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Surface Layer in General Relativity and Integral Form of Field Law

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Assuming that the metric tensor g_{ij} depends on a continuous parameter ϵ, we write Einstein's field equations of general relativity in the form of a divergence equation. The theory of surface layer in general relativity is then worked out. Finally, an integral flux law is proposed from which both the field equations and the boundary conditions for a surface layer can be deduced.

INTRODUCTION

In this paper we assume that the metric tensor g_{ij} depends on a continuous parameter ϵ; that is, instead of a single metric tensor we consider a family of metric tensors. Under this hypothesis we first write Einstein's field equations of general relativity in the form of a divergence equation. The theory of surface layers in general relativity is then worked out, which consists principally of the generalization of the classical boundary condition, viz., that the jump of the normal derivative of the potential is proportional to the surface density of matter together with the surface analog of the Bianchi identities. All these are found to give correct classical approximations. Finally, an integral flux law is proposed from which both the field equations and the boundary conditions for a surface layer can be deduced.

1. THE PARAMETER ϵ; DIVERGENCE FORM OF FIELD LAW

Postulate 1.1: We assume that the metric tensor g_{ij} depends on a real parameter ϵ such that $g_{ij} = g_{ij}(\epsilon)$ are analytic functions of ϵ (i.e., representable in power series of ϵ) in some neighborhood of $\epsilon = 0$, and for $\epsilon = 0$, $g_{ij} = {}^0g_{ij}$, where ${}^0g_{ij}$ denotes the Euclidean metric tensor in any coordinate system.

Notation: In the following discussions, an overhead bar will indicate differentiation with respect to ϵ and a left superscript zero the value of a quantity for $\epsilon = 0$. As usual, a comma in the subscript will denote partial differentiation, and a semicolon in the subscript will denote covariant differentiation.

Remark: $\bar{\Gamma}^k_{ij}$ is a mixed tensor of rank three.

The Ricci tensor is given by

$$R_{ij} = \Gamma^\alpha_{i\alpha,j} - \Gamma^\alpha_{ij,\alpha} + \Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij}. \quad (1.1)$$

The terms on the rhs containing the second derivatives only are

$$\Gamma^\alpha_{i\alpha,j} - \Gamma^\alpha_{ij,\alpha} = \delta^k_j\Gamma^\alpha_{i\alpha,k} - \Gamma^k_{ij,k},$$

which is in the form of an ordinary divergence. We note that $\delta^k_j\Gamma^\alpha_{i\alpha} - \Gamma^k_{ij}$ is not a tensor, but that $\delta^k_j\bar{\Gamma}^\alpha_{i\alpha} - \bar{\Gamma}^k_{ij}$ is a tensor.

Definition 1.1:

$$P^k_{ij} = \delta^k_j\bar{\Gamma}^\alpha_{i\alpha} - \bar{\Gamma}^k_{ij}.$$

The tensor P^k_{ij} has the following remarkable property:

$$P^k_{ij;k} = \bar{R}_{ij}. \quad (1.2)$$

P^k_{ij} is unsymmetric in i, j, and we define Q^k_{ij} as follows.

Definition 1.2:

$$Q^k_{ij} = \tfrac{1}{2}(P^k_{ij} + P^k_{ji}),$$

which is symmetric in i, j.

From (1.2) we have

$$Q^k_{ij;k} = \bar{R}_{ij}. \quad (1.3)$$

Definition 1.3: Let T_{ij} denote the energy-momentum tensor. Then

$$T^*_{ij} = T_{ij} - \tfrac{1}{2}g_{ij}T$$

will be called the modified energy–momentum tensor.

The field equations of general relativity are known to be

$$R_{ij} = -\kappa T^*_{ij}, \quad \kappa = \text{const.} \quad (1.4)$$

Postulate 1.2: For $\epsilon = 0$, $T^*_{ij} = {}^0T^*_{ij} = 0$.

Remark: If we neglect pressure or stresses, $T_{ij} = \rho_0 u_i u_j$, where $u^i = dx^i/ds$ and ρ_0 is the proper mass density. Then

$$T = \rho_0 = -T^*, \quad T^*_{ij} = \rho_0(u_i u_j - \tfrac{1}{2}g_{ij})$$

so that Postulate 1.2 amounts to saying that $\rho_0 = \rho_0(\epsilon)$ is such that $\rho_0(0) = 0$.

