AIP Journal of Mathematical Physics

# Some exact solutions of nonlinear chiral field equations

Pranab Krishna Chanda, Dipankar Ray, and Utpal Kumar De

Citation: Journal of Mathematical Physics **25**, 2557 (1984); doi: 10.1063/1.526441
View online: http://dx.doi.org/10.1063/1.526441
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/25/8?ver=pdfcov
Published by the AIP Publishing

# Some exact solutions of nonlinear chiral field equations

Pranab Krishna Chanda and Dipankar Ray
*Department of Applied Mathematics, Calcutta University, 92, A. P. C. Road, Calcutta 700 009, India*
Utpal Kumar De
*Department of Physics, Jadavpur University, Calcutta 700 032, India*

(Received 19 July 1983; accepted for publication 4 November 1983)

Some new exact solutions of the nonlinear field equations for the chiral invariant model of pion dynamics are presented here. These solutions are a further generalization of some previous works presented by one of the authors (Ray). It is interesting to note that equations in (3.2) obtained by Ray (1978) are conformally invariant. Hence from any solution of these equations one can immediately generate infinitely many other solutions of these equations simply by replacing ($x^1$, $x^2$) by ($y$, $z$), where $y$ and $z$ are any two mutually conjugate solutions of Laplace's equations. Further, a striking similarity in form of these equations with one of the two generalized Lund–Regge equations makes the study of the solutions of these equations more worthwhile with the view that the study of the solutions of these equations will eventually lead to the study of the solution of a larger class of equations that will include these equations and generalized Lund–Regge equations as special cases.

PACS numbers: 11.30.Rd, 11.10.Lm

## 1. INTRODUCTION

Under tangential parametrization (Charap, 1973)[1] the field equations for the Chiral invariant model of the pion dynamics take the form (Charap, 1976)[2]

$$\Box\phi = \eta^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\mu}},$$
$$\Box\psi = \eta^{\mu\nu}\frac{\partial\psi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}},$$
$$\Box\chi = \eta^{\mu\nu}\frac{\partial\chi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}}, \qquad (1)$$

where

$$\eta^{\mu\nu} = 0 \quad \text{for } \mu\neq\nu,$$
$$= 1 \quad \text{for } \mu=\nu\neq 4,$$
$$= -1 \quad \text{for } \mu=\nu=4,$$
$$\beta = \ln(f_\pi^2+\phi^2+\psi^2+\chi^2),$$
$$f_\pi = \text{const.}$$

Under the assumptions

$$\phi = \phi(x^1, x^2, x^3 - x^4),$$
$$\psi = \psi(x^1, x^2, x^3 - x^4), \qquad (2)$$
$$\chi = \chi(x^1, x^2, x^3 - x^4),$$

the set of differential equations in four variables, namely, Eqs. (1), reduces to a set of differential equations in two variables as follows (Ray, 1978)[3]:

$$\phi_{11} + \phi_{22} = \beta_1\phi_1 + \beta_2\phi_2, \qquad (3a)$$
$$\psi_{11} + \psi_{22} = \beta_1\psi_1 + \beta_2\psi_2, \qquad (3b)$$
$$\chi_{11} + \chi_{22} = \beta_1\chi_1 + \beta_2\chi_2, \qquad (3c)$$

where

$$\beta = \ln(f_\pi^2+\phi^2+\psi^2+\chi^2), \qquad (3d)$$
$$f_\pi = \text{const}, \qquad (3e)$$
$$\phi_1 = \frac{\partial\phi}{\partial\chi^1}, \quad \beta_1 = \frac{\partial\beta}{\partial\chi^1}, \quad \text{and so on.}$$

Apart from the physical significance of Eqs. (3) already described, Eqs. (3) seem to be of considerable mathematical interest as well. This is first because Eqs. (3) are conformally invariant, i.e., the form of Eqs. (3) remain invariant under transformation $(x^1, x^2)\to(y, z)$, where $y$ and $z$ are two mutually conjugate solutions of Laplace's equations in $x^1$ and $x^2$. Hence from any solution of Eqs. (3) one can immediately generate infinitely many other solutions of Eqs. (3) simply by replacing $(x^1, x^2)$ by $(y, z)$, where $y$ and $z$ are two mutually conjugate solutions of Laplace's equations.

