.013\,3547$ and $\nu = 0.670\,16$. The dashed line represents the exact mean field calculation of Zandi *et al* [9] with $\xi_0 = 1.4593$ Å, $\frac{1}{4} b \xi_0^4 k_B T_\lambda = 0.013\,8166$ and $\nu = 0.670\,16$. A few experimental points are taken from [1] for $\tau > 0$ and from [2] for $\tau < 0$.

If we plot equation (17) we must get almost the same result as obtained by Zandi *et al* [9]. However, we need to improve the mean field result (in equation (17)) by the confinement of the critical fluctuations as proposed by them.

5.3. Improvement of the mean field result

The net Casimir scaling function for $T < T_\lambda$ is obtained from equations (12) and (17), and is given by

$$\vartheta(\tau) = \vartheta_{\rm mf}(\tau) + \vartheta_{\rm cf}(\tau). \tag{18}$$

We plot the right-hand sides of equations (6) and (18) in figure 2. For $T > T_\lambda$, our theory matches very well with the experimental data of Garcia and Chan [1]. From figure 2 we also see that the ϑ approaches a constant ($-\zeta(3)/8\pi = -0.0478$) for $\tau \lesssim -75.6$ Å$^{1/\nu}$ (with $L = 238$ Å) and for $T \lesssim 2.13$ K as well.

Although our theory for $T < T_\lambda$ does not match very well with the experimental data, yet it predicts the basic nature of the critical Casimir force, which is characterized by a non-vanishing constant tail [2]. It is of course clear from figure 2 that inclusion of the effect of the critical fluctuations substantially improves the exact mean field result of Zandi *et al* [9].

6. Conclusion

Complementing the numerical [10, 11, 14] and analytical [9, 12, 13, 15, 16, 18] works, we have given a unified theory for the critical Casimir force below and above the λ point in a single framework. In particular, we have explored the effect of the critical fluctuations in the Gaussian level over the mean field contribution [9].

The tail of the scaling function approaches a non-vanishing constant $-\zeta(3)/8\pi = -0.0478$ owing to the consideration of the two-component (ψ_1, ψ_2) critical fluctuations. Although this constant is closer to the numerical simulation result obtained by Hucht [10], yet it is nearly one-fifth of that obtained by the experimentalists [2]. On the other hand, this constant is zero in the mean field level [9]. Hence, our calculation of the Casimir scaling function goes beyond that of Zandi *et al* [9], and compares favorably with the numerical simulation of Hucht [10] and the experimental data of Garcia and Chan [1] and Ganshin *et al* [2].

It should be mentioned that the theory of the Casimir force for $T < T_\lambda$ was also improved by Zandi *et al* [30] with consideration of the confinement of the Goldstone modes and surface fluctuations, and that the thinning of the liquid ^{4}He film was first (but admittedly not very precisely) observed by Dionne and Hallock [31].

While the Casimir force for the quantum fluctuations of the electromagnetic field is observed within 10^{-12} N [32], the critical Casimir force considered by us is observed within 10^{-3} N [1, 2]. The confinement of the classical (critical) fluctuations of course is much stronger than that of the quantum (vacuum) fluctuations.

The experimental dip of the ϑ has been adjusted with the value of b, which nobody has determined (for the film) so far from the theoretical point of view. How to determine the parameter b for the film and how to calculate the Casimir force by considering the coupling between the mean field and fluctuating parts remain, to this day, as open problems.

Acknowledgments

S Biswas acknowledges the hospitality and financial support of the Department of Theoretical Physics of IACS and the CNS of Hong Kong Baptist University, where he did integral parts of the research work of this paper. The final part of his research work was sponsored by the University Grants Commission (UGC) under the UGC-Dr D S Kothari Postdoctoral Fellowship Scheme (no. F.4-2/2006(BSR)13-280/2008(BSR)).

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