We note that for $\epsilon = 0$, $R_{ij} = {}^0R_{ij} = 0$, and, by virtue of Postulate 1.2, (1.4) is equivalent to

$$\bar{R}_{ij} = -\kappa \bar{T}^*_{ij}$$

or by (1.3) to

$$Q^k_{ij;k} = -\kappa \bar{T}^*_{ij}, \tag{1.5}$$

which is the required divergence form of the field equations.

2. SURFACE LAYER: CONSERVATION IDENTITY AND BOUNDARY CONDITION

We assume that there exists a coordinate system in which the Euclidean metric tensor

$${}^0g_{ij} = \mathrm{diag}\,(-1, -1, -1, 1).$$

Consider a smooth hypersurface

$$S\colon f(x^1, x^2, x^3, x^4) = 0 \tag{2.1}$$

(in any coordinate system), where the function f is independent of ϵ and the unit vector normal to S is spacelike, i.e.,

$$g^{\alpha\beta} n_\alpha n_\beta = -1 \tag{2.2}$$

so that

$$n_i = f_{,i}/(-g^{\alpha\beta} f_{,\alpha} f_{,\beta})^{\frac{1}{2}}. \tag{2.3}$$

Definition 2.1: A hypersurface S is said to be parametrically stationary if the covariant components of the unit normal vector n_i are independent of ϵ, i.e., $\bar{n}_i = 0$.

By (2.3) S will be parametrically stationary if $g_{\alpha\beta} f_{,\alpha} f_{,\beta}$ is independent of ϵ. In particular, the hypersurface $x^1 = 0$ will be parametrically stationary if g^{11} is independent of ϵ, i.e., $g^{11} = {}^0g^{11}$; in fact, we can always make a coordinate transformation $(x^i) \to (y^i)$ in which $y^1 = f(x^1, x^2, x^3, x^4)$.

Postulate 2.1: A hypersurface S represents a surface layer if and only if it is parametrically stationary.

Postulate 2.2: The energy-momentum tensor S_{ij} for a surface layer will be given by

$$S_{ij} = \sigma_0 u_i u_j,$$

where σ_0 is the proper surface density of mass, the pressure or stresses being neglected. Further, $\sigma_0 = \sigma_0(\epsilon)$ is such that $\sigma_0(0) = 0$.

Definition 2.2: The modified energy-momentum tensor for a surface layer is given by

$$S^*_{ij} = S_{ij} - \tfrac{1}{2} g_{ij} S.$$

By Postulate 2.2,

$$S = \sigma_0, \quad S^*_{ij} = \sigma_0(u_i u_j - \tfrac{1}{2} g_{ij}). \tag{2.4}$$

It follows that for $\epsilon = 0$,

$$S^*_{ij} = {}^0S^*_{ij} = 0. \tag{2.5}$$

In the following discussions the symbol Δ will denote the jump of a quantity across a given hypersurface.

Remark: $\Delta\Gamma^k_{ij}$ is a tensor.

Definition 2.3:

$$H^k_{ij} = \delta^k_j \Delta\Gamma^\alpha_{i\alpha} - \Delta\Gamma^k_{ij}.$$

The tensor H^k_{ij} satisfies the identity

$$H^k_{kj} = 0. \tag{2.6}$$

We assert that at a surface layer this identity would express the conservation law or the physical boundary condition and as such is the surface analog of the Bianchi identities. Noting that H^k_{ij} is unsymmetric in i, j, we make the following postulate.

Postulate 2.3: g_{ij} are continuous throughout 4-space and, at a surface layer S,

$$\frac{\partial}{\partial\epsilon}(\sigma_0^{-1} H^k_{ij}) = \kappa\left(2u_i \bar{u}_j n^k - \frac{1}{2}\frac{\partial}{\partial\epsilon}(g_{ij} n^k)\right).$$

Therefore,

$$\frac{\partial}{\partial\epsilon}(\sigma_0^{-1} H^k_{kj}) = \kappa(2u_k \bar{u}_j n^k - \tfrac{1}{2}\bar{n}_j) = 2\kappa \bar{u}_j u_k n^k,$$

since n_i are independent of ϵ. Hence the identity (2.6) leads to the condition at S

$$u_k n^k = 0 = u^k n_k. \tag{2.7}$$

Interchanging i, j in Postulate 2.3 and adding the result to the same postulate, we obtain

$$\frac{\partial}{\partial\epsilon}[\tfrac{1}{2}\sigma_0^{-1}(H^k_{ij} + H^k_{ji})] = \kappa\left(\frac{\partial}{\partial\epsilon}(u_i u_j) n^k - \frac{1}{2}\frac{\partial}{\partial\epsilon}(g_{ij} n^k)\right).$$

