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Signless Laplacian spectrum of power graphs of finite cyclic groups

Subarsha Banerjee[a,*,1], Avishek Adhikari[b]

[a] Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata 700019, India
[b] Department of Mathematics, Presidency University, India

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Abstract

In this paper, we have studied Signless Laplacian spectrum of the power graph of finite cyclic groups. We have showed that $n-2$ is an eigen value of Signless Laplacian of the power graph of $\mathbb{Z}_n, n \geq 2$ with multiplicity at least $\phi(n)$. In particular, using the theory of Equitable Partitions, we have completely determined the Signless Laplacian spectrum of power graph of $\mathbb{Z}_n$ for $n = pq$ where p, q are distinct primes.

Keywords: Finite groups; Power graphs; Signless Laplacian eigenvalues

1. Introduction

The directed power graph of a semigroup S was defined by Kelarav and Quinn [1] as the digraph $\mathcal{P}(S)$ with vertex set S, in which there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m.

Motivated by this, Chakrabarty et al. [2] defined the (undirected) power graph $\mathcal{P}(G)$ of a group G as an undirected graph whose vertex set is G and two vertices $u, v \in G$ are adjacent if and only if $u \neq v$ and $u = v^m$ or $v = u^k$ for positive integers m, k. In particular the power graph $\mathcal{P}(G)$ is connected for any finite group G. In [2], it was shown that the power graph $\mathcal{P}(G)$ of a finite group G is complete if and only if G is a cyclic group of order 1 or p^m for some prime p and for some positive integer m. For a complete survey on power graphs readers may refer to [3].

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *Adjacency* matrix of G denoted by $A(G) = (a_{ij})$ is given by $a_{ij} = 1$ if two vertices i, j are adjacent in G and 0 otherwise. The *Laplacian* matrix $L(G)$ of G is given by $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees.

The *Signless Laplacian* matrix $Q(G)$ of G is given by $Q(G) = D(G) + A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees. Clearly $Q(G)$ is a real symmetric matrix and

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* Corresponding author.
E-mail addresses: subarshabnrj@gmail.com (S. Banerjee), avishek.adh@gmail.com (A. Adhikari).
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hence all its eigen values are real. Also $Q(G)$ is a positive semidefinite matrix and hence all its eigen values are non-negative. More information on Signless Laplacian can be found in [4–6].

The paper is organized as follows:

In Section 3 we have studied the Signless Laplacian spectrum of finite cyclic groups. Sections 4 and 5 deal with the Signless Laplacian spectrum of the power graph of the finite cyclic group $\mathbb{Z}_{pq}$. In these sections we have completely determined the spectrum of $Q(\mathcal{P}(\mathbb{Z}_n))$ for $n = pq$ where p, q are primes with $p < q$.

2. Notation and prerequisites

By $J_{m\times n}$ we mean the matrix of order $m \times n$ whose entries are all one. We denote any vector as $\underline{\mathbf{v}}$. $N(A)$ denotes the Null Space of the matrix A i.e. $N(A) = \{\underline{\mathbf{v}} : A\underline{\mathbf{v}} = \underline{\mathbf{0}}\}$. By $\dim N(A)$ we mean the *Dimension* of the Null Space of A. The *geometric multiplicity* of an eigen value λ of A equals the dimension of $N(A - \lambda I)$. *Here by multiplicity of an eigen value we refer to its algebraic multiplicity i.e. the number of times it occurs as a root of the characteristic polynomial.* Here $\phi(n)$ denotes the number of positive integers prime to n and less than n. For basic definitions, theorems and results on Linear Algebra and Matrices readers are referred to [7] and [8]. We will use the following lemma in our theorems quite often whose proof can be found in [9].

Lemma 2.1. *If J denotes the square matrix of order n with all entries equal to one and I denotes the identity matrix of order n then the eigen values of $aI + bJ$ are a with multiplicity $n - 1$ and $a + nb$ with multiplicity* 1.

3. Signless Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_n)$

In this section we discuss about the spectrum of Signless Laplacian of Power Graph of $\mathbb{Z}_n$ for various $n \in \mathbb{N}$.

We first find out the eigen values of the matrix $Q(\mathcal{P}(\mathbb{Z}_n))$ for any integer of the form p^m where p is a prime and m is a positive integer. Then we show that $n-2$ is an eigen value of $Q(\mathcal{P}(\mathbb{Z}_n))$ for any integer $n \neq 1, p^m$ with multiplicity at least $\phi(n)$.

