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THE CONTROLLED CONVERGENCE THEOREM FOR THE GAP-INTEGRAL

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ABSTRACT. The concept of the GAP-integral was introduced by the authors [GANGULY, D. K.—MUKHERJEE, R.: *The generalized approximate Perron integral*, Math. Slovaca **58** (2008), 31–42]. In this paper we prove the controlled convergence theorem for the GAP-integral and deduce other convergence theorems as corollaries.

1. Introduction

Djvarsheishvili [1] proved a generalized convergence theorem for the Denjoy integral. Lee and Chew [8–10] also proved independently a series of generalized convergence theorems for ther Henstock integral, one of which is equivalent to Djvarsheishvili's result [7]. It is well-known that the Denjoy and Henstock integrals are equivalent [6].

We know that the Henstock integral is included in the approximately continuous Perron integral of Burkill [4,5]. Soeparna and Lee [12] proved the controlled convergence theorem for the approximately continuous Perron integral of Burkill and deduced other convergence theorems (for example, the dominated convergence theorem, the monotone convergence theorem and the uniform convergence theorem) as corollaries.

An attempt has been made in this paper is to establish the controlled convergence theorem which is the best convergence theorem in some sense for the Generalized Approximately Continuous Perron (GAP)-integral. The other convergence theorems for the GAP-integral [3] directly follows from the controlled convergence theorem.

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2. Preliminaries and definitions

Definition 2.1. A collection Δ of closed subintervals of $[a,b]$ is called an approximate full cover (AFC) if for every $x \in [a,b]$ there exists a measurable set $D_x \subset [a,b]$ such that $x \in D_x$ and D_x has density 1 at x, with $[u,v] \in \Delta$ whenever $u, v \in D_x$ and $u \le x \le v$.

A division of $[a,b]$ obtained by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \dots, \xi_n\}$ is called a Δ-division if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, if we have $x_{i-1} \le \xi_i \le x_i$ and $x_{i-1}, x_i \in D_{\xi_i}$ for all i. We call ξ_i the associated point of $[x_{i-1}, x_i]$ and x_i $(i = 0, 1, \dots, n)$ the division points.

A division of $[a,b]$ given by $a \le y_1 < z_1 \le y_2 < z_2 \cdots \le y_m < z_m \le b$ and $\{\zeta_1, \zeta_2, \dots, \zeta_m\}$ is called a Δ-partial division if $\bigcup\limits_{i=1}^{m} [y_i, z_i] \subset [a,b]$ and Δ is an approximate full cover with $[y_i, z_i]$ coming from Δ or more precisely, $y_i \le \zeta_i \le z_i$ and $y_i, z_i \in D_{\zeta_i}$ for all i.

Let $E \subset [a,b]$. Given a family of measurable sets $D_x \subset [a,b]$ for each $x \in E$ such that $x \in D_x$ and D_x has density 1 at x, if $u, v \in D_\zeta$ with $\zeta \in [u,v]$, we say that ζ is an associated point of $[u,v]$ and u, v the division points. The set of all $([u,v], \zeta)$, where $\zeta \in E$, is called an approximate full cover on E.

We say that a finite set of interval-point pairs $([u_i, v_i], \zeta_i) \in \Delta$, where $\zeta_i \in E$ for all $i = 1, \dots, p$, is a Δ-partial division on E.

In [2], the GAP-integral is defined as follows:

Definition 2.2. A function $U\colon [a,b] \times [a,b] \to R$ is said to be Generalized Approximate Perron (GAP)-integrable to a real number A if for every $\varepsilon > 0$ there is an AFC Δ of $[a,b]$ such that for every Δ-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$ we have

$$\Big| (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - A \Big| < \varepsilon$$

and we write $A = (GAP) \int\limits_a^b U$.

The set of all functions U which are GAP-integrable on $[a,b]$ is denoted by GAP$[a,b]$. We use the notation

$$S(U, D) = (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\}$$

for the Riemann-type sum corresponding to the function U and the Δ-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$.

Note that the integral is uniquely determined.

For a given function $U\colon [a,b] \times [a,b] \to R$ and a tagged interval (τ, J) with $\tau \in J = [\alpha, \beta] \subset [a,b]$ we will use the notation

$$U(\tau, J) = U(\tau, \beta) - U(\tau, \alpha)$$

With the notion of partial division we have proved the following theorem in [2].