In this context attention should also be drawn to another set of coupled equations, namely, the generalized Lund–Regge equations, which are[4,5]

$$\text{(a)}\quad \theta_{11} \pm \theta_{22} - 4g(\theta) + h(\theta)(\lambda_1^2 + \lambda_2^2) = 0,$$
$$\text{(b)}\quad \lambda_{11} \pm \lambda_{22} = 2p(\theta)(\lambda_1\theta_1 + \lambda_2\theta_2), \qquad (4)$$

where

$$\theta = \theta(x^1, x^2), \quad \lambda = \lambda(x^1, x^2),$$
$$\theta_1 = \frac{\partial\theta}{\partial x^1} \quad \text{and so on.}$$

Particular examples of the generalized Lund–Regge equations include a good number of physically interesting equations, e.g., equations of two-dimensional Heisenberg ferromagnets,[6,7] Ginzburg–Pitaevski equations for superfluids,[8,9] stationary wave envelope in nonlinear optics,[10] and so on. It is interesting to note that Eqs. (3) are of the same form as one of the two generalized Lund–Regge equations, namely, (4b). Since a good number of studies of the solutions of the generalized equations have been made it is quite possible that a study of the solutions of (3) will eventually lead to the study of the solution of a larger class of equations that will include (3) and generalized Lund–Regge equations as special cases. For all these reasons the study of the solution of (3) seems worthwhile.

## 2. SOLUTIONS

Equation (3) may be rewritten as

$$(e^{-\beta}\phi_1)_1 + (e^{-\beta}\phi_2)_2 = 0, \quad (5a)$$

$$\psi_{11} + \psi_{22} = \psi_1\beta_1 + \psi_2\beta_2, \quad (5b)$$

$$\chi_{11} + \chi_{22} = \chi_1\beta_1 + \chi_2\beta_2, \quad (5c)$$

where

$$\beta = \ln (f_\pi^2 + \phi^2 + \psi^2 + \chi^2). \quad (5d)$$

*Case I*: Assuming

$$\beta = \beta(\phi), \quad (6)$$

one can set

$$X = \int e^{-\beta} \quad d\phi. \quad (7)$$

Then Eq. (5a) with the help of Eq. (7) takes the form

$$X_{11} + X_{22} = 0, \quad (8)$$

which is the Laplace's equation and standard solutions for $X$ are obtainable. Now, one can write

$$\psi^2 + \chi^2 = \alpha^2, \quad (9a)$$

when (5d) reduces to

$$\beta = \ln(f_\pi^2 + \phi^2 + \alpha^2). \quad (9b)$$

Then one gets from (9b), (6), and (7),

$$\alpha = \alpha(X), \quad (9c)$$

and from (9a),

$$\psi = \alpha \cos \theta, \quad (9d)$$

$$\chi = \alpha \sin \theta. \quad (9e)$$

One can take

$$\theta = \theta(X, Y), \quad (10)$$

where $Y$ is the solution of the Laplace's equation conjugate to $X$, i.e., $X_1 = Y_2$ and $X_2 = -Y_1$. Using (6), (7), (9d), (9e), and (10) one may reduce (5b) and (5c) to the following forms:

$$\{\alpha_{XX} - \alpha(\theta_X^2 + \theta_Y^2) - \alpha_X\beta_X\} \cos \theta = \{2\alpha_X\theta_X + \alpha(\theta_{XX} + \theta_{YY}) - \alpha\beta_X\theta_X\} \sin \theta, \quad (11a)$$

$$\{\alpha_{XX} - \alpha(\theta_X^2 + \theta_Y^2) - \alpha_X\beta_X\} \sin \theta = -\{2\alpha_X\theta_X + \alpha(\theta_{XX} + \theta_{YY}) - \alpha\beta_X\theta_X\} \cos \theta. \quad (11b)$$

(11a) and (11b) are equivalent to

$$\alpha_{XX} - \alpha(\theta_X^2 + \theta_Y^2) - \alpha_X\beta_X = 0, \quad (12a)$$

$$2\alpha_X\theta_X + \alpha(\theta_{XX} + \theta_{YY}) - \alpha\beta_X\theta_X = 0. \quad (12b)$$

Equation (12) may be rewritten as

$$\theta_X^2 + \theta_Y^2 = \delta(X), \quad (13a)$$

$$\delta(X) = \frac{\alpha_{XX} - \alpha_X\beta_X}{\alpha} \neq 0, \quad (13b)$$

$$\alpha(\theta_{XX} + \theta_{YY}) + 2\alpha_X\theta_X - \alpha\beta_X\theta_X = 0. \quad (13c)$$

The above equation leads to

$$\theta_Y = \text{const} = B \quad \text{(say)}. \quad (14)$$

To verify it, let us consider three possibilities separately.

(i) If $\theta_X \neq 0$, $\theta_Y = 0$, then (14) is automatically satisfied with $B = 0$.

(ii) If $\theta_X = 0$, $\theta_Y \neq 0$, the left-hand side of (13a) is a function of $Y$ only and the right-hand side is a function of $X$ only. Since both are constants. Hence, $\theta_Y = $ const and (14) is satisfied.