Theorem 3.1. *If $n = p^m$ for some prime p and for some positive integer m then the eigen values of $Q(\mathcal{P}(\mathbb{Z}_n))$ are $2(n - 1)$ with multiplicity 1 and $n - 2$ with multiplicity $n - 1$.*

Proof. If $n = p^m$ for some prime p and for some positive integer m then $\mathcal{P}(\mathbb{Z}_n)$ is a complete graph (refer [2] for a proof) and hence $Q(\mathcal{P}(\mathbb{Z}_n))$ is of the form

$$Q = \begin{bmatrix} n-1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & n-1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & n-1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots & n-1 \end{bmatrix}.$$

Hence $Q = (n - 2)I + J$ where J denotes the square matrix of order n with all entries equal to one and I denotes the identity matrix of order n.

It follows by Lemma 2.1 the eigen values of Q are $2(n - 1)$ with multiplicity 1 and $n - 2$ with multiplicity $n - 1$. □

Theorem 3.2. *If $n \neq 1, p^m$ for any prime p and any positive integer m, then $n-2$ is an eigen value of $Q(\mathcal{P}(\mathbb{Z}_n))$ with multiplicity at least $\phi(n)$.*

Proof. The rows and columns of the matrix $Q(\mathcal{P}(\mathbb{Z}_n))$ are indexed in the following manner: Let us denote the elements of $\mathbb{Z}_n$ as $[0], [1], [2], \dots, [n-2], [n-1]$.

For indexing the rows and columns of the matrix Q we follow the following algorithmic process:

We start with the element $[0]$, we then list the elements which are generators in $\mathbb{Z}_n$, clearly we have $\phi(n)$ many generators in $\mathbb{Z}_n$, we then consider the non-generator elements and list them in the following way:

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If $[a]$ is any non-generator element of $\mathbb{Z}_n$ we first list the elements generated by $[a]$ like $[a], [2a], [3a], \ldots, [ka]$ such that $(k+1)[a] = [0]$.

We then consider the second non-generator element of $\mathbb{Z}_n$ and repeat the previous process. The process gets repeated until all the elements of $\mathbb{Z}_n$ are exhausted.

If $n \neq 1, p^m$ for any prime p and any positive integer m then $Q(\mathcal{P}(\mathbb{Z}_n))$ has the following form:

$$Q = \left[\begin{array}{c|c} ((n-2)I + J)_{l\times l} & J_{l\times n-l} \\ \hline J^T_{(n-l)\times l} & A_{(n-l)\times(n-l)} \end{array}\right].$$

where $l = \phi(n) + 1$ and $J_{m\times n}$ denotes a matrix of order $m \times n$ with all entries equal to 1. Then A is a matrix of order $(n-l) \times (n-l)$ whose entries depend on n.

Now consider $Q - (n-2)I$ which has the following form:

$$Q - (n-2)I = \left[\begin{array}{c|c} J_{l\times l} & J_{l\times n-l} \\ \hline J^T_{(n-l)\times l} & (A-(n-2)I)_{(n-l)\times(n-l)} \end{array}\right].$$

We find that $Q - (n-2)I$ has l identical rows as $[\underbrace{1, 1, 1, \ldots, 1}_{n \text{ times}}]$.

Applying the following elementary row operations on $Q - (n-2)I$
$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, \ldots, R_l \to R_l - R_1$ we find that
$Q - (n-2)I$ is *row-equivalent* to the following matrix

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \hline & & & & & \\ & J^T_{(n-l)\times l} & & & (A-(n-2)I)_{(n-l)\times(n-l)} & \end{array}\right].$$

Since $Q-(n-2)I$ has $\phi(n)$ zero rows, so $\text{Rank}(Q-(n-2)I) \leq n-\phi(n)$ and hence dimension of the null space of $(Q-(n-2)I) \geq \phi(n)$. The geometric multiplicity of 0 as an eigen value of $Q-(n-2)I$ equals $\dim N(Q-(n-2)I)$ which is at least $\phi(n)$. Since $Q-(n-2)I$ is symmetric so the geometric multiplicity and algebraic multiplicity of each of its eigen value is same and hence 0 is an eigen value of $Q-(n-2)I$ with multiplicity at least $\phi(n)$. Thus $n-2$ is an eigen value of Q with multiplicity at least $\phi(n)$. □

4. Signless Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_{pq})$

In this section we particularly study the Signless Laplacian spectrum of Power Graph of $\mathbb{Z}_{pq}$. We show that the spectrum of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ contains $n-2$ as an eigen value with multiplicity exactly $\phi(n)+1$. We also figure out some other eigen values in the spectrum of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$.

