

Scalar meson field in a conformally flat space.

Abhik Kumar Sanyal¹ and D. Ray²

March 8, 2021

¹ Dept of Physics, University College of Science,
92 A.P.C. Road, Calcutta-700009, India.

² Dept of Applied Mathematics, University College of Science,
92 A.P.C. Road, Calcutta-700009, India.

Abstract

Among several authors, who studied massive and massless scalar meson fields in general relativity, attempts to obtain a complete set of solutions for a conformally flat metric $e^\psi (dx^{12} + dx^{22} + dx^{32} - dx^{42})$ were made by Ray for massive and massless mesons and Gursay for massless mesons. Both of them concluded that ψ must be a function of $K_0 (dx^{12} + dx^{22} + dx^{32} - dx^{42}) + K_1 x^1 + K_2 x^2 + K_3 x^3 + K_4 x^4$, where, where K_0, K_1, K_2, K_3, K_4 are all constants. Both Ray and Gursay, however, overlooked an important particular case, which is studied here. As a by-product certain equations obtained by Auria and Regge in connection with ‘‘Gravitational theories with asymptotic flat Instantons,’’ are solved under less restrictive assumptions.

1 Introduction:

Einstein’s field equations for general theory of relativity considering a scalar meson field to be the source, is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = - \left[\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}(\phi_{,\alpha}\phi^{,\alpha} - m^2\phi^2) \right], \quad (1)$$

in the unit $8\pi G = c = 1$, where ϕ is the meson field variable, m is the associate mass, and $\phi_{,\mu} = \frac{\partial\phi}{\partial x^\mu}$ and so on. The above equation (1) may also be expressed as,

$$R_{\mu\nu} = -\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}g_{\mu\nu}m^2\phi^2. \quad (2)$$

For a conformally flat metric,

$$\begin{aligned} g_{\mu\nu} &= e^\psi \eta_{\mu\nu}, \quad \text{where} \\ \eta_{\mu\nu} &= 0, \quad \text{for } \mu \neq \nu, \\ &= 1, \quad \text{for } \mu = \nu = 1, 2, 3, \\ &= 1, \quad \text{for } \mu = \nu = 4. \end{aligned} \quad (3)$$

$R_{\mu\nu}$ takes the form,

$$R_{\mu\nu} = \psi_{,\mu\nu} - \frac{1}{2}\psi_{,\mu}\psi_{,\nu} + \frac{1}{2}\eta_{\mu\nu}\chi, \quad (4)$$

where,

$$\chi = \psi_{,11} + \psi_{,22} + \psi_{,33} - \psi_{,44} + \psi_{,1}^2 + \psi_{,2}^2 + \psi_{,3}^2 - \psi_{,4}^2. \quad (5)$$

The coupled equations (2) - (5) form the complete set of equations for a scalar meson field in a conformally flat space in general relativity. In order to simplify these equations and to study the properties of their solutions

⁰Electronic address:

¹ sanyal_ak@yahoo.com

Present address: Dept. of Physics, Jangipur College, India - 742213.

Ray [1] noted that for $\phi = \text{const}$ the solution of the above-mentioned set of coupled equations is found quite simply, while for $\phi \neq \text{const}$, the coupled equations lead to

$$\left(e^{-\frac{\psi}{2}}\right)_{,\mu} = \eta P_{,\mu} + \rho Q_{,\mu}, \quad (6)$$

$$\frac{1}{2}e^{-\frac{\psi}{2}}\phi_{,\mu} = \eta_{,\phi}P_{,\mu} + \rho_{,\phi}Q_{,\mu}, \quad (7)$$

$$2\eta_{,\phi\phi} + \eta = \xi(\phi)\eta_{,\phi}, \quad (8)$$

$$2\rho_{,\phi\phi} + \rho = \xi(\phi)\rho_{,\phi}, \quad (9)$$

where, $\eta = \eta(\phi)$ and $\rho = \rho(\phi)$, and [2]

$$P = Ax^1 + A_1x^4, \quad (10)$$

$$Q = \frac{1}{2}B(x^{12} + x^{22} + x^{32} - x^{42}) + B_1x^1 + B_x^2 + B_3x^3 + B_4x^4. \quad (11)$$