Theorem 2.1 (**Saks-Henstock Lemma**). *Let* $U\colon [a,b] \times [a,b] \to R$ *be GAP-integrable over* $[a,b]$. *Then, given* $\varepsilon > 0$, *there is an approximate full cover* Δ *of* $[a,b]$ *such that for every* Δ*-division* $D = \{([\alpha_{j-1}, \alpha_j], \tau_j) :\ j = 1, 2, \dots, q\}$ *of* $[a,b]$ *we have*

$$\Big|\sum_{j=1}^{q}\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_a^b U\Big| < \varepsilon.$$

Then, if $\{([\beta_j, \gamma_j], \zeta_j) :\ j = 1, 2, \dots, m\}$ *represents a* Δ*-partial division of* $[a,b]$, *we have*

$$\Big|\sum_{j=1}^{m}\Big[\{U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)\} - (GAP)\int_{\beta_j}^{\gamma_j} U\Big]\Big| < \varepsilon.$$

In [2], the indefinite GAP-integral is defined as follows:

Definition 2.3. Let $U \in GAP[a,b]$. The function $\phi\colon [a,b] \to R$ defined by $\phi(s) = (GAP)\int_a^s U$, $a < s \le b$, $\phi(a) = 0$ is called the indefinite GAP-integral of U.

Given a function $\phi\colon [a,b] \to R$ then for $[\alpha, \beta] \subset [a,b]$, we put $\phi(\alpha, \beta) = \phi(\beta) - \phi(\alpha)$.

We need the following definitions to establish the main theorem in the consequence:

Definition 2.4. A function ϕ is said to satisfy the approximate strong Lusin condition (ASL), on a subset X of $[a,b]$, if for every set E of measure zero and for every $\varepsilon > 0$ there exists an approximate full cover Δ on X such that for any Δ-partial division $D = ([\alpha, \beta], \tau)$ on $E \cap X$ i.e. $\tau \in E \cap X$, we have $(D)\sum |\phi(\beta) - \phi(\alpha)| < \varepsilon$.

A sequence of functions $\{\phi_n\}$ is said to satisfy the uniformly approximate strong Lusin condition, briefly, UASL, if for every $\varepsilon > 0$ and every set E of measure zero there exists an approximate full cover Δ of $[a,b]$, independent of n, such that for any Δ-partial division $D = \{([\alpha, \beta], \tau)\}$ on E and for all n we have $(D)\sum |\phi_n(\alpha, \beta)| < \varepsilon$.

DEFINITION 2.5. ([6]) A function ϕ defined on $[a, b]$ is said to be ACG^* on $X \subset [a, b]$ if X is the union of a sequence of subsets X_i, $i = 1, 2, \dots$, such that ϕ is $AC^*(X_i)$ for each i, that is, for every $\varepsilon > 0$ there is $\eta > 0$ such that for any finite or infinite sequence of non-overlapping intervals $\{[a_k, b_k]\}$ with at least one of a_k or b_k belonging to X_i satisfying $\sum_k |b_k - a_k| < \eta$ we have $\sum_k |\phi(a_k, b_k)| < \varepsilon$.

A sequence of functions $\{\phi_n\}$ is said to be $UACG^*$ on $X \subset [a, b]$ if X is the union of a sequence of subsets X_i, $i = 1, 2, \dots$, such that $\{\phi_n\}$ is $UAC^*(X_i)$ for each i, that is, the η in the definition of $AC^*(X_i)$ with ϕ replaced by ϕ_n is independent of n.

DEFINITION 2.6. A function ϕ defined on $[a, b]$ is said to be ACG^*_{ap} on $E \subset [a, b]$ if E is the union of a sequence of subsets X_k, $k = 1, 2, \dots$, having each X_k measurable such that ϕ is $AC^*_{ap}(X_k)$ for each k, that is, for every $\varepsilon > 0$ there exists an approximate full cover Δ and $\eta > 0$ such that for any Δ-partial division $D = \big\{([\alpha, \beta], \tau)\big\}$ on X_k satisfying $(D) \sum |\beta - \alpha| < \eta$ we have $(D) \sum |\phi(\beta) - \phi(\alpha)| < \varepsilon$.

A sequence of functions $\{\phi_n\}$ is said to be $UACG^*_{ap}$ on $E \subset [a, b]$ if E is the union of a sequence of subsets X_k, $k = 1, 2, \dots$, having each X_k measurable such that $\{\phi_n\}$ is $UAC^*_{ap}(X_k)$ for each k, that is, the η and the approximate full cover Δ in the definition of $AC^*_{ap}(X_k)$ with ϕ replaced by ϕ_n are independent of n.