Lemma 4.1. *Consider the sets* $\{[p], [2p], \ldots, [(q-1)p]\}$ *and* $\{[q], [2q], \ldots, [(p-1)q]\}$ *in the group* $(\mathbb{Z}_{pq}, +)$. *There does not exist positive integers* r, s *such that* $r[a] = s[b] (\neq [0])$ *where* $[a] \in \{[p], [2p], \ldots, [(q-1)p]\}$ *and* $[b] \in \{[q], [2q], \ldots, [(p-1)q]\}$.

Proof. Since $[a] \in \{[p], [2p], \ldots, [(q-1)p]\}$ we have $[a] = [lp]$ where $1 \leq l \leq q-1$ and $[b] = [kq]$ where $1 \leq k \leq p-1$.

Assume that there exist positive integers r, s such that $r[a] = s[b]$ which in turn implies that $[rlp] = [skq]$.

From $[rlp] = [skq]$, we get that $rlp - skq \equiv 0 \pmod{pq}$. Said differently, we have $rlp - skq = pqt$. This means $skq \equiv 0 \pmod{p}$. Since $p \neq q$, we get $s \equiv 0 \pmod{p}$ (because $1 \leq k < p$), likewise $r \equiv 0 \pmod{q}$. But then both $r[a]$ and $s[b]$ are the zero class $[0]_{pq}$, a contradiction. □

Theorem 4.2. *If* $n = pq$ *for some primes* p, q *with* $p < q$ *then* $n-2$ *is an eigen value of* $Q(\mathcal{P}(\mathbb{Z}_n))$ *with multiplicity exactly* $\phi(n)+1$.

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Proof. The rows and columns of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ are indexed in the same way as in Theorem 3.2. So $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ will take the following form:

$$Q = \left[\begin{array}{c|c} ((n-2)I+J)_{l\times l} & J_{l\times n-l} \\ \hline J^T_{(n-l)\times l} & A_{(n-l)\times(n-l)} \end{array}\right].$$

The square matrix A of order $n-l$ contains the rows and columns which are indexed by the non-generator elements of $\mathbb{Z}_{pq}$. The non-generator elements of $\mathbb{Z}_{pq}$ are $[p], [2p], [3p], \ldots, [(q-1)p]$ and $[q], [2q], [3q], \ldots, [(p-1)q]$.

Now using Lemma 4.1 we find the matrix A will take the following form:

$$A = \left[\begin{array}{c|c} C_{(q-1)\times(q-1)} & 0 \\ \hline 0 & D_{(p-1)\times(p-1)} \end{array}\right],$$

where,

$$C = \begin{bmatrix} p(q-1) & 1 & 1 & \ldots & 1 \\ 1 & p(q-1) & 1 & \ldots & 1 \\ 1 & 1 & p(q-1) & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & p(q-1) \end{bmatrix}$$

and

$$D = \begin{bmatrix} q(p-1) & 1 & 1 & \ldots & 1 \\ 1 & q(p-1) & 1 & \ldots & 1 \\ 1 & 1 & q(p-1) & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & q(p-1) \end{bmatrix}.$$

Clearly $C = (p(q-1)-1)I+J$ where both I and J are square matrices of order $q-1$ and $D = (q(p-1)-1)I+J$ where both I and J are square matrices of order $p-1$.

The eigen values of C by Lemma 2.1 are $p(q-1)-1$ with multiplicity $q-2$ and $pq-p+q-2$ with multiplicity 1. Similarly the eigen values of D by Lemma 2.1 are $q(p-1)-1$ with multiplicity $p-2$ and $pq-q+p-2$ with multiplicity 1.