Multiplying by n_k, which are independent of ϵ, we have

$$\frac{\partial}{\partial\epsilon}[\tfrac{1}{2}\sigma_0^{-1}(H^k_{ij} + H^k_{ji}) n_k] = -\kappa\left(\frac{\partial}{\partial\epsilon}(u_i u_j) - \tfrac{1}{2}\bar{g}_{ij}\right)$$

$$= -\kappa \frac{\partial}{\partial\epsilon}(\sigma_0^{-1} S^*_{ij})$$

or

$$\frac{\partial}{\partial\epsilon}[\sigma_0^{-1}(\tfrac{1}{2} n_j \Delta\Gamma^\alpha_{i\alpha} + \tfrac{1}{2} n_i \Delta\Gamma^\alpha_{j\alpha} - n_k \Delta\Gamma^k_{ij} + \kappa S^*_{ij})] = 0,$$

which are satisfied if

$$\tfrac{1}{2} n_j \Delta\Gamma^\alpha_{i\alpha} + \tfrac{1}{2} n_i \Delta\Gamma^\alpha_{j\alpha} - n_k \Delta\Gamma^k_{ij} = -\kappa S^*_{ij}. \tag{2.8}$$

These are the required boundary conditions at a surface layer S. We note that (2.8) is free from derivatives with respect to ϵ and as such provides a satisfactory criterion.

Since $\Delta^0\Gamma^k_{ij} = 0$, ${}^0S^*_{ij} = 0$, and n_i are independent of ϵ, Eq. (2.8) is equivalent to

$$n_k \Delta Q^k_{ij} = -\kappa S^*_{ij}. \tag{2.9}$$

3. CLASSICAL APPROXIMATIONS

In the linear approximation we assume

$$g_{ij} \simeq {}^0g_{ij} + \epsilon h_{ij}, \tag{3.1}$$

where ϵ is a small quantity whose square is negligible, and we take that coordinate system in which

$${}^0g_{ij} = {}^0g^{ij} = \mathrm{diag}\,(-1, -1, -1, 1). \tag{3.2}$$

We set

$$h^{ij} = {}^0g^{i\alpha}\,{}^0g^{j\beta}h_{\alpha\beta}, \quad h^i_j = {}^0g^{i\alpha}h_{\alpha j}, \quad h = {}^0g^{\alpha\beta}h_{\alpha\beta}, \tag{3.3}$$

$$g = \det(g_{ij}) \simeq -1 - \epsilon h, \tag{3.4}$$

$$g^{ij} \simeq {}^0g^{ij} - \epsilon h^{ij}. \tag{3.5}$$

If $(ij, k)_h$ denote the Christoffel symbols of the first kind by treating h_{ij} as the metric tensor, then

$$\Gamma^k_{ij} \simeq \epsilon\, {}^0g^{k\alpha}(ij, \alpha)_h, \tag{3.6}$$

$$R_{ij} \simeq \tfrac{1}{2}\epsilon({}^0g^{\alpha\beta}h_{ij,\alpha\beta} + h_{,ij} - h^\alpha_{i,\alpha j} - h^\alpha_{j,\alpha i}). \tag{3.7}$$

For convenience, let us write $(x^i) = (x, y, z, t)$. In the classical approximation, in addition to the linear approximation, we have to make two other approximations, viz., (i) the quasisteady approximation in which the derivatives with respect to t are treated as small compared with the derivatives with respect to x, y, z and (ii) the approximation that the velocity of a particle is small compared with the velocity of light which is taken to be unity, so that we may take u^1, u^2, u^3 to be small quantities and $u^4 \simeq 1$.