In the above, $A, A_1, B, B_1, B_2, B_3, B_4$ are constants. From (6) and (7) Ray [1] obtained

$$2e^{-\frac{\epsilon}{2}} \left(\frac{\eta\rho_{,\phi}}{\eta_{,\phi}} - \rho \right) = \frac{f(Q)}{e^{\frac{f(Q)}{2}}}, \quad (12)$$

where $\epsilon = \int \frac{nd\phi}{\eta_{,\phi}}$, and f is a function of Q . Since the left-hand side of Eq. (12) is a function of ϕ and the right-hand side of Eq. (12) is a function of Q , it was argued by Ray [?] that from Eq. (12) one gets $\phi = \phi(Q)$. However, it is obvious from Eq. (12) that there arises another possibility which is

$$2e^{-\frac{\epsilon}{2}} \left(\frac{\eta\rho_{,\phi}}{\eta_{,\phi}} - \rho \right) = \frac{f(Q)}{e^{\frac{f(Q)}{2}}} = K_1, \quad (13)$$

where K_1 is a constant. In this case ϕ need not be a function of Q . In the present paper we shall see whether this previously left out possibility of ϕ not being a function of Q leads to some new solutions. Also we explore whether the results obtained here can be suitably modified so as to be applied in the case of gravitational instantons (Sec. 3).

2 New solutions for scalar meson field:

In search of solutions that may have been left out by Ray [1] we require that

$$\eta_{,\phi} \neq 0, \quad \text{and} \quad \frac{\rho_{,\phi}}{\eta_{,\phi}} \neq \text{constant}, \quad (14)$$

and also

$$Q_{,\mu} \neq 0, \quad \text{and} \quad \frac{P_{,\mu}}{Q_{,\mu}} \neq \text{constant}. \quad (15)$$

The reason for making these requirements is that, if Eq. (14) is not satisfied from Eq. (7), one gets

$$\phi = \phi(KP + K_1Q), \quad (16)$$

where K and K_1 are constants. In view of the set of Eqs. (1.7), the above equation is essentially of the same form as $\phi = \phi(Q)$. Similarly if Eq. (15) holds, then from Eq. (7) one gets $\phi = \phi(Q)$. Since all the solutions of the form $\phi = \phi(Q)$ are already given by Ray [1], there is no need to look for such solutions. Now from the set of Eqs. (1.5) one can write,

$$\begin{aligned} \psi &= \psi(P, Q), & \phi &= \phi(P, Q), \\ \left(e^{-\frac{\psi}{2}}\right)_{,P} &= \eta, & \left(e^{-\frac{\psi}{2}}\right)_{,Q} &= \rho, \\ \frac{1}{2}e^{-\frac{\psi}{2}}\phi_{,P} &= \eta_{,\phi}, & \frac{1}{2}e^{-\frac{\psi}{2}}\phi_{,Q} &= \rho_{,\phi}. \end{aligned} \quad (17)$$

Now, the $\phi = \text{constant}$ surfaces are given by,

$$\eta_{,\phi}dP + \rho_{,\phi}dQ = 0. \quad (18)$$

Integrating, treating ϕ as a constant, we find

$$\eta_{,\phi}P + \rho_{,\phi}Q = \lambda_{,\phi}. \quad (19)$$

From Eqs. (14) and (15) we get

$$(\eta P + \rho Q)_{,\mu} = \lambda_{,\mu} \phi_{,\mu} + \left(e^{-\frac{\psi}{2}} \right)_{,\mu}. \quad (20)$$

Integrating and absorbing the constant of integration we get

$$\eta P + \rho Q = \lambda + e^{-\frac{\psi}{2}}. \quad (21)$$

Differentiating Eq. (19) with respect to P , we get

$$(\eta_{,\phi\phi}P + \rho_{,\phi\phi} - \lambda_{,\phi\phi})\phi_{,P} + \eta_{,\phi} = 0. \quad (22)$$

From Eq. (17) we obtain

$$(2\lambda_{,\phi\phi} + \lambda) - (2\eta_{,\phi\phi} + \eta)P - (2\rho_{,\phi\phi} + \rho)Q = 0. \quad (23)$$

From Eqs. (8), (9) and (15) we get

$$2\lambda_{,\phi\phi} + \lambda = \xi(\phi)\lambda_{,\phi}. \quad (24)$$

Clearly, from Eqs. (8), (9) and (24), η , ρ , p , and λ are found to be the solutions of the same ordinary linear second-order differential equation, and since in view of Eq. (14) η , ρ , and P are linearly independent, there exists two constants D and D_1 such that $\lambda = \eta D + \rho D_1$. Now from Eqs. (19) and (24), we find

$$\eta_{,\phi}(P - D) + \rho_{,\phi}(Q - D_1) = 0, \quad (25)$$

$$\eta(P - D) + \rho(Q - D_1) = e^{-\frac{\psi}{2}}. \quad (26)$$

Further, Eq. (13) gives

$$2e^{-\frac{\psi}{2}} \left(\frac{\eta\rho_{,\phi}}{\eta_{,\phi}} - \rho \right) = K_1. \quad (27)$$