Since all eigen values of A are non zero so

$$\text{Rank}(A) = p+q-2. \tag{4.1}$$

Now as given in Theorem 3.2, after elementary row and column operations, $Q-(n-2)I$ takes the following form:

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \hline & & & & & \\ & J^T_{(n-l)\times l} & & & & (A-(n-2)I)_{(n-l)\times(n-l)} \end{array}\right].$$

For $n = pq$, the matrix $(A-(n-2)I)_{(n-l)\times(n-l)}$ takes the following form:

$$\left[\begin{array}{c|c} (C-(pq-2)I)_{(q-1)\times(q-1)} & 0 \\ \hline 0 & (D-(pq-2)I)_{(p-1)\times(p-1)} \end{array}\right]$$

where,

$$C-(pq-2)I=\begin{bmatrix} 2-p & 1 & 1 & \dots & 1 \\ 1 & 2-p & 1 & \dots & 1 \\ 1 & 1 & 2-p & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2-p \end{bmatrix}$$

and

$$D-(pq-2)I=\begin{bmatrix} 2-q & 1 & 1 & \dots & 1 \\ 1 & 2-q & 1 & \dots & 1 \\ 1 & 1 & 2-q & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2-q \end{bmatrix}.$$

Claim. Rank $(Q-(n-2)I)=n-\phi(n)-1$.

Proof. We already know from Theorem 3.2 that for any $n\in\mathbb{N}$,

Rank $(Q-(n-2)I)\le n-\phi(n)$.

In order to show that Rank $(Q-(n-2)I)=n-\phi(n)-1$ we consider the following square matrix P of order $n-\phi(n)$ which is a submatrix of $Q-(n-2)I$, where,

$$P=\left[\begin{array}{c|c|c} 1 & 1 \;\dots\; 1 & 1 \;\dots\; 1 \\ 1 & & \\ \vdots & C & 0 \\ 1 & & \\ \hline 1 & & \\ \vdots & 0 & D \\ 1 & & \end{array}\right]$$

$$=\left[\begin{array}{cccccc|cccccc} 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2-p & 1 & \dots & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 2-p & \dots & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 2-p & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & \dots & 1 & 2-p & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 1 & 0 & 0 & \dots & 0 & 0 & 2-q & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 2-q & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 2-q & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \dots & 0 & 0 & 1 & 1 & 1 & \dots & 2-q & 1 \\ 1 & 0 & \dots & \dots & 0 & 0 & 1 & 1 & 1 & \dots & 1 & 2-q \end{array}\right].$$

Consider the vector

$$\underline{\mathbf{v}}=(q-p,\underbrace{-1,\dots,-1}_{q-1\ \text{times}},\underbrace{1,\dots,1}_{p-1\ \text{times}}) \tag{4.2}$$

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Then we have $P\underline{\mathbf{v}} = \mathbf{0}$.
Since $\underline{\mathbf{v}} \neq \underline{\mathbf{0}}$, Hence

$$\mathrm{Rank}(P) \leq p + q - 2. \tag{4.3}$$

Now P contains A as a submatrix whose rank is exactly $p + q - 2$ (By (4.1)).
So

$$\mathrm{Rank}(P) \geq p + q - 2. \tag{4.4}$$

Combining (4.3) and (4.4) we have

$$\mathrm{Rank}(P) = p + q - 2. \tag{4.5}$$

Since P is obtained by adjoining the vector $[\underbrace{1, 1, 1, \ldots, 1}_{p+q-1\text{times}}]$ as both row and column with the matrix A and $\mathrm{Rank}(P) = p + q - 2$ we find that

$$\mathrm{Rank}(Q - (n-2)I) = p + q - 2 = n - \phi(n) - 1. \quad \square \tag{4.6}$$

Since $\mathrm{Rank}(Q - (n-2)I) = p + q - 2 = n - \phi(n) - 1$, so the geometric multiplicity of 0 as an eigen value of $Q - (n-2)I$ equals $\phi(n) + 1$ and since the matrix is symmetric, the algebraic multiplicity of 0 equals $\phi(n) + 1$ which in turn implies that the multiplicity of $n - 2$ as an eigen value of Q is exactly $\phi(n) + 1$.