Neglecting pressure or stresses which are usually small compared with the matter-density, we have $T^{ij} = \rho_0 u^i u^j$. Since $\rho_0 = 0$ for $\epsilon = 0$, $\rho_0 = O(\epsilon)$. The only component of T^{ij} of order ϵ is $T^{44} \simeq \rho_0$; all other components are negligible. Hence, $T = \rho_0$, $T_{44} \simeq \rho_0$, and

$$T^*_{44} \simeq T_{44} - \tfrac{1}{2}\,{}^0g_{44}T \simeq \tfrac{1}{2}\rho_0.$$

From (3.7)

$$R_{44} \simeq -\tfrac{1}{2}\epsilon\nabla^2 h_{44}.$$

Hence, the equation $R_{44} = -\kappa T^*_{44}$ approximates to $\nabla^2\phi = \frac{1}{2}\kappa\rho_0$ if

$$\phi = \tfrac{1}{2}\epsilon h_{44}. \tag{3.8}$$

This equation reduces to the classical Poisson equation if

$$\kappa = 8\pi G, \tag{3.9}$$

G being the Newtonian constant of gravitation. This deduction, however, is well known.

In the case of a surface layer we similarly have $S^*_{44} \simeq \frac{1}{2}\sigma_0$. In the quasisteady approximation f_t will be taken to be small compared with the space derivatives f_x, f_y, f_z. By (2.3)

$$n_i \simeq f_{,i}/(f_x^2 + f_y^2 + f_z^2)^{\frac{1}{2}}. \tag{3.10}$$

If $\mathbf{n} = (n_x, n_y, n_z)$ denotes unit vector normal to the spatial surface $f(x, y, z, t) = 0$ at time t, then $\mathbf{n} \simeq (n_1, n_2, n_3)$ and n_4 is small compared with n_1, n_2, n_3. For $k = 1, 2, 3$,

$$\Gamma^k_{44} \simeq \tfrac{1}{2}\epsilon h_{44,k} = \phi_{,k}.$$

Hence, the 44-component of (2.8) reduces to the approximate equation

$$\Delta(n_x\phi_x + n_y\phi_y + n_z\phi_z) = \tfrac{1}{2}\kappa\sigma_0$$

or

$$\Delta\left(\frac{\partial\phi}{\partial n}\right) = 4\pi G\sigma_0,$$

which is the classical boundary condition at a surface layer.

We set

$$\mathbf{q} = (q_x, q_y, q_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

$$\mathbf{q} \simeq (u^1, u^2, u^3), \quad u^4 \simeq 1.$$

The condition (2.7) approximates to

$$q_x f_x + q_y f_y + q_z f_z + f_t = 0,$$

which is the well-known classical condition that the equation of the surface layer at any time is $f(x, y, z, t) = 0$. We note that q_x, q_y, q_z, and f_t are small quantities.

4. INTEGRAL FIELD LAW

The field equations (1.5) and the boundary conditions (2.8) can be simultaneously deduced from an integral flux law.

Definition 4.1: Let V be a region of 4-space. Then the integral

$$-\int_V T^*_{ij}\, dV,$$

where dV denotes the invariant 4-volume element, will be called the parametric rate of decrease or simply the parametric decay of energy-momentum in the region V.

Similarly for a surface layer S, the parametric decay of energy-momentum in the layer S is defined to be

$$-\int_S \bar{S}^*_{ij}\, dS,$$

where dS is the invariant hypersurface element.

Postulate 4.1 (Integral Field Law): Let Σ be any closed hypersurface given by $F(x^1, x^2, x^3, x^4) = 0$ where the function F does not involve the parameter ϵ and N_i, the unit normal to Σ, is spacelike, i.e., $g^{\alpha\beta}N_\alpha N_\beta = -1$. Let V denote the region of 4-space bounded by Σ and N_i point out of V across Σ. We postulate that

$$\int_\Sigma Q^k_{ij} N_k\, d\Sigma$$

($d\Sigma$-invariant hypersurface element), which represents the flux of the tensor Q^k_{ij} across Σ, is κ times the parametric decay of energy-momentum in V.

Thus, for any region V,

$$\int_\Sigma Q^k_{ij} N_k\, d\Sigma = -\kappa \int_V \bar{T}^*_{ij}\, dV.$$

By Gauss's theorem we get

$$\int_V (Q^k_{ij;k} + \kappa \bar{T}^*_{ij})\, dV = 0,$$

which gives (1.5).