(iii) If $\theta_X \neq 0$, $\theta_Y \neq 0$, one can proceed as follows: Differentiating (13a) with respect to $Y$,

$$\theta_X\theta_{XY} + \theta_Y\theta_{YY} = 0.$$

Eliminating $\theta_{YY}$ from (13c) with the use of the above, dividing throughout by $[\theta_Y e^{-(2\ln\alpha - \beta)}]$ and integrating, $\alpha^2\theta_X e^{-\beta}/\theta_Y = \gamma(Y)$, where $\gamma(Y)$ is an unspecified function of $Y$.

This readily gives $\theta = \theta(w)$, where

$$w = u + v,$$

$$u = \int \frac{e^\beta}{\alpha^2} dX,$$

$$v = \int \frac{dY}{\gamma(Y)}.$$

Using the above relations, we get from (13a)

$$\frac{u_X^2}{\delta(X)} + \frac{v_Y^2}{\delta(X)} = \frac{1}{\theta_w^2}.$$

Differentiating the above expression separately with respect to $u$ and $v$, respectively, and comparing the results,

$$\delta\left(\frac{u_X^2}{\delta}\right)_u + \delta\left(\frac{1}{\delta}\right)_u v_Y^2 = (v_Y^2)_v.$$

Differentiating this successively with respect to $u$ and $v$,

$$[\delta(1/\delta)_u]_u (v_Y^2)_v = 0.$$

Hence,

$$\delta(1/\delta)_u = \text{const},$$

or

$$v_Y = \text{const}.$$

That $\delta(1/\delta)_u = $ const is not possible will be shown in Appendix A.

Differentiating $[u_X^2/\delta(X) + v_Y^2/\delta(X)] = 1/\theta_w^2$ with respect to $v$ and using $v_Y = $ const, one can get

$$\left(\frac{1}{\theta_w^2}\right)_w = 0,$$

which gives $\theta_w = $ const. Hence, $\theta_Y = \theta_w \cdot w_Y = \theta_w \cdot v_Y$ = const. Thus, in this case of $\theta_X \neq 0$, $\theta_Y \neq 0$ too we see that (14) is satisfied. Using (14) in (13c) and after integrating once, one may get

$$\theta_X = Ae^\beta/\alpha^2, \quad \text{where } A = \text{const}. \quad (15)$$

Generalizing (15) and (14), one gets

$$\theta = \int \frac{Ae^\beta}{\alpha^2} dX + BY + C, \quad (16)$$

where $B$ and $C$ are also constants. Using (16) in (13a) and (13b),

$$(\alpha_{XX} - \alpha_X\beta_X)/\alpha = A^2e^{2\beta}/\alpha^4 + B^2,$$

or $(e^{-\beta}\alpha_X)_X = (A^2e^\beta/\alpha^4 + B^2/e^\beta)\alpha$, which may be rewritten with the use of (9b) and (7) as

$$\alpha_{\phi\phi} = \frac{A^2}{\alpha^3} + \frac{B^2\alpha}{(f_\pi^2 + \phi^2 + \alpha^2)^2}. \quad (17)$$

Solution of Eq. (17) gives the dependence of $\alpha$ in terms of $\phi$. It may be pointed out that Eq. (17) is easily solvable with the use of $\alpha_{\phi\phi} = \frac{1}{2}(\alpha_\phi^2)_\alpha$ in the particular case $B = 0$. Then all the unknown quantities like $\psi, \chi, \phi, \beta$ become a function of $X$ only, which in turn will be a function of $(x^1, x^2, x^3 - x^4)$.

We also get from (9b) and (7),

$$\phi_X = f_\pi^2 + \phi^2 + \alpha^2. \tag{18}$$

Thus, we can conclude that Eqs. (18) and (17) give the solution of $\phi$ in terms of $X$. As $e^\beta = \phi_X$, $\beta$ is also known in terms of $X$. The use of the expression for $\phi$ in Eq. (17) will let us determine $\alpha$ in terms of $X$. Then Eq. (16) gives $\theta$ in terms of $X$ and $Y$, where $Y$ is conjugate Laplace solution to $X$. Lastly, $X$ is obtained as a solution of Laplace's equation $X_{11} + X_{22} = 0$. These completely solve the unknown quantities of Eq. (1).

Another case of interest may be added, when $\delta(X) = 0$. Then $\theta$ becomes a constant. Since now $(\alpha_{XX} - \alpha_X \beta_X)/\alpha = 0$, one finally gets, with the help of Eq. (7), $\alpha_{\phi\phi} = 0$. On integration, we get $\alpha$ in terms of $\phi$. Using this in (18), and on integration, which is easily possible, one may get $\phi$ in terms of $X$, which is a solution of Laplace's equation $X_{11} + X_{22} = 0$. Now, the unknown quantities of Eq. (1) become known.

