SOME SOLUTIONS FOR YANG'S EQUATIONS FOR SELF-DUAL SU(2) GAUGE FIELDS

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It is shown that a large class of solutions of Yang's equations for self-dual SU(2) gauge fields can be obtained from the solutions of two-dimensional and four-dimensional Laplace equations.

Yang [1] has reduced the equations for self-dual SU(2) gauge fields on euclidean space to the following equations:

$$\phi(\phi_{y\bar{y}}+\phi_{z\bar{z}})-\phi_y\phi_{\bar{y}}-\phi_z\phi_{\bar{z}}+\rho_y\bar{\rho}_{\bar{y}}+\rho_z\bar{\rho}_{\bar{z}}=0, \quad \phi(\rho_{y\bar{y}}+\rho_{z\bar{z}})-2\rho_y\phi_{\bar{y}}-2\rho_z\phi_{\bar{z}}=0, \tag{1a,b}$$

where an over bar denotes the complex conjugate. ϕ and ρ are functions of $y, \bar{y}, z$ and $\bar{z}$. ϕ is real and ρ is complex. $\sqrt{2}\,y = x_1 + \mathrm{i}x_2$, $\sqrt{2}\,z = x_3 - \mathrm{i}x_4$, x_1, x_2, x_3 and x_4 are real. $F_{\mu\nu}$ is given by

$$F_{\mu\nu}=B_{\mu,\nu}-B_{\nu,\mu}-B_\mu B_\nu+B_\nu B_\mu, \quad B_\mu=-\tfrac{1}{2}\mathrm{i}b^k_\mu\sigma_k, \quad b^k_y=\phi^{-1}(\mathrm{i}\rho_y,\rho_y,-\mathrm{i}\phi_y), \quad b^k_z=\phi^{-1}(\mathrm{i}\rho_z,\rho_z,-\mathrm{i}\phi_z),$$

σ_k are Pauli matrices, $\mu = 1, 2, 3, 4$ and $k = 1, 2, 3$. When written in terms of real variables, eqs. (1) read:

$$\tfrac{1}{2}\phi(\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44})-\tfrac{1}{2}(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)+\tfrac{1}{2}[(\alpha_1+\beta_2)^2+(\alpha_2-\beta_1)^2+(\alpha_3-\beta_4)^2+(\alpha_4+\beta_3)^2]=0, \tag{1a$'$}$$

$$\tfrac{1}{2}\phi(\alpha_{11}+\alpha_{22}+\alpha_{33}+\alpha_{44})-[(\alpha_1+\beta_2)\phi_1+(\alpha_2-\beta_1)\phi_2+(\alpha_3-\beta_4)\phi_3+(\alpha_4+\beta_3)\phi_4]=0, \tag{1b$'$}$$

$$\tfrac{1}{2}\phi(\beta_{11}+\beta_{22}+\beta_{33}+\beta_{44})-[(\beta_1-\alpha_2)\phi_1+(\alpha_1+\beta_2)\phi_2+(\alpha_4+\beta_3)\phi_3+(\beta_4-\alpha_3)\phi_4]=0, \tag{1c$'$}$$

where

$$\rho=\alpha+\mathrm{i}\beta. \tag{2}$$

This letter presents two classes of solutions of eqs. (1′) as follows.

Case 1

$$\rho=\rho(\phi), \quad \text{i.e. } \alpha=\alpha(\phi) \text{ and } \beta=\beta(\phi), \tag{3}$$

where ϕ is not a constant and α and β are not both constants. Here eqs. (1) reduce to

$$\phi(\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44})-(1-\alpha'^2-\beta'^2)(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)=0, \tag{4a}$$

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$$\phi\alpha'(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}) - (2\alpha' - \phi\alpha'')(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = 0, \tag{4b}$$

$$\phi\beta'(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}) - (2\beta' - \phi\beta'')(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = 0, \tag{4c}$$

where $\alpha' \equiv \mathrm{d}\alpha/\mathrm{d}\phi$ and $\beta' \equiv \mathrm{d}\beta/\mathrm{d}\phi$.

Eliminating $(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44})$ and $(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)$ from (4b) and (4c) one gets after a little calculation

$$\alpha = A\gamma, \quad \beta = B\gamma, \tag{5}$$

where A and B are constants, γ is a function of ϕ and physically insignificant additive constants are omitted. Eliminating $\phi\,(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44})$ and $(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)$ from eqs. (4a) and (4b) and using (5) one gets

$$1 + (A^2 + B^2)\gamma'^2 - \phi\gamma''/\gamma' = 0,$$

which on integration gives

$$\gamma = [C(A^2 + B^2)]^{-1}\,[1 - C^2(A^2 + B^2)\phi^2]^{1/2}, \tag{6}$$

where C is a constant. Using (5) and (6) one can reduce (4a) to

$$V_{11} + V_{22} + V_{33} + V_{44} = 0, \tag{7}$$

where

$$\phi = [C(A^2 + B^2)^{1/2}]^{-1} \operatorname{sech} V. \tag{8}$$

It is now easy to check that if V satisfies (7) then α, β and ϕ as given by (5), (6) and (8) satisfy (1′).

Case 2. Here one assumes

$$\phi(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}) = (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2). \tag{9}$$

Eq. (1a′) then gives

$$\alpha_1 + \beta_2 = 0, \quad \alpha_2 - \beta_1 = 0, \quad \alpha_3 - \beta_4 = 0, \quad \alpha_4 + \beta_3 = 0. \tag{10}$$

Eqs. (10) give

$$\alpha_{11} + \alpha_{22} = 0, \quad \beta_{11} + \beta_{22} = 0, \quad \alpha_{33} + \alpha_{44} = 0, \quad \beta_{33} + \beta_{44} = 0.$$

One then sees that eqs. (9) and (10) together satisfy (1′). Eq. (9) can be integrated to give

$$\phi = \mathrm{e}^V, \quad \text{where } V \text{ satisfies (7)} \tag{11}$$

and (10) is satisfied by

$$\alpha = U(x_1, x_2) + \psi(x_3, x_4), \quad \beta = V(x_1, x_2) + \chi(x_3, x_4), \tag{12}$$

where

$$\partial U(x_1, x_2)/\partial x_1 + \partial V(x_1, x_2)/\partial x_2 = 0, \quad \partial U(x_1, x_2)/\partial x_2 - \partial V(x_1, x_2)/\partial x_1 = 0,$$

$$\partial\psi(x_3, x_4)/\partial x_3 - \partial\chi(x_3, x_4)/\partial x_4 = 0, \quad \partial\psi(x_3, x_4)/\partial x_4 + \partial\chi(x_3, x_4)/\partial x_3 = 0.$$

In summary, from solutions of eq. (7) which is a (3 + 1) dimensional Laplace equation one gets two classes of solutions of (1). One class is given by (2), (5), (6) and (8) and the other by (2), (11) and (12).

Reference

[1] C.N. Yang, Phys. Rev. Lett. 38 (1977) 1377.