We take a fixed point on a surface layer S, at which the unit normal is n_i, and take Σ to be a small closed cylinder across S, enclosing the point whose generators are parallel to n_i. Applying the integral law to Σ, we get

$$n_k \Delta Q^k_{ij}\, dS = -\kappa \bar{S}^*_{ij}\, dS,$$

which gives (2.9).

If S is any hypersurface which is not a surface layer (and, hence, S is not necessarily parametrically stationary), then the integral law gives the following boundary condition at S:

$$n_k \Delta Q^k_{ij} = 0$$

or, by (2.3),

$$f_{,k} \Delta Q^k_{ij} = 0.$$

Integrating with respect to ϵ from 0 to ϵ, we have

$$\tfrac{1}{2} f_{,j} \Delta\Gamma^\alpha_{i\alpha} + \tfrac{1}{2} f_{,i} \Delta\Gamma^\alpha_{j\alpha} - f_{,k} \Delta\Gamma^k_{ij} = 0$$

or

$$\tfrac{1}{2} n_j \Delta\Gamma^\alpha_{i\alpha} + \tfrac{1}{2} n_i \Delta\Gamma^\alpha_{j\alpha} - n_k \Delta\Gamma^k_{ij} = 0. \quad (4.1)$$

5. APPLICATIONS TO SCHWARZSCHILD'S SPHERE

We shall now apply the criterion (4.1) to the field of a stationary sphere of perfect fluid[1] and show that it leads to physically significant results.

The field being static and spherically symmetric, we assume

$$ds^2 = -e^\lambda\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^\nu\, dt^2, \quad (5.1)$$

where $\lambda = \lambda(r)$, $\nu = \nu(r)$.

Let $r = r_1$ be the boundary of the sphere at which we have $\Delta g_{ij} = 0$. Further $\lambda, \nu \to 0$ as $r \to \infty$.

The nonvanishing Christoffel symbols have the following values:

$$\begin{aligned} &\Gamma^1_{11} = \tfrac{1}{2}\lambda', && \Gamma^2_{12} = \Gamma^3_{13} = r^{-1}, \\ &\Gamma^4_{14} = \tfrac{1}{2}\nu', && \Gamma^3_{23} = \cot\theta, \\ &\Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, && \Gamma^2_{33} = -\sin\theta\cos\theta, \\ &\Gamma^1_{22} = -re^{-\lambda}, && \Gamma^1_{44} = \tfrac{1}{2}\nu' e^{\nu-\lambda}, \end{aligned} \quad (5.2)$$

where the prime denotes differentiation with respect to r.

External region: $r > r_1$: The equations are

$$\begin{aligned} e^{-\lambda}(r^{-1}\nu' + r^{-2}) - r^{-2} &= 0, \\ e^{-\lambda}(\tfrac{1}{2}\nu'' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 + \tfrac{1}{2}r^{-1}\nu' - \tfrac{1}{2}r^{-1}\lambda') &= 0, \\ e^{-\lambda}(r^{-1}\lambda' - r^{-2}) + r^{-2} &= 0. \end{aligned} \quad (5.3)$$

The solution of these equations under the given condition at infinity is the well-known Schwarzschild's exterior solution

$$ds^2 = -(1 - 2Gm/r)^{-1}\, dr^2 - r^2\, d\theta^2 - r^2\sin^2\theta\, d\phi^2 + (1 - 2Gm/r)\, dt^2, \quad (5.4)$$

where m is a constant.

Internal region: $r < r_1$: The energy-momentum tensor is given by

$$T^{ij} = (\rho_0 + p_0)u^i u^j - g^{ij} p_0,$$

where ρ_0 and p_0 are the proper density and pressure of the fluid and we take ρ_0 to be constant. The equations are

$$\begin{aligned} e^{-\lambda}(r^{-1}\nu' + r^{-2}) - r^{-2} &= \kappa p_0, \\ e^{-\lambda}(\tfrac{1}{2}\nu'' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 + \tfrac{1}{2}r^{-1}\nu' - \tfrac{1}{2}r^{-1}\lambda') &= \kappa p_0, \\ e^{-\lambda}(r^{-1}\lambda' - r^{-2}) + r^{-2} &= \kappa \rho_0. \end{aligned} \quad (5.5)$$

It follows that

$$p_0' + \tfrac{1}{2}(\rho_0 + p_0)\nu' = 0. \quad (5.6)$$

The interior solution is of the form

$$ds^2 = -(1 - r^2/R^2)^{-1}\, dr^2 - r^2\, d\theta^2 - r^2\sin^2\theta\, d\phi^2 + [A - B(1 - r^2/R^2)^{\frac{1}{2}}]^2\, dt^2, \quad (5.7)$$

where $R^2 = 3/\kappa\rho_0$.