DEFINITION 2.7. A sequence of GAP-integrable functions $U_n \colon [a, b] \times [a, b] \to R$, $n = 1, 2, \dots$ is said to be equi-GAP integrable on $[a, b]$, if for every $\varepsilon > 0$, there is an approximate full cover Δ of $[a, b]$, independent of n, such that for any Δ-division $D = \{([\alpha, \beta], \tau)\}$ of $[a, b]$ and for every n

$$\left|(D) \sum \{U_n(\tau, \beta) - U_n(\tau, \alpha)\} - (GAP) \int_a^b U_n\right| < \varepsilon.$$

DEFINITION 2.8. Let the functions $U, U_n \colon [a, b] \times [a, b] \to R$, $n = 1, 2, \dots$ be given where $U_n \in GAP[a, b]$ for $n = 1, 2, \dots$. We say that $U_n \to U$ almost everywhere in $[a, b]$, i.e., except perhaps a set S of measure zero where S is given by $S = \big\{\tau \in [a, b] : U_n(\tau, [\alpha, \beta])$ does not converge to $U(\tau, [\alpha, \beta])$ where $\alpha, \beta \in D_\tau$ with $\alpha \le \tau \le \beta\big\}$, whenever $D_\tau \subseteq [a, b]$ is a measurable set of density 1 at τ and $U(\tau, [\alpha, \beta])$ denotes $\{U(\tau, \beta) - U(\tau, \alpha)\}$.

3. Main theorem

Now we state the Controlled Convergence Theorem for the GAP-integral as follows:

Theorem 3.1 (**Controlled Convergence Theorem**). *Let*

(i) $U, U_n\colon [a,b]\times[a,b]\to R$, $n=1,2,\dots$ *be such that* $U_n\in GAP[a,b]$ *for all* $n=1,2,\dots$ *with* $U_n\to U$ *almost everywhere in* $[a,b]$ *as* $n\to\infty$,

(ii) *the primitives* ϕ_n *of* U_n *be* $UACG^*_{ap}$ *on* $[a,b]$.

Then $U\in GAP[a,b]$ *and*

$$\lim_{n\to\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U.$$

We shall prove Theorem 3.1 by four lemmas which run as follows:

Lemma 3.1. *Let* X *be a closed subset of* $[a,b]$. *If* $\{\phi_n\}$ *is* $UAC^*_{ap}(X)$, *then for every* $\varepsilon>0$ *there exists a closed set* $Y\subset X$, *independent of* n, *such that* $|X\setminus Y|<\varepsilon$ *and* $\{\phi_n\}$ *is* $UAC^*(Y)$.

Proof. The proof is same as that of [13: Lemma 6]. □

Lemma 3.2. *Let*

(i) $U, U_n\colon [a,b]\times[a,b]\to R$, $n=1,2,\dots$ *be such that* $U_n\in GAP[a,b]$ *for all* $n=1,2,\dots$ *with* $U_n\to U$ *almost everywhere in* $[a,b]$ *as* $n\to\infty$,

(ii) *the primitives* ϕ_n *of* U_n *be* $UACG^*_{ap}$ *on* $[a,b]$.

Then $\{\phi_n\}$ *satisfies UASL on* $[a,b]$ *and is* $UACG^*(X)$ *where* X *is a closed subset of* $[a,b]$ *such that* $[a,b]\setminus X$ *is of measure zero.*

Proof. We first show that $\{\phi_n\}$ satisfies ASL uniformly in n on $[a,b]$.

Since $\{\phi_n\}$ is $UACG^*_{ap}$ on $[a,b]$, there exists a sequence of measurable sets $\{X_k\}$ where $[a,b]=\bigcup_{k=1}^{\infty}X_k$ such that $\{\phi_n\}$ is $AC^*_{ap}(X_k)$ uniformly in n for each k and let E be a set of measure zero with $E\subset[a,b]$.

Then for every $\varepsilon>0$, there exists $\eta_k>0$ and an approximate full cover Δ_k on X_k such that for any Δ_k-partial division $D=\{([\alpha,\beta],\tau)\}$ on X_k with $\tau\in X_k$ satisfying $(D)\sum|\beta-\alpha|<\eta_k$ we have

$$(D)\sum|\phi_n(\beta)-\phi_n(\alpha)|<\varepsilon/2^k \qquad \text{for all } n \tag{3.1}$$

Choose an open set $G_k\supseteq X_k\cap E$ such that $|G_k|<\eta_k$, for each k.

We may modify Δ_k, so that $\tau\in[\alpha,\beta]\subset G_k$ whenever $\tau\in X_k\cap E$.

We now define an approximate full cover Δ on $[a,b]$ as follows:

$([\alpha,\beta],\tau)\in\Delta_k$ if $\tau\in(X_k\setminus\bigcup_{i=0}^{k-1}X_i)$ with $X_0=\phi$.

Then for any Δ-partial division $D=\{([\alpha,\beta],\tau)\}$ on E we have $(D)\sum|\phi_n(\alpha,\beta)|$ $<\sum_{k=1}^{\infty}\varepsilon/2^k=\varepsilon$, by (3.1). Hence $\{\phi_n\}$ satisfies ASL uniformly in n on $[a,b]$.

We now verify that if $\{\phi_n\}$ is $UAC^*_{ap}(X'_i)$, $i = 1, 2$ then $\{\phi_n\}$ is $UAC^*_{ap}(X'_1 \cup X'_2)$.