*Case II*: Without loss of generality one can write, from Eq. (5a)

$$e^{-\beta}\phi_1 = \sigma_2, \tag{19a}$$

$$e^{-\beta}\phi_2 = -\sigma_1. \tag{19b}$$

Now, we introduce an assumption that

$$\psi^2 + \chi^2 = (f_\pi^2 + \phi^2)\xi^2(\sigma), \tag{20a}$$

where $\xi(\sigma)$ = some unspecified function of $\sigma$. Then it turns out that

$$\psi = (f_\pi^2 + \phi^2)^{1/2}\xi(\sigma)\cos\Theta, \tag{20b}$$

$$\chi = (f_\pi^2 + \phi^2)^{1/2}\xi(\sigma)\sin\Theta, \tag{20c}$$

$$\Theta = \Theta(\sigma, \phi), \tag{20d}$$

$$\beta = \ln(f_\pi^2 + \phi^2) + \ln(1 + \xi^2). \tag{20e}$$

Using (20b), (20c), (20d), (20e), and (19), one may reduce (5b) and (5c) to the following forms:

$$\begin{aligned} &[\Phi_{\phi\phi}\xi\Phi^4(1+\xi^2)^2 + \Phi\xi_{\sigma\sigma} - \xi\Theta_\phi^2\Phi^5(1+\xi^2)^2 \\ &\quad - \Phi\xi\Theta_\sigma^2 - 2\Phi\xi_\sigma(1+\xi^2)_\sigma/(1+\xi^2)]\cos\Theta \\ &= [2\Phi\xi_\sigma\Theta_\sigma + \xi\Theta_{\phi\phi}\Phi^5(1+\xi^2)^2 + \Phi\xi\Theta_{\sigma\sigma} \\ &\quad + 2\Phi^4\xi\Theta_\phi\Phi_\phi(1+\xi^2)^2 \\ &\quad - 2\Phi\xi\Theta_\sigma(1+\xi^2)_\sigma/(1+\xi^2)]\sin\Theta, \end{aligned} \tag{21a}$$

and

$$\begin{aligned} &[\Phi_{\phi\phi}\xi\Phi^4(1+\xi^2)^2 + \Phi\xi_{\sigma\sigma} - \xi\Theta_\phi^2\Phi^5(1+\xi^2)^2 \\ &\quad - \Phi\xi\Theta_\sigma^2 - 2\Phi\xi_\sigma(1+\xi^2)_\sigma/(1+\xi^2)]\sin\Theta \\ &= -[2\Phi\xi_\sigma\Theta_\sigma + \xi\Theta_{\phi\phi}\Phi^5(1+\xi^2)^2 + \Phi\xi\Theta_{\sigma\sigma} \\ &\quad + 2\Phi^4\xi\Theta_\phi\Phi_\phi(1+\xi^2)^2 \\ &\quad - 2\Phi\xi\Theta_\sigma(1+\xi^2)_\sigma/(1+\xi^2)]\cos\Theta, \end{aligned} \tag{21b}$$

where

$$\Phi = (f_\pi^2 + \phi^2)^{1/2}. \tag{22}$$

(21a) and (21b) are equivalent to

$$\begin{aligned} &\Phi_{\phi\phi}\xi\Phi^4(1+\xi^2)^2 + \Phi\xi_{\sigma\sigma} - \xi\Theta_\phi^2\Phi^5(1+\xi^2)^2 \\ &\quad - \Phi\xi\Theta_\sigma^2 - 2\Phi\xi_\sigma(1+\xi^2)_\sigma/(1+\xi^2) = 0 \end{aligned} \tag{23a}$$

and

$$\begin{aligned} &2\Phi\xi_\sigma\Theta_\sigma + \xi\Theta_{\phi\phi}\Phi^5(1+\xi^2)^2 + \Phi\xi\Theta_{\sigma\sigma} \\ &\quad + 2\Phi^4\xi\Theta_\phi\Phi_\phi(1+\xi^2)^2 \\ &\quad - 2\Phi\xi\Theta_\sigma(1+\xi^2)_\sigma/(1+\xi^2) = 0, \end{aligned} \tag{23b}$$

which may be rewritten with the use of (22) as

$$\begin{aligned} &f_\pi^2 + \frac{\xi_{\sigma\sigma}}{\xi(1+\xi^2)^2} - \Phi^4\Theta_\phi^2 - \frac{\Theta_\sigma^2}{(1+\xi^2)^2} \\ &\quad - \frac{2\xi_\sigma(1+\xi^2)_\sigma}{\xi(1+\xi^2)^3} = 0, \end{aligned} \tag{24a}$$

and

$$\begin{aligned} &\frac{\xi^2(\xi^2\Theta_\sigma)_\sigma}{\xi^4(1+\xi^2)^2} + (f_\pi^2 + \phi^2)[(f_\pi^2 + \phi^2)\Theta_\phi]_\phi \\ &\quad - \frac{2(1+\xi^2)_\sigma}{(1+\xi^2)^3}\Theta_\sigma = 0. \end{aligned} \tag{24b}$$