Substituting η and $\eta_{,\phi}$, from Eq. (25) and (26) and simplifying we get

$$e^{\frac{\psi}{2}} = \left[\frac{2}{K_1}(D_1 - Q) \right] e^{-\frac{\psi}{2}}. \quad (28)$$

Now differentiating Eq. (25) with respect to ϕ and using (7), we get

$$\frac{1}{2}e^{-\frac{\psi}{2}} + (P - D)\eta_{,\phi\phi} = (D_1 - Q)\rho_{,\phi\phi}. \quad (29)$$

Now substituting $e^{-\frac{\psi}{2}}$ from Eq. (28) we have

$$(P - D)\eta_{,\phi\phi} = (D_1 - Q) \left[\rho_{,\phi\phi} - \frac{K_1 e^{\frac{\psi}{2}}}{4} \right]. \quad (30)$$

From Eqs. (30), (25) and (26) we get

$$\eta_{,\phi\phi}\rho_{,\phi} = \eta_{,\phi}\rho_{,\phi\phi} - \eta_{,\phi}K_1 \left(\frac{e^{\frac{\psi}{2}}}{4} \right). \quad (31)$$

Let us now choose

$$P - D = P_1, \quad \text{and} \quad Q - D_1 = Q_1, \quad (32)$$

so that

$$\alpha = \frac{P_1}{Q_1}, \quad (33)$$

where,

$$P = Ax^1 + A_1x^4 - D, \quad (34)$$

and

$$Q_1 = \frac{1}{2}B(x^{12} + x^{22} + x^{32} - x^{42}) + B_1x^1 + B_2x^2 + B_3x^3 + B_4x^4 - D_1, \quad (35)$$

also

$$\phi = \phi(\alpha). \quad (36)$$

Now,

$$\phi_{,\mu} = \frac{\phi_{,\alpha}P_{1,\mu}}{Q_1} - \left(\frac{P_1}{Q_1^2}\right)\phi_{\alpha}Q_{1,\mu}. \quad (37)$$

Comparing the above equation with Eq. (7) and noting that P and P_1 differ by a constant and likewise Q and Q_1 we get

$$2e^{\frac{\psi}{2}}\eta_{,\phi} = \frac{\phi_{,\alpha}}{Q_1}, \quad (38)$$

$$2e^{\frac{\psi}{2}}\rho_{,\phi} = \phi_{,\alpha}\left(\frac{P_1}{Q_1^2}\right). \quad (39)$$

The general scalar meson field equation is

$$-2e^{\frac{\psi}{2}}\left(e^{-\frac{\psi}{2}}\right)_{,\mu\nu} = -\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}g_{\mu\nu}m^2\phi^2. \quad (40)$$

From Eq. (38) we get

$$\left(e^{-\frac{\psi}{2}}\right)_{,\mu\nu} = \frac{Q_{1,\mu\nu}}{F} - \frac{F_{,\alpha}}{F^2}(Q_{1,\mu\alpha,\nu} + Q_{1,\nu\alpha,\mu} + Q_{1,\alpha,\mu\nu}) + \frac{Q_{1\alpha,\mu\alpha,\nu}}{F^2}\left(2\frac{F_{,\alpha}}{F} - F_{,\alpha\alpha}\right), \quad (41)$$

where,

$$F = \frac{\phi_{,\alpha}}{2\eta_{,\phi}} = Q_1e^{\frac{\psi}{2}}. \quad (42)$$

Case-I:

Let $\mu \neq \nu$. Equation (40) therefore becomes

$$\alpha_{,\mu\alpha,\nu}\left[\frac{Q_1}{F^2}\left(2\frac{F_{,\alpha}}{F} - F_{,\alpha\alpha}\right) - \phi_{,\alpha}^2\frac{Q_1}{2F}\right] = \frac{F_{,\alpha}}{F^2}(Q_{1,\mu\alpha,\nu} + Q_{1,\nu\alpha,\mu} + Q_{1\alpha,\mu\nu}). \quad (43)$$

Again, from the set of Eqs. (2.11) we get

$$(\alpha Q_1)_{,\mu\nu} = P_{1,\mu\nu} = 0, \quad for \quad \mu \neq \nu, \quad (44)$$

which gives

$$Q_{1,\mu\alpha,\nu} + Q_{1,\nu\alpha,\mu} + Q_{1\alpha,\mu\nu} = 0, \quad for \quad \mu \neq \nu. \quad (45)$$