Since $\{\phi_n\}$ satisfies $UAC^*_{ap}(X'_i)$, $i = 1, 2$, then given $\varepsilon > 0$, there exists $\eta_i > 0$ and approximate full cover Δ_i $(i = 1, 2)$ such that for any Δ_i-partial division $D_i = \{([\alpha, \beta], \tau)\}$ on X'_i ((i=1, 2)) satisfying $(D_i) \sum |\beta - \alpha| < \eta_i$ we have

$$(D_i) \sum |\phi_n(\beta) - \phi_n(\alpha)| < \varepsilon/2 \qquad \text{for all} \quad n \tag{3.2}$$

Let $\eta = \min(\eta_1, \eta_2)$ and $\Delta' = \Delta_1 \cap \Delta_2$.

Then for any Δ'-partial division $D' = \{([\alpha, \beta], \tau)\}$ on $X'_1 \cup X'_2$ satisfying $(D') \sum |\beta - \alpha| < \eta$ we split D' into D'_1 and D'_2 in which $\tau \in X'_1$ and $\tau \in X'_2 \setminus X'_1$ respectively.

Then

$$\begin{aligned}
&(D') \sum |\phi_n(\beta) - \phi_n(\alpha)| \\
&< (D'_1) \sum |\phi_n(\beta) - \phi_n(\alpha)| + (D'_2) \sum |\phi_n(\beta) - \phi_n(\alpha)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon,
\end{aligned}$$

by (3.2), for all n.

Hence $\{\phi_n\}$ is $AC^*_{ap}(X'_1 \cup X'_2)$ uniformly in n.

Again since $\{\phi_n\}$ is $UACG^*_{ap}$ on $[a, b]$, there exists a sequence of measurable sets $\{X_k\}$ where $[a, b] = \bigcup_{k=1}^{\infty} X_k$ such that $\{\phi_n\}$ is $AC^*_{ap}(X_k)$ uniformly in n for each k.

We are now ready to assume that $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k \subseteq \ldots$.

Let $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n \cdots > 0$ with $\varepsilon_n \to 0$ as $n \to \infty$.

Since X_k is measurable for each k, we can choose a closed set Y_k such that $Y_k \subseteq X_k$ with $|X_k \setminus Y_k| < \varepsilon_k$ for each k.

We see that

$$|[a, b] \setminus \bigcup_{k=1}^{\infty} Y_k| = \left| \bigcap_{k=1}^{\infty} ([a, b] \setminus Y_k) \right| \le |[a, b] \setminus X_k| + |X_k \setminus Y_k|$$

for all k.

Hence $|[a, b] \setminus \bigcup_{k=1}^{\infty} Y_k| = 0$.

Then $[a, b] = \bigcup_{k=1}^{\infty} Y_k \cup Z$ where $Z = [a, b] \setminus \bigcup_{k=1}^{\infty} Y_k$ having $|Z| = 0$.

As $|\bigcup_{k=1}^{\infty} Y_k| = b - a$, hence given $\varepsilon > 0$, there exists a positive integer m such that

$$|[a, b] \setminus \bigcup_{k=1}^{m} Y_k| < \varepsilon/2$$

Let $Y = \bigcup_{k=1}^{m} Y_k$.

Hence Y is a closed set and $\{\phi_n\}$ is $AC^*_{ap}(Y)$ uniformly in n. Then by Lemma 3.2, $\{\phi_n\}$ is $AC^*(Y)$ uniformly in n. □

Lemma 3.3. *Let*

(i) $U, U_n\colon [a,b] \times [a,b] \to R$, $n = 1, 2, \dots$ *be such that* $U_n \in GAP[a,b]$ *with primitives* ϕ_n *for all* $n = 1, 2, \dots$ *and* $U_n \to U$ *almost everywhere in* $[a,b]$,

(ii) *the primitives* ϕ_n *satisfy UASL on* $[a,b]$ *and be* $UACG^*$ *on* X *where* X *is a closed subset of* $[a,b]$ *such that* $[a,b] \setminus X$ *is of measure zero.*

Then there is a sequence $\{U_n^*\}$ *such that* $U_n^*(\tau,t) = U_n(\tau,t)$ *almost everywhere in* $[a,b]$ *for all* n *and for* $\tau, t \in [a,b]$ *and a subsequence of* $\{U_n^*\}$ *is equi-GAP integrable on* $[a,b]$.

Proof. Since $\{\phi_n\}$ is $UACG^*$ on X, the family $\{\phi_n\}$ is equi-continuous and therefore uniformly bounded on X. It follows from Ascoli theorem that there is a subsequence of $\{\phi_n\}$ uniformly convergent to a continuous function ϕ on X. For convenience, let $\{\phi_n\}$ be itself the subsequence.

