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$SO(2,1)$, Supersymmetry and D-Dimensional Radial Schrödinger Equation

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We point out a connection between the underlying function of the representations of the $SO(2,1)$ algebra and the superpotential of SUSY quantum mechanics. We also write down the expressions of the generators of $SO(2,1)$ in D-dimensions.

In a recent paper Barut, Beker and Rador[1] have explored a class of realization of $SO(2,1)$ algebra within the framework of dynamical $O(4,2)$ to study the radial Schrödinger equation

$$[-(d^2/dr^2+(2/r)d/dr)+V_E(r)-E_{n,l}]\chi_{n,l}(r)=0\,, \tag{1}$$

where $V_E(r)=l(l+1)/r^2+V(r)$ represents the effective potential.

Since it is well-accepted [Refs. 2), 3) and references therein] that superpotential resides in the radial equation in an obvious manner, a natural question arises whether a link could be set up connecting supersymmetry (SUSY) with $SO(2,1)$. The purpose of this paper is to establish this by expressing the superpotential in terms of the underlying function of $SO(2,1)$. Our results turn out to be generalizable to systems of D-dimensions also.

The $SO(2,1)$ algebra defined by the generators (Γ_0, Γ_4, T) satisfies the commutation relations

$$[\Gamma_4, T]=-i\Gamma_0\,, \quad [T, \Gamma_0]=i\Gamma_4\,, \quad [\Gamma_0, \Gamma_4]=iT\,. \tag{2}$$

The Casimir operator C^2 is given by $\Gamma_0{}^2-\Gamma_4{}^2-T^2$. In terms of an arbitrary function $G(r)$ the algebra (2) admits the following representations:

$$\Gamma_0=(G/G'^2)K^2+C^2/G-GG'''/2G'^3+3GG''^2/4G'^4+G/4\,,$$

$$\Gamma_4=(G/G'^2)K^2+C^2/G-GG'''/2G'^3+3GG''^2/4G'^4-G/4\,,$$

$$T=(G/G')K-iGG''/2G'^2\,, \tag{3}$$

where $K=-i(d/dr+1/r)$.

Taking the eigenvalue of C^2 to be $(1/2)\gamma((1/2)\gamma-1)$ and the spectrum Γ_0 given by $\nu+(1/2)\gamma$, $\nu=0, 1, 2\cdots$, Barut et al.[1] have compared Γ_0 with (1) to arrive at the following relation:

$$G'''/2G'-3G''^2/4G'^2-\frac{1}{2}\gamma\left(\frac{1}{2}\gamma-1\right)G'^2/G^2+\left(\nu+\frac{\gamma}{2}\right)G'^2/G-G'^2/4$$

$$=-V_E(r)+E_{n,l}\,, \tag{4}$$

where $\hbar=2m=1$ has been set.

We wish to point out that the above relation is manifestly supersymmetric if the superpotential is chosen as

$$W(r)=(G'-\gamma G'/G+G''/G')/2\,. \tag{5}$$

Indeed for such a choice of $W(r)$ we find

$$V_E(r)-E_0=W^2-W'\,, \tag{6}$$

where E_0 stands for the ground state energy ($\nu=0$). As is well-known with a superpotential at hand one can define the supercharges $Q=(p-iW)\sigma_+$, $Q^+=(p+iW)\sigma_-$. One is thus led to a supersymmetric Hamiltonian $H_S=\{Q, Q^+\}/2$ whose structure induces a pair of partner Hamiltonians $H_\pm=-d^2/dr^2\pm(W^2-W')$. It is clear from (6) that solvability of W depends on the functional form of the potential and involves solving an equation of the Ricatti type.

The advantage with (5) is that given $G(r)$ the superpotential $W(r)$ can be immediately determined. The converse problem, however, may not be easy to solve since it may not always be possible to obtain $W(r)$ in a closed form from the knowledge of $G(r)$. Further not all $G(r)$ will produce a constant on the lhs of (4) to match with the energy term on the rhs (for example, $G(r)=\log r$). One should note that having obtained a $W(r)$ care is to be taken to ensure that the ground state wave function is suitably normalizable. It may be mentioned that in the unbroken version of SUSY the ground state is non-degenerate and may be chosen to be associated with H_+.