Defining

$$\int \frac{d\sigma}{\xi^2} = p \tag{25a}$$

and

$$\int \frac{d\phi}{f_\pi^2 + \phi^2} = q, \tag{25b}$$

Eqs. (24) may be reduced to

$$\begin{aligned} \frac{\Theta_p^2}{\xi^4(1+\xi^2)^2} + \Theta_q^2 = f_\pi^2 &+ \frac{\xi_{\sigma\sigma}}{\xi(1+\xi^2)^2} \\ &- \frac{2\xi_\sigma(1+\xi^2)_\sigma}{\xi(1+\xi^2)^3} \end{aligned} \tag{26a}$$

and

$$\frac{\Theta_{pp}}{\xi^4(1+\xi^2)^2} + \Theta_{qq} = \frac{2(1+\xi^2)_\sigma}{(1+\xi^2)^3\xi^2}\Theta_p. \tag{26b}$$

Again defining

$$\frac{1}{\xi^4(1+\xi^2)^2} = U(p), \tag{27a}$$

$$f_\pi^2 + \frac{\xi_{\sigma\sigma}}{\xi(1+\xi^2)^2} - \frac{2\xi_\sigma(1+\xi^2)_\sigma}{\xi(1+\xi^2)^3} = V(p), \tag{27b}$$

and

$$\frac{2(1+\xi^2)_\sigma}{(1+\xi^2)^3\xi^2} = W(p), \tag{27c}$$

Eqs. (26) may be reduced to

$$U\Theta_p^2 + \Theta_q^2 = V, \tag{28a}$$

$$U\Theta_{pp} + \Theta_{qq} = W\Theta_p. \tag{28b}$$

From Eq. (28a) one can examine four cases separately. However, the following three cases, i.e., (i) $\Theta_p = \Theta_q = 0$, (ii) $\Theta_p \neq 0$ and $\Theta_q = 0$, and (iii) $\Theta_p = 0$ and $\Theta_q \neq 0$, can be grouped under one head of $\Theta_q$ = const, where the constant may even take the value zero.

To study the fourth case, i.e., $\Theta_p \neq 0$, $\Theta_q \neq 0$, one can proceed as follows.

Differentiating (28a) with respect to $q$,

$$U\Theta_p\Theta_{pq} + \Theta_q\Theta_{qq} = 0.$$

Eliminating $\Theta_{qq}$ from (28b) with the use of the above, dividing throughout by $[U\Theta_q]$, expressing $U$ and $W$ in terms of $\xi$ with the help of (27), and integrating,

$$\frac{1}{(1+\xi^2)^2}\cdot\frac{\Theta_p}{\Theta_q} = \epsilon(q),$$

where $\epsilon(q)$ is an unspecified function of $q$. This readily gives

$$\Theta = \Theta(\lambda),$$

where

$$\lambda = m + n,$$

$$m = \int \frac{dq}{\epsilon},$$

$$n = \int (1+\xi^2)^2\, dp.$$

Using the above relations one gets from (28a)

$$n_p^2\left(\frac{U}{V}\right) + \frac{m_q^2}{V} = \frac{1}{\Theta_\lambda^2}.$$

Differentiating the above expression separately with respect to $n$ and $m$, respectively, and comparing the results, one gets

$$V\left(\frac{U}{V}n_p^2\right)_n + V\left(\frac{1}{V}\right)_n m_q^2 = (m_q^2)_m.$$

Differentiating this successively with respect to $n$ and $m$, one can conclude that

$$[V(1/V)_n]_n (m_q^2)_m = 0.$$

Hence,

$$V(1/V)_n = \text{const},$$

or,

$$m_q = \text{const}.$$

At first let us take, $m_q = \text{const}$. Then, differentiating $[n_p^2(U/V) + m_q^2/V] = 1/\Theta_\lambda^2$ with respect to $m$ and using $m_q = \text{const}$, one gets

$$(1/\Theta_\lambda^2)_\lambda = 0,$$

which gives $\Theta_\lambda = \text{const}$. Hence, $\Theta_q = \Theta_\lambda \cdot \lambda_q = \Theta_\lambda \cdot m_q = \text{const}$. Thus, in the case $\Theta_p \neq 0$, $\Theta_q \neq 0$, we immediately find that again the $\Theta_q = \text{const}$ case is satisfied.