Theorem 4.3. *For $n = pq$ where p, q are primes and $p < q$, $p(q-1) - 1$ and $q(p-1) - 1$ belongs to the spectrum of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ with multiplicity $q - 2$ and $p - 2$ respectively.*

Proof. From Theorem 4.2, we know that $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ will take the following form:

$$Q = \left[\begin{array}{c|c} ((n-2)I + J)_{l \times l} & J_{l \times n-l} \\ \hline J^T_{(n-l)\times l} & A_{(n-l)\times(n-l)} \end{array}\right]$$

where,

$$A = \left[\begin{array}{c|c} C_{(q-1)\times(q-1)} & 0 \\ \hline 0 & D_{(p-1)\times(p-1)} \end{array}\right].$$

Also

$$C = \begin{bmatrix} p(q-1) & 1 & 1 & \ldots & 1 \\ 1 & p(q-1) & 1 & \ldots & 1 \\ 1 & 1 & p(q-1) & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & p(q-1) \end{bmatrix}$$

and

$$D = \begin{bmatrix} q(p-1) & 1 & 1 & \ldots & 1 \\ 1 & q(p-1) & 1 & \ldots & 1 \\ 1 & 1 & q(p-1) & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & q(p-1) \end{bmatrix}.$$

Consider the $(q-1) \times 1$ vector

$$\underline{\mathbf{1}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.7}$$

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Since the sum of the entries in each row of C is $p(q-1)+q-2$,
so,

$$C.\underline{\mathbf{1}} = (p(q-1)+q-2)\underline{\mathbf{1}}$$

i.e. $p(q-1)+q-2$ is an eigen value of C corresponding to the eigen vector $\underline{\mathbf{1}}$.

Also since $p(q-1)-1$ is an eigen value of C with multiplicity $q-2$ (Refer Theorem 4.2) hence,

$$C.\underline{\mathbf{v}} = (p(q-1)-1)\underline{\mathbf{v}},$$

where $\underline{\mathbf{v}}$ is an eigen vector of C corresponding to the eigen value $p(q-1)-1$ and $\underline{\mathbf{v}}$ is a $q-1\times 1$ vector.

Since C is a symmetric matrix and the eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal hence we have

$$\underline{\mathbf{v}}.\underline{\mathbf{1}} = 0. \tag{4.8}$$

Now consider the vector $\underline{\mathbf{w}}$ which is an $n\times 1$ vector given by,

$$\underline{\mathbf{w}} = \begin{bmatrix} \mathbf{0}_{l\times 1} \\ \mathbf{v}_{(q-1)\times 1} \\ \mathbf{0}_{(p-1)\times 1} \end{bmatrix}. \tag{4.9}$$

Now using Eq. (4.8),

$$Q.\underline{\mathbf{w}} = \left[\begin{array}{c|c} ((n-2)I+J)_{l\times l} & J_{l\times n-l} \\ \hline J^T_{(n-l)\times l} & A_{(n-l)\times(n-l)} \end{array}\right] \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{v}} \\ \underline{\mathbf{0}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} \\ C.\underline{\mathbf{v}} \\ \underline{\mathbf{0}} \end{bmatrix} = (p(q-1)-1)\underline{\mathbf{w}}.$$

Hence $p(q-1)-1$ is an eigen value of Q.

Also,

$$D.\underline{\mathbf{1}} = (q(p-1)+p-2)\underline{\mathbf{1}}$$

i.e. $q(p-1)+p-2$ is an eigen value of D corresponding to the eigen vector $\underline{\mathbf{1}}$.

Again since $q(p-1)-1$ is an eigen value of D with multiplicity $p-2$ (Refer Theorem 4.2) hence,

$$D.\underline{\mathbf{x}} = (q(p-1)-1)\underline{\mathbf{x}},$$

where $\underline{\mathbf{x}}$ is an eigen vector of D corresponding to the eigen value $q(p-1)-1$ and $\underline{\mathbf{x}}$ is a $p-1\times 1$ vector.

$$\underline{\mathbf{x}}.\underline{\mathbf{1}} = 0. \tag{4.10}$$

Now consider the vector $\underline{\mathbf{y}}$ which is an $n\times 1$ vector given by,

$$\underline{\mathbf{y}} = \begin{bmatrix} \mathbf{0}_{l\times 1} \\ \mathbf{0}_{(q-1)\times 1} \\ \mathbf{x}_{(p-1)\times 1} \end{bmatrix}. \tag{4.11}$$

Now using Eq. (4.10),

$$Q.\underline{\mathbf{y}} = \left[\begin{array}{c|c} ((n-2)I+J)_{l\times l} & J_{l\times n-l} \\ \hline J^T_{(n-l)\times l} & A_{(n-l)\times(n-l)} \end{array}\right] \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{0}} \\ \underline{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{0}} \\ D.\underline{\mathbf{x}} \end{bmatrix} = (q(p-1)-1)\underline{\mathbf{y}}.$$

Hence $q(p-1)-1$ is an eigen value of Q.