Then by [6: Theorem 9.8], X is the union of a sequence of closed sets X_i, $i = 1, 2, \dots$, and for every i, j and $\varepsilon > 0$ there is an integer $n(i,j)$ such that for any partial division of X given by $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \dots \le \alpha_p < \beta_p$ with at least one of α_k, β_k belonging to X_i, for each k, we have

$$\Big| \sum_{k=1}^{p} \big\{ \phi_n(\alpha_k, \beta_k) - \phi_m(\alpha_k, \beta_k) \big\} \Big| < \varepsilon 2^{-i-j} \qquad \text{whenever} \quad n, m \ge n(i,j) \tag{3.3}$$

We may assume that, for each i, the sequence $\big\{n(i,j) : \ j = 1, 2, \dots\big\}$ is strictly increasing. Consequently, given i, we may choose $\big\{n(i+1,j) : \ j = 1, 2, \dots\big\}$ as a subsequence of $\big\{n(i,j) : \ j = 1, 2, \dots\big\}$. Next we replace ϕ_n by $\phi_{n(i,i)}$. For convenience we denote $\phi_{n(i,i)}$ by ϕ_n again.

Since each $U_n \in GAP[a,b]$, by the Saks-Henstock Lemma, there is an approximate full cover Δ_n of $[a,b]$ such that for any Δ_n-partial division $D = \big\{([\alpha,\beta],\tau)\big\}$ of $[a,b]$ we have

$$(D) \sum \big|\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)\big| < \varepsilon/2^n \tag{3.4}$$

Let $P = \big\{\tau \in [a,b] : \ U_n(\tau,[\alpha,\beta])$ does not converge to $U(\tau,[\alpha,\beta])$ where $\alpha, \beta \in D_\tau$, with $\alpha \le \tau \le \beta\big\}$, whenever $D_\tau \subseteq [a,b]$ is a measurable set of density 1 at τ with $\tau \in D_\tau$.

We write $Y_1 = X_1$ and $Y_i = X_i \setminus (X_1 \cup X_2 \cup \dots \cup X_{i-1})$ for $i = 2, 3, \dots$ where $\bigcup_{i=1}^{\infty} Y_i = X$.

Then for every $\tau \in ([a,b]\setminus P)\cap Y_i$ we choose $m(\tau) \geq n(i,i)$ such that whenever $n, m \geq m(\tau)$ we have

$$\left|\left\{U_n(\tau,\beta) - U_n(\tau,\alpha)\right\} - \left\{U_m(\tau,\beta) - U_m(\tau,\alpha)\right\}\right| < \varepsilon/2^{m(\tau)} \qquad (3.5)$$

whenever $D_\tau \subseteq [a,b]$ is a measurable set of density 1 at τ with $\alpha, \beta \in D_\tau$.

The family $\left\{D_\tau : \ \tau \in ([a,b] \setminus P) \cap X\right\}$ constitutes an approximate full cover Δ'.

Without any loss of generality, we may assume that $\Delta' = \Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_{m(\tau)}$.

Since $P \cap X$ is of measure zero, we may choose an open set G such that $G \supset P \cap X$ and $m(G) < \varepsilon$.

If $\tau \in P \cap X$, then $\tau \in G$ and hence we can find $[\alpha,\beta]$ such that $\tau \in [\alpha,\beta] \subset G$.

The family $\left\{([\alpha,\beta],\tau) : \ \tau \in P \cap X\right\}$ constitutes an approximate full cover Δ''.

For each $\tau \in X$, we choose $\Delta_0 = \Delta' \cap \Delta''$.

We show that $\{U_n\}$ is equi-GAP integrable on X.

Let $D = \{([\alpha,\beta],\tau)\}$ be any Δ_0-division of X, i.e., $\tau \in X$. We split D into D_1 and D_2 in which $\tau \in P \cap X$ and $\tau \in ([a,b] \setminus P) \cap X$ respectively.

Then we have

$$\begin{aligned}
&(D)\sum |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)| \\
&\leq (D_1)\sum |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)| \\
&\quad + (D_2)\sum |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)| \\
&< \varepsilon + (D_2) \sum_{m(\tau)\geq n} |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)| \\
&\quad + (D_2) \sum_{m(\tau)< n} |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)| \\
&< 2\varepsilon + (D_2)\sum |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}| \\
&\quad + (D_2)\sum |\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \phi_{m(\tau)}(\alpha,\beta)| \\
&\quad + (D_2)\sum |\phi_{m(\tau)}(\alpha,\beta) - \phi_n(\alpha,\beta)| < 5\varepsilon,
\end{aligned}$$

using (3.3), (3.4) and (3.5) in the reverse order.

That is, $\{U_n\}$ is equi-GAP integrable on X.

If we define $U_n^* = U_n\chi_X$ and $U^* = U\chi_X$, where χ_X is the characteristic function of X.

Then $U_n^*(\tau,t) = U_n(\tau,t)$ almost everywhere in $[a,b]$ for all n, since $[a,b] \setminus X$ is of measure zero.