Finally, from Eq. (43) we therefore get

$$\frac{1}{F}\left(2\frac{F_{,\alpha}}{F} - F_{,\alpha\alpha}\right) - \frac{1}{2}\phi_{,\alpha}^2 = 0. \quad (46)$$

Case-II:

Let $\mu = \nu$. Using equation (40) and (46) it can easily be shown that the field equation for $\mu = \nu$ turns out to be

$$\frac{F_{,\alpha}}{F}(A - B\alpha) = B + \frac{F^2m^2\phi^2}{4Q_1}. \quad (47)$$

Now, since A , B and m are constants, F and ϕ are functions of α , while Q_1 is not a function of α . So Eq. (47) can hold only if $m = 0$. So Eqs. (47) and (46) give

$$F = \frac{C}{A - B\alpha}, \quad (48)$$

$$\phi = 2\sqrt{2}\ln(A - B\alpha),$$

where,

$$\alpha = \frac{P_1}{Q_1} = \frac{Ax^1 + A_1x^4 - D}{\frac{1}{2}B(x^{12} + x^{22} + x^{32} - x^{42}) + B_1x^1 + B_2x^2 + B_3x^3 + B_4x^4 - D_1}. \quad (49)$$

In the above, A , B and C are all constants.

3 Application to the case of gravitational theories with asymptotic flat Instantons:

In a recent paper in connection with “Gravitational theories with asymptotic flat instantons” Auria and Regge [3] have sought to obtain a conformally flat solution of

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{3}{8} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi \right) - \frac{3}{8}g_{\mu\nu}M(\phi), \quad (50)$$

where,

$$\phi_{;\mu}{}^{;\mu} = \frac{1}{2} \frac{\partial M}{\partial \phi}, \quad (51)$$

with an Euclidean metric in the form $(+, +, +, +)$. Auria and Regge [3] assumed

$$g_{\mu\nu} = e^\psi \left(dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2} \right), \quad (52)$$

where,

$$\psi = \psi \left(x^{1^2} + x^{2^2} + x^{3^2} + x^{4^2} \right). \quad (53)$$

However, proceeding as in the paper by Ray [1] and the present paper, it is easy to see that for solutions of Eq. (50) of the form Eq. (52) unless $M(\phi) = 0$, ψ can be expressed as a function of either $(x^{1^2} + x^{2^2} + x^{3^2} + x^{4^2})$ or $(K_1x^1 + K_2x^2 + K_3x^3 + K_4x^4)$ where K_1, K_2, K_3, K_4 etc. are constants. However, if ψ is a function of $(K_1x^1 + K_2x^2 + K_3x^3 + K_4x^4)$, it is obvious that the solution cannot be asymptotically flat. Hence ψ must be a function of $(x^{1^2} + x^{2^2} + x^{3^2} + x^{4^2})$. So Eq. (53) is not an assumption but can be obtained automatically from other conditions.

4 Conclusion:

In summary, we find that Ray’s conclusion that for massive and massless scalar meson fields in general relativity with a conformally flat metric $e^\psi (dx^{1^2} + dx^{2^2} + dx^{3^2} - dx^{4^2})$ that one must have ψ as a function of $K_0(dx^{1^2} + dx^{2^2} + dx^{3^2} - dx^{4^2}) + K_1x^1 + K_2x^2 + K_3x^3 + K_4x^4$, is true except that another solution given by Eq. (2.19) is also possible for massless mesons. It should be noted that the case of massless mesons was also studied by Gürsay [4] and he also overlooked the above possibility. However, the calculations presented by Gürsay [4] are too sketchy for us to find where the oversight occurred. Furthermore we find that the above result has significant application in the study of the “Gravitational theories with asymptotically flat instantons” by Auria and Regge [3], i.e., Eq. (3.3) is not an assumption but it naturally follows from Eqs. (3.1) and (3.2).

References

- [1] D. Ray, J. Math. Phys. 18, 1899 (1977).
- [2] In the above work of Ray [1], the expression for P was given by $P = \frac{1}{2}A(x^{1^2} + x^{2^2} + x^{3^2} - x^{4^2}) + A_1x^1 + A_2x^2 + A_3x^3 + A_4x^4$. But this expression for P can be reduced to the form given by Eq. (10) by suitable choice of coordinates.
- [3] R. D’Auria and T. Regge, Nucl. Phys. B 195, 2, 308 (1982).
- [4] M. Gürses, Phys. Rev. D 15, 2731 (1977).