We now turn to some applications of our equation (5). The Coulomb and harmonic oscillator potential are the potentials which exibit accidental degeneracies and solve the radial Schrödinger equation. From SUSY quantum mechanics we know[2] that the corresponding superpotentials are $W=1/2(l+1)-(l+1)/r$ (Coulomb) and $W=r/2-(l+1)/r$ (isotropic oscillator). Inserting these forms for W in our equation (5) we find on integrating $G=\lambda r$ (Coulomb) and $G=\lambda r^2/2$ (isotropic oscillator) where λ is a constant. These agree with the G's used in Ref. 1) as potential applications of Eq. (4).

More generally we may take the following representation[4] for the superpotential of the type:

$$W=ar+1/2r-ar\gamma/[ar^2+c]\,. \tag{7}$$

It implies $G(r)=ar^2+c$ and corresponds to the Coulomb and cutoff Coulomb potential[5] perturbed by a polynomial in r.

The extension to D-dimensions (D arbitrary) of the representations may be obtained by taking advantage of the following proposition.

Proposition For an arbitrary function $G(r)$ the commutation relations

$$[K, G/G']=-i(1-GG''/G'^2)\,,$$

$$[K, G''/G']=-i(G'''/G'-G''^2/G'^2)$$

hold, where $K=-i[d/dr+(D-1)/2r]$.

Proof These are easy to verify by direct evaluation of the commutators.

As a result we can write down the following representations for the generators in D-dimensions:

$$\Gamma_0=(G/G'^2)K^2+C^2/G-GG''/2G'^3+3GG''^2/4G'^4+G/4\,,$$

$$\Gamma_4=(G/G'^2)K^2+C^2/G-GG''/2G'^3+3GG''^2/4G'^4-G/4\,,$$

$$T=(G/G')K-iGG''/2G'^2\,,$$

where $K^2=-(d^2/dr^2+\{(D-1)/r\}d/dr+(D-1)(D-3)/4r^2)$. It may be checked that these representations satisfy the $SO(2,1)$ algebra.

Noting that the radial Schrödinger equation in D-dimension reads as

$$[-(d^2/dr^2+\{(D-1)/r\}d/dr)+l(l+D-2)/r^2+V(r)-E_{n,l}]=0\,, \tag{8}$$

the superpotential in D-dimension may be worked out readily. We have

$$G'''/2G'-3G''^2/4G'^2-(\gamma/2)((\gamma/2)-1)G'^2/G^2+(\nu+(\gamma/2))G'^2/G-G'^2/4$$
$$=-(D-1)(D-3)/4r^2-l(l+d-2)/r^2-V(r)+E_{n,l}\,. \tag{9}$$

For the Coulomb and the isotropic oscillator problems $W(r)$ assumes the forms

$$W_{\text{Col}}=1/(2l+D-1)-(2l+D-1)/2r\,,\quad E_{n,l}=-[2n+2l+D-1]^{-2}\,, \tag{10a}$$

$$W_{\text{osc}}=r/2-(2l+D-1)/2r\,,\quad E_{n,l}=[2n+l+D/2] \tag{10b}$$

for $G(r)=\lambda r$ (Coulomb, $\gamma=2l+D-1$) and $G(r)=\lambda r^2/2$ (isotropic oscillator, $\gamma=l+D/2$).

To conclude we have examined the possibility of expressing the underlying function $G(r)$ of the $SO(2,1)$ algebra in terms of the superpotential within the framework of SUSY quantum mechanics. We have applied our results to some physical problems. We have also sought generalizations and, to this end, have written down the repesentations of the $SO(2,1)$ algebra in D-dimensions. As applications we have considered the Coulomb and isotropic oscillator problems for which we have determined the superpotentials for specific choices of $G(r)$.

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