Also each U_n^* is GAP-integrable with primitive ϕ_n and $U_n^* \to U^*$ everywhere in $[a,b]$.

Again since the primitives ϕ_n satisfy UASL condition on $[a,b]$, for any $\varepsilon > 0$ there is an approximate full cover Δ^* of $[a,b]$, independent of n, such that for any Δ^*-partial division $D = \{([\alpha,\beta],\tau)\}$ on $[a,b] \setminus X$, i.e., $\tau \in [a,b] \setminus X$ we have $(D)\sum |\phi_n(\alpha,\beta)| < \varepsilon$.

For each $\tau \in [a,b]$, we choose $\Delta = \Delta_0 \cap \Delta^*$.

The functions U_n^* are also equi-GAP integrable on $[a,b]$ since for any Δ-division $D = \big\{([\alpha,\beta],\tau)\big\}$ of $[a,b]$ we have $(D)\sum \big|\big\{U_n^*(\tau,\beta) - U_n^*(\tau,\alpha)\big\} - \phi_n(\alpha,\beta)\big| = (D)\sum_{\tau\in X} \big|\big\{U_n(\tau,\beta) - U_n(\tau,\alpha)\big\} - \phi_n(\alpha,\beta)\big| + (D)\sum_{\tau\in[a,b]\setminus X} |\phi_n(\alpha,\beta)| < 2\varepsilon$. □

Lemma 3.4. *Let*

(i) $U, U_n : [a,b] \times [a,b] \to R$, $n = 1,2,\dots$ *be such that* $U_n \in GAP[a,b]$ *for all* $n = 1,2,\dots$ *with* $U_n \to U$ *almost everywhere in* $[a,b]$,

(ii) *the functions* U_n *be equi-GAP integrable on* $[a,b]$,

(iii) *the primitives* ϕ_n *of* U_n *satisfy UASL on* $[a,b]$.

Then $U \in GAP[a,b]$ *and*

$$\lim_{n\to\infty} (GAP) \int_a^b U_n = (GAP) \int_a^b U.$$

P r o o f. Since the functions U_n are equi-GAP integrable on $[a,b]$, for every $\varepsilon > 0$ there exists an approximate full cover Δ of $[a,b]$, independent of n, such that for all n

$$(D)\sum |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)| < \varepsilon$$

whenever $D = \big\{([\alpha,\beta],\tau)\big\}$ is any Δ-division of $[a,b]$.

By the UASL condition on the primitives ϕ_n, for any set $X \subset [a,b]$ of measure zero and for any $\varepsilon > 0$ there is an approximate full cover Δ_0, independent of n, such that $(D)\sum |\phi_n(\alpha,\beta)| < \varepsilon$ for all n whenever $D = \{([\alpha,\beta],\tau)\}$ is a Δ_0-partial division of $[a,b]$ with $\tau \in X$.

Let us choose $\Delta \subseteq \Delta_0$.

Also let $X = \big\{\tau \in [a,b] :\ U_n(\tau,[\alpha,\beta])$ does not converge to $U(\tau,[\alpha,\beta])$ where $\alpha,\beta \in D_\tau$ with $\alpha \le \tau \le \beta\big\}$, whenever $D_\tau \subseteq [a,b]$ is a measurable set of density 1 at τ including τ. Then the measure of X is zero.

Suppose that $U_n^* = U_n \chi_{[a,b]\setminus X}$ and $U^* = U\chi_{[a,b]\setminus X}$, where $\chi_{[a,b]\setminus X}$ is the characteristic function of $[a,b] \setminus X$.

Then each U_n^* is GAP-integrable with primitive ϕ_n and $U_n^* \to U^*$ everywhere in $[a,b]$.

The functions U_n^* are also equi-GAP integrable on $[a,b]$ since for any Δ-division $D = \big\{([\alpha,\beta],\tau)\big\}$ we have $(D)\sum \big|\big\{U_n^*(\tau,\beta) - U_n^*(\tau,\alpha)\big\} - \phi_n(\alpha,\beta)\big| = (D)\sum_{\tau\in[a,b]\setminus X} \big|\big\{U_n(\tau,\beta) - U_n(\tau,\alpha)\big\} - \phi_n(\alpha,\beta)\big| + (D)\sum_{\tau\in X} |\phi_n(\alpha,\beta)| < 2\varepsilon$.

Since the functions U_n^* are equi-GAP integrable on $[a,b]$, it is true that for every $\varepsilon > 0$ there exists an approximate full cover Δ of $[a,b]$, independent of n, such that for all n

$$|S(U_n^*, D) - (GAP)\int_a^b U_n^*| < \varepsilon/2$$

whenever D is any Δ-division of $[a,b]$.