Since multiplicity of $p(q-1)-1$ as an eigen value of C is $q-2$ and multiplicity of $q(p-1)-1$ as an eigen value of D is $p-2$ so $p(q-1)-1$ and $q(p-1)-1$ are eigen values of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ with multiplicity $q-2$ and $p-2$ respectively. □

5. Using equitable partition to determine remaining eigen values of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$

In this section we introduce and utilize the idea of equitable partition to find the remaining two eigen values of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$, thus determining the complete spectrum of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$.

5.1. Equitable Partition — Introduction and Definition

Given a graph G, a partition π of $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is an equitable partition if every vertex in V_i has the same number of neighbours in V_j for all $i, j \in \{1, 2, \ldots, k\}$. In other words, for a graph G a partition π of $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is equitable if the number of neighbours in V_j of a vertex u in V_i is a constant say b_{ij} irrespective of $u \in V_i$. Interested readers may refer to [10,11] and [12] for more information on equitable partition.

5.2. Matrix corresponding to an equitable partition π

Now given an equitable partition π of a graph G we can form a matrix $Q_\pi = (q_{ij})$ in the following manner:

$$q_{ij} = \begin{cases} b_{ij}, & i \neq j \\ b_{ii} + \sum_{j=1}^{k} b_{ij}, & i = j, \end{cases} . \tag{5.1}$$

where b_{ij} is the number of neighbours a vertex $v \in V_i$ has in V_j and b_{ii} is the number of neighbours a vertex $v \in V_i$ has in V_i.

We refer the matrix Q_π as the matrix corresponding to the partition π of a graph G.

We prove the following fact regarding Q_π:

Lemma 5.1. *Let Q be the signless Laplacian matrix associated with a graph G and let π be a partition of the graph G. If λ is an eigen value of Q_π then λ is also an eigen value of Q. In other words the spectrum of Q_π is contained in Q.*

Proof. Let $V_1, V_2, \ldots, V_k$ is a partition of the vertex set of G corresponding to π.
Suppose λ is an eigen value of Q_π, then
$Q_\pi \mathbf{v} = \lambda \mathbf{v}$ for some vector $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_k)$.
Now consider the vector $\mathbf{w} = (w_v)_{v \in V}$ where $w_v = v_i$ if $v \in V_i$, $1 \leq i \leq k$
Hence,

$$Q\mathbf{w} = (Q(w_v))_{v \in V} = (Q_\pi v_i)_i = (Q_\pi \mathbf{v}) = \lambda \mathbf{v} = (\lambda v_i)_i = \lambda (w_v)_{v \in V} = \lambda \mathbf{w}.$$

Thus λ is an eigen value of Q corresponding to eigen vector $\mathbf{w}$. □

5.3. Complete spectrum of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$

In this section we use the definition of equitable partitions given in Section 5.1 to find an equitable partition π of $\mathcal{P}(\mathbb{Z}_{pq})$ and then use the construction of matrix shown in Section 5.2 to construct Q_π corresponding to π.
Let $V(G) = V_1 \cup V_2 \cup V_3$ where
V_1 is the set of generators of the group $\mathbb{Z}_{pq}$ together with the zero element [0]. Since $\mathbb{Z}_{pq}$ has $\phi(pq) = (p-1)(q-1)$ generators, so V_1 has $(p-1)(q-1)+1$ elements.
V_2 is the set of elements of the group $\mathbb{Z}_{pq}$ generated by the element $[p]$ i.e. $V_2 = \{[p], [2p], [3p], \ldots, [(q-1)p]\}$. Thus V_2 has $q-1$ elements.
V_3 is the set of elements of the group $\mathbb{Z}_{pq}$ generated by the element $[q]$ i.e. $V_3 = \{[q], [2q], [3q], \ldots, [(p-1)q]\}$. Thus V_3 has $p-1$ elements.
Note that each member of V_1 has exactly $q-1$ neighbours in V_2 and $p-1$ neighbours in V_3. Each member of V_2 has exactly $(p-1)(q-1)+1$ neighbours in V_1 and no neighbours in V_3. Each member of V_3 has exactly $(p-1)(q-1)+1$ neighbours in V_1 and no neighbours in V_2.
Hence the partition $V(G) = V_1 \cup V_2 \cup V_3$ given above is an equitable partition of $\mathcal{P}(\mathbb{Z}_{pq})$. We denote it by π.