So, let us proceed to find the solution in the general case when

$$\Theta_q = \text{const}. \tag{29}$$

Using $\Theta_q = \text{const}$ in (28b) and after integration once, one gets

$$\Theta_p = E(1+\xi^2)^2, \tag{30}$$

where $E$ is a const. Generalizing (29) and (30), one gets

$$\Theta = E\int (1+\xi^2)\, dp + Fq + G, \tag{31}$$

where $F$ and $G$ are also constants. The use of (25) reduces (31) to

$$\Theta = E\int \frac{(1+\xi^2)^2}{\xi^2}\, d\sigma + \frac{F}{f_\pi}\tan^{-1}\left(\frac{\phi}{f_\pi}\right) + G. \tag{32}$$

Using (31) in (28a), expressing $U$ and $V$ with the help of (27) and rearranging, one can get

$$\xi_{\sigma\sigma} - [4\xi/(1+\xi^2)](\xi_\sigma^2)$$
$$= E^2(1+\xi^2)^4\xi^{-3} + \xi(1+\xi^2)^2(F^2 - f_\pi^2),$$

which with the use of $\xi_{\sigma\sigma} = \frac{1}{2}(\xi_\sigma^2)_\xi$ can easily be integrated to give

$$\sigma = \frac{1}{2}\int \frac{dZ}{[HZ(1+Z)^4 - E^2(1+Z)^4 - (F^2 - f_\pi^2)(1+Z)^3 Z]^{1/2}} + \text{const}, \tag{33}$$

where $H = \text{const}$ and $Z = \xi^2$. With the use of (20e), Eqs. (19a) and (19b) may be reduced to

$$\phi_1/(f_\pi^2 + \phi^2) = (1+\xi^2)\sigma_2$$

and

$$\phi_2/(f_\pi^2 + \phi^2) = -(1+\xi^2)\sigma_1.$$

One may define

$$\int (1+\xi^2)\, d\sigma = r. \tag{34}$$

Then using (25b) and (34), the above two relations may be reduced to $q_1 = r_2$ and $q_2 = -r_1$, which indicate that $q$ and $r$ are mutually conjugate solutions of Laplace's equation.

Further, (25b) may be integrated to give

$$\phi = f_\pi \tan(qf_\pi + Kf_\pi), \tag{35}$$

where $K$ is a constant and $q$ is a solution of Laplace's equation. Thus, one may conclude that (35) gives the solution of $\phi$ in terms of $q$, where $q$ is obtained as a solution of Laplace's equation $q_{11} + q_{22} = 0$. Expressing $\sigma$ in terms of $\xi$ from (33), then $r$ in terms of $\xi$ from (34), and remembering that $r$ is obtained as a solution of Laplace's equation $r_{11} + r_{22} = 0$, $\xi$ becomes known. With $\phi$ and $\xi$ known, one gets immediately $\beta$ and $\Theta$ from (20e) and (32), respectively. With all these being known, solutions of $\psi$ and $\chi$ are readily available through Eqs. (20b) and (20c), respectively. These completely solve the unknown quantities of Eq. (1). That the other possibility of the situation $\Theta_p \neq 0$, $\Theta_q \neq 0$, i.e., $V(1/V)_n = \text{const}$ does not give rise to any new case will be shown in Appendix B.

## 3. CONCLUSION

In our case the solutions of our basic equations given by (3), can be classified under two broad cases, depending on the assumptions being made.

*Case I*: Here it was only assumed that $\beta$ is a function of the single variable $\phi$. The entire problem can be solved exactly if the solution of a single-variable second-order differential equation could be performed [namely, Eq. (17)]. However, in two special cases, the complete solution could be presented in very simple forms.

*Case II*: Here it was assumed that $\psi^2 + \chi^2 = (f_\pi^2 + \phi^2)\xi^2(\sigma)$, where $\sigma$ is a new variable. Then the problem can be solved completely with $\sigma$ being expressed in terms of $\xi$ in the form of an integral equation. However, one can easily find the solution of $\sigma$ in terms of the usual variables like $x^1$, $x^2$ if the exact relation of $\xi$ in terms of $\sigma$ could be established. This is possible in principle.

It may be noted that, due to the absence of $x^3$ and $x^4$ in the reduced form (3) of (1), all the constants involved in the solutions are functions of $(x^3 - x^4)$.