Since $U_n^* \to U^*$ everywhere in $[a,b]$ as $n \to \infty$, there exists a positive integer N_1 such that for $n > N_1$, we get

$$\left|\left\{U_n^*(\tau,\beta) - U_n^*(\tau,\alpha)\right\} - \left\{U^*(\tau,\beta) - U^*(\tau,\alpha)\right\}\right| < \varepsilon/2^n$$

for $\tau \in [a,b]$ with $\alpha, \beta \in D_\tau$, whenever $D_\tau \subseteq [a,b]$ is a measurable set of density 1 at τ.

The family $\{D_\tau : \ \tau \in [a,b]\}$ constitutes an approximate full cover Δ'. We take any Δ'-division $D = \{([\alpha,\beta],\tau)\}$ of $[a,b]$.

So, for all $n > N_1$, we have $|S(U_n^*, D) - S(U^*, D)| = \left|\sum[\{U_n^*(\tau,\beta) - U_n^*(\tau,\alpha)\} - \{U^*(\tau,\beta) - U^*(\tau,\alpha)\}]\right| < \varepsilon/2$.

That is, $\lim\limits_{n\to\infty} S(U_n^*, D) = S(U, D)$.

We choose an approximate full cover Δ such that $\Delta \subseteq \Delta'$.

Therefore, for any Δ-division D of $[a,b]$ we have

$$\begin{aligned} &|S(U^*, D) - (GAP)\int_a^b U_n^*| \\ &\le |S(U^*, D) - S(U_n^*, D)| + |S(U_n^*, D) - (GAP)\int_a^b U_n^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{for all} \quad n > N_1. \end{aligned} \tag{3.6}$$

First, we get from (3.6) that for all positive integers $n, p > N_1$

$$\begin{aligned} &\left|(GAP)\int_a^b U_n^* - (GAP)\int_a^b U_p^*\right| \\ \le &\left|(GAP)\int_a^b U_n^* - S(U^*, D)\right| + \left|S(U^*, D) - (GAP)\int_a^b U_p^*\right| \\ < &\ \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

This means that $\{(GAP)\int\limits_a^b U_n^*\}$ is a Cauchy sequence in R and therefore $\lim\limits_{n\to\infty}(GAP)\int\limits_a^b U_n^*$ exists. Let

$$\lim_{n\to\infty}(GAP)\int\limits_a^b U_n^* = A. \tag{3.7}$$

Then, given $\varepsilon > 0$, there exists a positive integer N_2 such that

$$|(GAP)\int\limits_a^b U_n^* - A| < \varepsilon \qquad \text{for all} \quad n > N_2$$

Let $N = \max(N_1, N_2)$. Then, we get from (3.6) and (3.7) for $n > N$, that

$$|S(U^*, D) - A| \le \left|S(U^*, D) - (GAP)\int\limits_a^b U_n^*\right| + \left|(GAP)\int\limits_a^b U_n^* - A\right| < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence $U^* \in GAP[a, b]$ and

$$\lim_{n\to\infty}(GAP)\int\limits_a^b U_n^* = (GAP)\int\limits_a^b U^*.$$

□

Theorem 3.1 follows combining Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Corollary 3.1.1 (**Uniform Integrability Theorem**). ([3]) *Let*

(i) $U, U_n \colon [a, b] \times [a, b] \to R$, $n = 1, 2, \dots$ *be such that* $U_n \in GAP[a, b]$ *for all* $n = 1, 2, \dots,$

(ii) *there be an approximate full cover* Δ_0 *of* $[a, b]$ *such that*

$$\lim_{n\longrightarrow\infty}[U_n(\tau, t_2) - U_n(\tau, t_1)] = U(\tau, t_2) - U(\tau, t_1)$$

for each $\tau \in [a, b]$ *and for every interval-point pair* $([t_1, t_2], \tau) \in \Delta_0$,

(iii) *for every* $\eta > 0$ *there be an approximate full cover* Δ *of* $[a, b]$ *such that*

$$|S(U_n, D) - (GAP)\int\limits_a^b U_n| < \eta$$

for every Δ*-division* D *of* $[a, b]$ *and every* $n = 1, 2, \dots$

Then $(GAP)\int_a^b U$ *exists and*

$$\lim_{n\longrightarrow\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U.$$

The Uniform Integrability Theorem follows from the Controlled Convergence Theorem if the condition (ii) in Theorem 3.1 is replaced by the equi-GAP integrability condition i.e., for every $\eta > 0$ there be an approximate full cover Δ of $[a,b]$ such that

$$|S(U_n, D) - (GAP)\int_a^b U_n| < \eta$$

for every Δ-division D of $[a,b]$ and every $n = 1, 2, \dots$

Corollary 3.1.2 (Monotone Convergence Theorem). ([3]) *Let*

(i) $U, U_n\colon [a,b]\times[a,b] \to R$, $n = 1, 2, \dots$ *be such that* $U_n \in GAP[a,b]$ *for all* $n = 1, 2, \dots$ *with* $\sup (GAP)\int_a^b U_n < \infty$,