$\Delta g_{11} = 0$ at $r = r_1$ gives

$$m = \tfrac{4}{3}\pi r_1^3 \rho_0. \quad (5.8)$$

The first equation is of (5.5) gives

$$\kappa R^2 p_0 = \frac{3B(1 - r^2/R^2)^{\frac{1}{2}} - A}{A - B(1 - r^2/R^2)^{\frac{1}{2}}}. \quad (5.9)$$

From the first equations of (5.3) and (5.5), we get

$$\Delta\nu' = -\kappa r_1 e^{\lambda(r_1)} p_0(r_1) \quad \text{at} \quad r = r_1. \quad (5.10)$$

By (5.2), the only nonvanishing components of $\Delta\Gamma^k_{ij}$ at $r = r_1$ are

$$\Delta\Gamma^1_{11} = \tfrac{1}{2}\Delta\lambda', \quad \Delta\Gamma^4_{14} = \tfrac{1}{2}\Delta\nu', \quad \Delta\Gamma^1_{44} = \tfrac{1}{2}e^{\nu-\lambda}\Delta\nu'. \quad (5.11)$$

For the boundary $r = r_1$, n_2, n_3, n_4 are all zero. Then (4.1) simply gives $\Delta\nu' = 0$ and, hence, by (5.10),

$$p_0(r_1) = 0. \quad (5.12)$$

This is, in fact, a physical necessity and was assumed by Schwarzschild on physical grounds.

Now (5.12) and $\Delta g_{44} = 0$ at $r = r_1$ give

$$A = \tfrac{3}{2}(1 - r_1^2/R^2)^{\frac{1}{2}}, \quad B = \tfrac{1}{2}, \quad (5.13)$$

and the solution is complete.

Remark: Note that in the above solution the conditions (4.1) are satisfied although dg_{11}/dr is discontinuous at the boundary.

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[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford U. P., London, 1934).

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Symmetric Dirac Bracket in Classical Mechanics

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It is known that, in classical systems that have second-class constraints which relate the canonical coordinates and momentum, the ordinary skew-symmetric Poisson bracket must be replaced by the skew-symmetric Dirac bracket. It is also known that in the process of quantization of such systems, the Dirac bracket replaces the Poisson bracket in its correspondence with the quantum commutators. In this paper we obtain the symmetric partner of the Dirac bracket, which is of interest not only to classical mechanics but also in regard to the quantization procedure; i.e., the quantization rules for systems which are restricted by second-class constraints such that the commutation rules involve anticommutators (instead of commutators, as in certain fields with Fermi–Dirac statistics) can be given in terms of this new symmetric bracket. This symmetric bracket is related to the Poisson–Droz-Vincent symmetric bracket.

1. INTRODUCTION

There are important physical systems that are restricted by second-class constraints.[1,2] In the quantization of these systems, the Poisson bracket *does not* correspond to the classical limit of the commutator and, therefore, the quantization procedure must be modified. This was first shown by Dirac in the skew-symmetric case.[1–4] There are important systems such as the vector field[5] and the gravitational field[4,6] which belong to this case.

As a review, let us remember that usually in Hamiltonian mechanics the generalized coordinates q_r and their canonically conjugate momenta

$$p_r = \frac{\partial L}{\partial(dq_r/dt)} \quad (1.1)$$

are independent variables. Also there are some less known cases in which it fails and the Eqs. (1.1) are identities in q and p only, even before Lagrange equations are used. Equations (1.1) and others derived from it by consistency relations are called constraints and denoted by $f_a(q, p) \approx 0$.[1–7] The weak equality sign, $\approx$ introduced by Dirac, means that the expression can be used only after computing all the Poisson brackets and partial differentiations in an equation (see, e.g., Ref. 2). The above mentioned second-class constraints[1–3] belong to a certain subset of the former set (where, eventually, some constraints are replaced by linear combinations of them[1,2,8]). We denote the second-class constraints as[9]

$$\theta_a(q, p) \approx 0, \quad a = 1, 2, \cdots, N_\theta. \quad (1.2)$$