## APPENDIX A

With

$$\delta(1/\delta)_u = \text{const} = L \quad \text{(say)}, \tag{A1}$$

it is evident from

$$\delta\left(\frac{u_X^2}{\delta}\right)_u + \delta\left(\frac{1}{\delta}\right)_u v_Y^2 = (v_Y^2)_v$$

that

$$\delta\left(\frac{u_X^2}{\delta}\right)_u = \text{const} = M \quad \text{(say)} \tag{A2}$$

and

$$(v_Y^2)_v = Lv_Y^2 + M. \tag{A3}$$

From (A1), we get

$$\delta = Ne^{-Lu}, \tag{A4}$$

where $N = \text{const} \neq 0$.

In the following it will be shown that the above equations are not satisfied simultaneously and hence $\delta(1/\delta)_u = \text{const}$ is not permissible.

### Case (i): $L \neq 0$

Expanding (A2) and then using (A1), one gets

$$(u_X^2)_u = M - L(u_X^2).$$

Integrating the above expression and with the help of $u = \int (e^\beta/\alpha^2)\, dX$, one finally gets

$$e^{2\beta}/\alpha^4 = (1/P)(Q - e^{-Lu}), \tag{A5}$$

where $P = \text{const} \neq 0$, $Q = \text{const}$. Using (7) and $u = \int (e^\beta/\alpha^2)\, dX$, we have

$$\phi_u = \phi_X X_u = \alpha^2. \tag{A6}$$

Using (9b), we have $e^\beta = f_\pi^2 + \phi^2 + \alpha^2$. Eliminating $e^\beta$ from the above with the use of (A5), one gets

$$\alpha^2 = \frac{\phi^2 + f_\pi^2}{[(Q - e^{-Lu})/P]^{1/2} - 1}. \tag{A7}$$

Eliminating $\alpha^2$ from (A7) with the use of (A6) and then on integration, one gets

$$\phi = f_\pi \tan\left\{ f_\pi \int \frac{du}{[(Q - e^{-Lu})/P]^{1/2} - 1} - Rf_\pi \right\}, \tag{A8}$$

where $R = \text{const}$. Using (A8) in (A7)

$$\frac{1}{\alpha} = \frac{1}{f_\pi}\left\{\left[\frac{(Q - e^{-Lu})}{P}\right]^{1/2} - 1\right\}^{1/2} \times \cos\left\{ f_\pi \int \frac{du}{[(Q - e^{-Lu})/P]^{1/2} - 1} - Rf_\pi \right\}. \tag{A9}$$

Further, considering (A4) and (13b), one gets

$$\frac{\alpha_{XX} - \alpha_X \beta_X}{\alpha} = Ne^{-Lu},$$

which may be written as

$$\frac{e^\beta (e^{-\beta}\alpha_X)_X}{\alpha} = Ne^{-Lu}.$$

With the change of variable from $X$ to $u$ with the help of $u = \int (e^\beta/\alpha^2)\, dX$, the above is reduced to

$$\frac{e^{2\beta}}{\alpha^4}\left(\frac{1}{\alpha}\right)_{uu} + \frac{1}{\alpha} Ne^{-Lu} = 0,$$

which may be rewritten with the use of (A5) as

$$(Q - e^{-Lu})(1/\alpha)_{uu} + (1/\alpha)PNe^{-Lu} = 0. \tag{A10}$$

The integration in (A9) may be done easily. Then the elimination of $1/\alpha$ from (A10) with the help of (A9) indicates as if the tangent of an angle is expressible in terms of a rational function in amalgamation with roots. This is not permissible in a physical situation.

### Case (ii): $L = 0$

From (A3) and (A4), respectively, one can get

$$(v_Y^2)_v = M, \tag{A11}$$

$$\delta = N. \tag{A12}$$

Expanding (A2) and then using (A12), one gets $(u_X^2)_u = M$. On integration and then with the use of $u = \int (e^\beta/\alpha^2)\, dX$ the above reduces to

$$e^{2\beta}/\alpha^4 = Mu + S, \tag{A13}$$

where $S = \text{const}$. As in (A6),

$$\phi_u = \phi_X X_u = \alpha^2. \tag{A14}$$

Using (9b), we have $e^\beta = f_\pi^2 + \phi^2 + \alpha^2$. Eliminating $e^\beta$ from the above with the use of (A13), one gets

$$\alpha^2 = (\phi^2 + f_\pi^2)/[(Mu + S)^{1/2} - 1]. \tag{A15}$$

Eliminating $\alpha^2$ from (A15) with the use of (A14) and then on integration, one gets

$$\phi = f_\pi \tan\left\{ f_\pi \int \frac{du}{(Mu + S)^{1/2} - 1} - Jf_\pi \right\}, \tag{A16}$$

where $J = \text{const}$. Using (A15) and (A16),

$$\frac{1}{\alpha} = \frac{1}{f_\pi}[(Mu + S)^{1/2} - 1]^{1/2} \times \cos\left\{ f_\pi \int \frac{du}{(Mu + S)^{1/2} - 1} - Jf_\pi \right\}. \tag{A17}$$

