Bol. Soc. Paran. Mat. (3s.) v. 37 3 (2019): 119–130.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i3.33027

Ricci Almost Solitons And Gradient Ricci Almost Solitons In (k,μ)-Paracontact Geometry

U. C. De and Krishanu Mandal

ABSTRACT: The purpose of this paper is to study Ricci almost soliton and gradient Ricci almost soliton in (k,μ)-paracontact metric manifolds. We prove the non-existence of Ricci almost soliton in a (k,μ)-paracontact metric manifold M with $k < -1$ or $k > -1$ and whose potential vector field is the Reeb vector field ξ. Further, if the metric g of a (k,μ)-paracontact metric manifold M^{2n+1} with $k \neq -1$ is a gradient Ricci almost soliton, then we prove that either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or, M^{2n+1} is an Einstein manifold. Finally, an illustrative example is given.

Key Words: (k,μ)-paracontact manifold, Ricci almost soliton, Gradient Ricci almost soliton, Einstein manifold.

Contents

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric [2]. In a pseudo-Riemannian manifold (M,g) a Ricci soliton is defined by

$$\pounds_V g + 2S - 2\lambda g = 0, \tag{1.1}$$

where $\pounds_V g$ is the Lie derivative of g along a vector field V (called the potential vector field), S the Ricci tensor of type $(0,2)$ and λ a constant. Naturally, a Ricci soliton with V zero or Killing is an Einstein metric. A Ricci soliton on a compact manifold is a gradient Ricci soliton [18]. For details, we refer to Chow and Knopf

2010 *Mathematics Subject Classification:* Primary 53B30, 53C15, 53C25, 53C50; Secondary 53D10, 53D15.
Submitted August 09, 2016. Published April 18, 2017

[10], Bejan and Crasmareanu [1] about Ricci solitons and gradient Ricci solitons. Ricci solitons and gradient Ricci solitons on several types of (almost) contact metric manifolds were studied by several authors such as Cho [8,9], De and Matsuyama [11], Deshmukh [12], Hamilton [14], Turan et al [21], Wang [22], Wang et al [23], Yildiz et al [24] and many others.

Recently, Calvaruso and Perrone [4] studied Ricci solitons in three-dimensional paracontact geometry. Also in [5], Calvaruso and Perrone proved that a paracontact metric manifold is H-paracontact if and only if the Reeb vector field is a Ricci eigenvector.

On the other hand, Pigola et al [19] introduced the notion of Ricci almost soliton in the study of Riemannian manifolds, defined by the same equation (1.1) in which λ is a smooth function. The Ricci almost soliton is said to be shrinking, steady or expanding according as λ is positive, zero or negative, respectively. If the complete vector field V is the gradient of a potential function f, then g is said to be a gradient Ricci almost soliton and equation (1.1) takes the form

$$\mathrm{Hess}f + S = \lambda g, \tag{1.2}$$

where Hess$f$ denotes the Hessian of a smooth function f on M and defined by $\mathrm{Hess}f = \nabla\nabla f$.

Sharma [20] studied Ricci almost solitons in K-contact geometry. Also Ghosh [13] studied Ricci almost solitons and gradient Ricci almost solitons in (k,μ)-contact geometry.

Motivated by these circumstances in this paper we study Ricci almost solitons and gradient Ricci almost solitons in (k,μ)-paracontact metric manifolds. The present paper is organized as follows: Section 2 contains some preliminary results of (k,μ)-paracontact metric manifolds. In Section 3, we prove the non-existence of Ricci almost soliton in a (k,μ)-paracontact metric manifold M with $k < -1$ or $k > -1$ and whose potential vector field is the Reeb vector field ξ. In the next section, we study gradient Ricci almost soliton in a (k,μ)-paracontact metric manifold M^{2n+1} with $k \neq -1$ and prove that if the metric g of M^{2n+1} is a gradient Ricci almost soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or, M^{2n+1} is an Einstein manifold. Finally, an illustrative example is given.