Also note that each member of V_1 has $\phi(pq) = (p-1)(q-1)$ neighbours in V_1, each member of V_2 has $q-2$ neighbours in V_2 and each member of V_3 has $p-2$ neighbours in V_3.
From Section 5.2 we find that $Q_\pi(\mathbb{Z}_{pq})$ takes the following form:

$$\begin{bmatrix} 2pq-p-q & q-1 & p-1 \\ pq-p-q+2 & pq+q-p-2 & 0 \\ pq-p-q+2 & 0 & p+pq-q-2 \end{bmatrix}.$$

We observe the following facts regarding $Q_\pi(\mathbb{Z}_{pq})$:

Lemma 5.2. *pq-2 is an eigen value of $Q_\pi(\mathbb{Z}_{pq})$.*

Proof. Consider $Q_\pi(\mathbb{Z}_{pq}) - ((pq-2)I)$

$$\begin{bmatrix} pq-p-q+2 & q-1 & p-1 \\ pq-p-q+2 & -p+q & 0 \\ pq-p-q+2 & 0 & p-q \end{bmatrix}.$$

Applying the elementary row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get the following matrix,

$$\begin{bmatrix} pq-p-q+2 & q-1 & p-1 \\ 0 & 1-p & 1-p \\ 0 & 1-q & 1-q \end{bmatrix}.$$

$\det(Q_\pi(\mathbb{Z}_{pq}) - ((n-2)I)) = 0$.
Hence 0 is an eigen value of $Q_\pi(\mathbb{Z}_{pq}) - ((pq-2)I)$ which implies $pq-2$ is an eigen value of $Q_\pi(\mathbb{Z}_{pq})$. □

Let λ_1, λ_2 be the other two eigen values of $Q_\pi(\mathbb{Z}_{pq})$.

We know that trace of a matrix equals sum of its eigen values and determinant of a matrix equals product of its eigen values. By Lemma 5.2 $Q_\pi(\mathbb{Z}_{pq})$ has $pq-2$ as one of its eigen value hence we have,

$$\lambda_1 + \lambda_2 = 3pq - p - q - 2. \tag{5.2}$$

and

$$\lambda_1\lambda_2 = 2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p - 2q + 4. \tag{5.3}$$

Since λ_1, λ_2 satisfies the following equation:
$x^2 - \text{trace}(Q_\pi(\mathbb{Z}_{pq})) + \det(Q_\pi(\mathbb{Z}_{pq})) = 0$ we have,

$$x^2 - (3pq - p - q - 2)x + (2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p - 2q + 4) = 0. \tag{5.4}$$

Lemma 5.3. *The eigen values λ_1, λ_2 of $(Q_\pi(\mathbb{Z}_{pq}))$ are not contained in $\{pq-2, p(q-1)-1, q(p-1)-1\}$, i.e. $\lambda_i, i \in \{1,2\} \notin \{pq-2, p(q-1)-1, q(p-1)-1\}$.*

Proof. If $\lambda_i, i \in \{1,2\}$ is one of $pq-2$, $p(q-1)-1$, $q(p-1)-1$ then either $pq-2$ or $p(q-1)-1$ or $q(p-1)-1$ must satisfy Eq. (5.4).

We consider the following 3 cases:

Case 1: Checking if $pq-2$ satisfies Eq. (5.4).

$$\begin{aligned} &(pq-2)^2 - (3pq-p-q-2)(pq-2)+ \\ &(2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p + 2pq - 2p - 2q + 4) \\ &= 6pq - p^2q - pq^2 - 4p - 4q + 4 \\ &= -pq(p+q-6) - 4(p+q) + 4. \end{aligned}$$

Here p, q are primes with $p < q$ and hence we consider the following two sub-cases:

Case I: $p = 2, q = 3$
Then $4 - 4(p+q) - pq(p+q-6) = 4 - 4(5) - 6(-1) = -10 \neq 0$.