Further, considering (A12) and (13b),

$$(\alpha_{XX} - \alpha_X \beta_X)/\alpha = N,$$

which may be reduced as in case (i) to

$$\frac{e^{2\beta}}{\alpha^4}\left(\frac{1}{\alpha}\right)_{uu} + \frac{N}{\alpha} = 0.$$

This may further be rewritten with the use of (A13) as

$$(Mu + S)(1/\alpha)_{uu} + N/\alpha = 0. \quad \text{(A18)}$$

The integration of (A17) may be done easily. The elimination of $1/\alpha$ from (A18) with the help of (A17) indicates if the tangent of an angle is expressible in terms of a rational function in amalgamation with roots. As in the previous case, this is not permissible.

## APPENDIX B

With

$$V(1/V)_n = \text{const} = \Gamma \,(\text{say}), \quad \text{(B1)}$$

it becomes evident from

$$V\left(\frac{U}{V}n_p^2\right)_n + V\left(\frac{1}{V}\right)_n m_q^2 = (m_q^2)_m$$

that

$$V\left(\frac{U}{V}n_p^2\right)_n = \text{const} = \Sigma \quad (\text{say}). \quad \text{(B2)}$$

Expanding (B2) and then using (B1), (27a), (25a), and

$$n = \int (1 + \xi^2)^2 \, dp,$$

one gets

$$\xi_\sigma = \tfrac{1}{4}[(\Gamma - \Sigma)\xi^5 + (3\Gamma - \Sigma)\xi^3 + 3\Gamma\xi + \Gamma\xi^{-1}]. \quad \text{(B3)}$$

When $\Gamma$ and $\Sigma$ are simultaneously zero, (B3) leads to $\xi = \text{const}$. When $\Gamma$ and $\Sigma$ are not simultaneously zero one can separate the situation into two cases depending on whether $\Gamma \neq 0$ or $\Gamma = 0$.

*Case (i)*: $\Gamma \neq 0$

Dividing (B2) by (B1), integrating, using (27), and $n = \int (1 + \xi^2)^2 \, dp$ one gets

$$\Gamma \cdot d \cdot \xi^3 (1 + \xi^2)\xi_{\sigma\sigma} - 4\Gamma \cdot d \cdot \xi^4 (\xi_\sigma)^2 + (\Sigma + \Gamma \cdot d \cdot f_\lambda^2)\xi^4 (1 + \xi^2)^3 - \Gamma (1 + \xi^2)^5 = 0. \quad \text{(B4)}$$

Using (B3) in (B4) one arrives at an equation in $\xi$ which is satisfied for discrete values of $\xi$ only.

*Case (ii)*: $\Gamma = 0$

From (B1) one can now write

$$V = \text{const} = k \,(\text{say}) \neq 0. \quad \text{(B5)}$$

Using (27b) one can reduce (B5) to

$$(1 + \xi^2)\xi_{\sigma\sigma} - 4\xi (\xi_\sigma)^2 + \xi (1 + \xi^2)^3 (f_\lambda^2 - k) = 0. \quad \text{(B6)}$$

Using (B3), with $\Gamma = 0$, in (B6) one arrives at an equation in $\xi$ which is satisfied for discrete values of $\xi$ only.

Thus, for any value of $\Gamma$ and $\Sigma$ one gets $\xi = \text{const}$. It is easy to check from (20e) that $\xi = \text{const}$ represents a special situation of case I [$\beta = \beta(\phi)$] and hence need not be considered separately.


[1] J. M. Charap, J. Phys. A: Math, Nucl. Gen. **6**, 987 (1973).
[2] J. M. Charap, J. Phys. A: Math. Gen. **9**, 1331 (1976).
[3] D. Ray, J. Phys. A: Math. Gen. **11**, (5), 995 (1978).
[4] J. Corones, J. Math. Phys. **19**, 2431 (1978).
[5] D. Ray, J. Math. Phys. **23**, 2155 (1982).
[6] S. Trimper, Phys. Lett. A **70**, 114 (1979).
[7] D. Ray, Phys. Lett. A **76**, 274 (1980).
[8] V. L. Ginzburg and L. V. Pitaevski, Zh. Eksp. Theor. Fiz. **34**, 1240 (1958). [Sov. Phys. JETP **7**, 858 (1958)].
[9] L. V. Kinknade and Yu. G. Mamaladze, Phys. Lett. A **72**, 351 (1979).
[10] P. K. Chanda and D. Ray, J. Math. Phys. **24**, 1923 (1983).