(ii) *there be an approximate full cover* Δ_0 *of* $[a,b]$ *such that*

$$U_n(\tau, t) - U_n(\tau,\tau) \le U_{n+1}(\tau, t) - U_{n+1}(\tau,\tau)$$

for every interval-point pair $([\tau, t], \tau) \in \Delta_0$ *where* $\tau < t$ *and*

$$U_n(\tau,\tau) - U_n(\tau, t) \le U_{n+1}(\tau,\tau) - U_{n+1}(\tau, t)$$

for every interval-point pair $([t,\tau],\tau) \in \Delta_0$ *where* $t < \tau$ $(n = 1, 2, \dots)$,

(iii) *there be an approximate full cover* Δ' *of* $[a,b]$ *such that*

$$\lim_{n\longrightarrow\infty}[U_n(\tau, t_2) - U_n(\tau, t_1)] = U(\tau, t_2) - U(\tau, t_1)$$

for each $\tau \in [a,b]$ *and every interval-point pair* $([t_1, t_2], \tau) \in \Delta'$

Then, $U \in GAP[a,b]$ *and*

$$\lim_{n\longrightarrow\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U.$$

The Monotone Convergence Theorem follows from the Controlled Convergence Theorem if the condition (ii) in Theorem 3.1 is replaced by the condition: there be an approximate full cover Δ_0 of $[a,b]$ such that

$$U_n(\tau, t) - U_n(\tau,\tau) \le U_{n+1}(\tau, t) - U_{n+1}(\tau,\tau)$$

for every interval-point pair $([\tau, t], \tau) \in \Delta_0$ where $\tau < t$ and

$$U_n(\tau, \tau) - U_n(\tau, t) \le U_{n+1}(\tau, \tau) - U_{n+1}(\tau, t)$$

for every interval-point pair $([t, \tau], \tau) \in \Delta_0$ where $t < \tau$ $(n = 1, 2, \dots)$.

Corollary 3.1.3 (**Dominated Convergence Theorem**). *Let*

(i) $U, U_n \colon [a, b] \times [a, b] \to R$, $n = 1, 2, \dots$ *be such that* $U_n \in GAP[a, b]$ *for all* $n = 1, 2, \dots,$

(ii) *there be an approximate full cover* Δ_0 *of* $[a, b]$ *such that for every* $\varepsilon > 0$ *there exist a function* $p(\tau)$ *defined on* $[a, b]$ *taking integer values and a positive superadditive interval function* ψ *defined for closed intervals* $J \subset [a, b]$ *with* $\psi([a, b]) < \varepsilon$ *such that for every* $\tau \in [a, b]$ *we have*

$$|U_n(\tau, J) - U(\tau, J)| < \psi(J)$$

provided $n > p(\tau)$ *and* $(\tau, J) \in \Delta_0$ *with* $\tau \in J \subset [a, b]$,

(iii) *there be two functions* $V, W : [a, b] \times [a, b] \to R$ *such that* $V, W \in GAP[a, b]$ *and there be an approximate full cover* Δ' *of* $[a, b]$ *such that for all* $n \in N$, $\tau \in [a, b]$ *we have*

$$V(\tau, J) \le U_n(\tau, J) \le W(\tau, J)$$

for any point-interval pair $(\tau, J) \in \Delta'$.

Then $U \in GAP[a, b]$ *and*

$$\lim_{n \longrightarrow \infty} (GAP) \int_a^b U_n = (GAP) \int_a^b U.$$

The Dominated Convergence Theorem follows from the Controlled Convergence Theorem if the condition $U_n \to U$ almost everywhere in $[a, b]$ in (i) in Theorem 3.1 is replaced by the condition:
there be an approximate full cover Δ_0 of $[a, b]$ such that for every $\varepsilon > 0$ there exist a function $p(\tau)$ defined on $[a, b]$ taking integer values and a positive superadditive interval function ψ defined for closed intervals $J \subset [a, b]$ with $\psi([a, b]) < \varepsilon$ such that for every $\tau \in [a, b]$ we have

$$|U_n(\tau, J) - U(\tau, J)| < \psi(J)$$

provided $n > p(\tau)$ and $(\tau, J) \in \Delta_0$ with $\tau \in J \subset [a, b]$.

And if the condition (ii) in Theorem 3.1 is replaced by the condition:
there be two functions $V, W \colon [a, b] \times [a, b] \to R$ such that $V, W \in GAP[a, b]$ and there be an approximate full cover Δ' of $[a, b]$ such that for all $n \in N$, $\tau \in [a, b]$ we have

$$V(\tau, J) \le U_n(\tau, J) \le W(\tau, J)$$

for any point-interval pair $(\tau, J) \in \Delta'$.

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