Case II: $(p, q) \neq (2, 3)$

Note that $(p+q) > 1$ for all primes p, q which implies that
$\{4 - 4(p+q)\} < 0$.
Also for all primes p, q where $(p, q) \neq (2, 3)$
$p + q - 6 > 0 \implies -pq(p+q-6) < 0$.
Hence we get that $\{4 - 4(p+q)\} - pq(p+q-6) < 0$ for all primes $(p, q) \neq (2, 3)$.

Hence $pq - 2$ is not a root of Eq. (5.4).

Case 2: Checking if $p(q-1) - 1$ satisfies Eq. (5.4).

$$
\begin{aligned}
&\{p(q-1)-1\}^2 - \{(3pq - p - q - 2)(pq - p - 1)\} + \\
&(2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p + 2pq - 2p - 2q + 4) \\
&= p^2q^2 + p^2 - 2pq^2 + 1 - 2pq + 2p - 3p^2q^2 + 3p^2q + 3pq + p^2q - p^2 - p \\
&+ pq^2 - pq - q + 2pq - 2p - 2 + 2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p + 2pq - 2p - 2q + 4 \\
&= 4pq - 3p - pq^2 + 3 - 3q \\
&= pq(4-q) - 3(p+q-1).
\end{aligned}
$$

Again considering two sub-cases as in Case 1 we can conclude that
$p(q-1) - 1$ does not satisfy Eq. (5.4).

Case 3: Checking if $q(p-1) - 1$ satisfies Eq. (5.4).

$$
\begin{aligned}
&\{q(p-1)-1\}^2 - \{(3pq - p - q - 2)(pq - q - 1)\} + \\
&(2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p - 2q + 4) \\
&= qp^2 + q - 2pq + 1 - 2pq + 2q - 3p^2q^2 + 3pq^2 + 3pq + p^2q - pq - p + \\
&q^2p - q^2 - q + 2pq - 2q - 2 + 2p^2q^2 - 2p^2q - 2pq^2 + 2pq - 2p - 2q + 4 \\
&-p^2q^2 + 2pq^2 + 2pq - 3p - q^2 - 2q - 3 \\
&= -q^2(p-1)^2 - 3(p+1) + 2q(p-1).
\end{aligned}
$$

Now $2q(p-1) < q^2(p-1)^2$ for all primes p, q with $p < q$.
Hence, $-q^2(p-1)^2 - 3(p+1) + 2q(p-1) < 0$.
Thus, $q(p-1) - 1$ is not a root of Eq. (5.4). □

Theorem 5.4. *The eigen values of $Q(\mathbb{Z}_{pq})$ are $pq-2$ with multiplicity $pq-p-q+2$, $p(q-1)-1$ with multiplicity $q-2$, $q(p-1)-1$ with multiplicity $p-2$, λ_1 and λ_2 each with multiplicity 1.*

Proof. By Lemma 5.1, the spectrum of $Q_\pi(\mathbb{Z}_{pq})$ is contained in $Q(\mathbb{Z}_{pq})$. Hence λ_1, λ_2 are eigen values of $Q(\mathbb{Z}_{pq})$. Also Lemma 5.3 shows that λ_1, λ_2 are different from the eigen values which we have found in Section 4. Thus the eigen values of $Q_\pi(\mathbb{Z}_{pq})$ are $pq - 2$ with multiplicity $pq - p - q + 2$, $p(q-1) - 1$ with multiplicity $q - 2$, $q(p-1) - 1$ with multiplicity $p - 2$, λ_1 and λ_2 where λ_1, λ_2 are roots of Eq. (5.4). □

6. Conclusion

In this paper we have studied the Signless Laplacian spectrum of the power graph of $\mathbb{Z}_n$ and in particular we have found out the Signless Laplacian spectrum of $\mathbb{Z}_{pq}$. We have determined all the eigenvalues of $Q(\mathcal{P}(\mathbb{Z}_{pq}))$ which are $pq - 2$ with multiplicity $pq - (p+q) + 2$, $p(q-1) - 1$ with multiplicity $q - 2$ and $q(p-1) - 1$ with multiplicity $p - 2$, λ_1 and λ_2 each with multiplicity 1.

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