2. Preliminaries on (k,μ)-paracontact Metric Manifolds

A $(2n+1)$-dimensional smooth manifold M is said to be an almost paracontact manifold if it admits an almost paracontact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ a vector field and its dual 1-form η and for any vector field X on M satisfying [15]

(i) $\phi^2 X = X - \eta(X)\xi$,

(ii) $\phi(\xi) = 0,\ \eta\circ\phi = 0,\ \eta(\xi) = 1$,

(iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n.

An almost paracontact structure is said to be normal [25] if and only if the $(1, 2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. If an almost paracontact manifold M equipped with a pseudo-Riemannian metric g of signature $(n + 1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on the manifold M, then (M, g) is called an almost paracontact metric manifold. An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g [25]. For any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ-basis [25], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, ..., E_n, \phi E_1, \phi E_2, ..., \phi E_n\}$, where $\xi, E_1, E_2, ..., E_n$ are space-like vector fields and then, by (2.1) vector fields $\phi E_1, \phi E_2, ..., \phi E_n$ are time-like. In a paracontact metric manifold there exists a symmetric, trace-free $(1, 1)$-tensor $h = \frac{1}{2}\pounds_\xi \phi$ satisfying [25]

$$\phi h + h\phi = 0,\ h\xi = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.3}$$

where ∇ is Levi-Civita connection of the pseudo-Riemannian manifold and for all $X \in \chi(M)$. It is clear that the tensor h satisfies $h = 0$ if and only if ξ is a Killing vector field and then (ϕ, ξ, η, g) is said to be a K-paracontact manifold. An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [25]

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.4}$$

for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.5}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.5) does not imply that the paracontact manifold is para-Sasakian manifold. Every para-Sasakian manifold is a K-paracontact manifold, but the converse is not always true, as it is shown in three dimensional case [3]. Paracontact metric manifolds have been studied by Cappelletti-Montano et al [6,7], Martin-Molina [16,17] and many others.

According to Cappelletti-Montano et al [6] we have the following definition.

Definition 2.1. *A paracontact metric manifold is said to be (k, μ)-paracontact manifold if the curvature tensor R satisfies*

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.6}$$

for all vector fields $X, Y \in \chi(M)$ and k, μ are real constants.

In a (k,μ)-paracontact manifold $(M^{2n+1},\phi,\xi,\eta,g), n>1$, the following relations hold [6]:

$$h^2=(k+1)\phi^2, \tag{2.7}$$

$$(\nabla_X\phi)Y=-g(X-hX,Y)\xi+\eta(Y)(X-hX), \qquad \text{for } k\neq -1, \tag{2.8}$$

$$\begin{aligned} QY &= [2(1-n)+n\mu]Y+[2(n-1)+\mu]hY \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \qquad \text{for } k\neq -1, \end{aligned} \tag{2.9}$$

$$S(X,\xi)=2nk\eta(X), \tag{2.10}$$

$$Q\xi=2nk\xi, \tag{2.11}$$

$$\begin{aligned} (\nabla_X h)Y &= -[(1+k)g(X,\phi Y)+g(X,\phi hY)]\xi \\ &\quad +\eta(Y)\phi h(hX-X)-\mu\eta(X)\phi hY, \qquad \text{for } k\neq -1, \end{aligned} \tag{2.12}$$

$$Q\phi-\phi Q=2[2(n-1)+\mu]h\phi, \tag{2.13}$$

for any vector fields $X,Y\in\chi(M)$, where Q is the Ricci operator defined by $g(QX,Y)=S(X,Y)$. Making use of (2.3) we have

$$(\nabla_X\eta)Y=g(X,\phi Y)+g(\phi hX,Y), \tag{2.14}$$

for all vector fields $X,Y\in\chi(M)$.
First of all we recall some useful results:

Lemma 2.2. *([26], Theorem 3.3) Let $M^{2n+1}, n>1$, be a paracontact metric manifold satisfies $R(X,Y)\xi=0$, for all $X,Y\in\chi(M)$. Then M^{2n+1} is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

Lemma 2.3. *([6], p.683, 687) Let (M,ϕ,ξ,η,g) be a (k,μ)-paracontact metric manifold. Then for any vector fields $X,Y\in\chi(M)$ we have*

$$\begin{aligned} (\nabla_X\phi h)Y &= g(h^2X-hX,Y)\xi+\eta(Y)(h^2X-hX) \\ &\quad -\mu\eta(X)hY, \quad \text{for } k>-1 \end{aligned} \tag{2.15}$$

$$\begin{aligned} (\nabla_X\phi h)Y &= (1+k)g(X,Y)\xi-g(hX,Y)\xi+\eta(Y)(h^2X-hX) \\ &\quad -\mu\eta(X)hY, \quad \text{for } k<-1. \end{aligned} \tag{2.16}$$

3. Ricci Almost Solitons in (k,μ)-paracontact Metric Manifolds

In this section we discuss about Ricci almost solitons in (k,μ)-paracontact manifolds. We prove the following:

Theorem 3.1. *There does not exist Ricci almost soliton in a (k,μ)-paracontact metric manifold M^{2n+1} $(n>1)$ whose potential vector field is the Reeb vector field ξ with $k<-1$ or $k>-1$.*

Proof: Suppose a (k, μ)-paracontact metric manifold admits a Ricci almost soliton (g, ξ). Then we have from (1.1)

$$(\pounds_\xi g)(Y, Z) + 2S(Y, Z) - 2\lambda g(X, Y) = 0. \tag{3.1}$$

This is equivalent to

$$g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2S(Y, Z) - 2\lambda g(Y, Z) = 0. \tag{3.2}$$

Using (2.3) in the above equation implies

$$g(\phi hY, Z) + S(Y, Z) - \lambda g(Y, Z) = 0, \tag{3.3}$$

that is,

$$\phi hY + QY - \lambda Y = 0. \tag{3.4}$$

Taking covariant differentiation of (3.4) with respect to an arbitrary vector field X, we have

$$(\nabla_X \phi h)Y + (\nabla_X Q)Y - (X\lambda)Y = 0. \tag{3.5}$$

Now we split our discussions into two cases:
`Case 1.` Let $k > -1$. Applying (2.9) and (2.15) in (3.5) we have

$$\begin{aligned}&(1+k)g(X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)X - (k+1)\eta(X)\eta(Y)\xi\\&-\eta(Y)hX - \mu\eta(X)hY + \{2(n-1)+\mu\}(\nabla_X h)Y\\&+\{2(n-1)+n(2k-\mu)\}\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\} - (X\lambda)Y = 0.\end{aligned} \tag{3.6}$$

With the help of (2.3), (2.12) and (2.14) we obtain from the above equation

$$\begin{aligned}&(1+k)g(X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)X - (k+1)\eta(X)\eta(Y)\xi\\&-\eta(Y)hX - \mu\eta(X)hY - \{2(n-1)+\mu\}\{(k+1)g(X, \phi Y)\xi + g(X, \phi hY)\xi\\&-(k+1)\eta(Y)\phi X + \eta(Y)\phi hX + \mu\eta(X)\phi hY\} + \{2(n-1)+n(2k-\mu)\}\\&\{g(X, \phi Y)\xi + g(\phi hX, Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX\} - (X\lambda)Y = 0.\end{aligned} \tag{3.7}$$

Contracting X in (3.7) we get

$$(2n+1)(k+1)\eta(Y) = Y\lambda. \tag{3.8}$$

Also putting $Y = Z = \xi$ in (3.3) yields $\lambda = 2nk$, which is a constant. Applying this in (3.8) we have $k = -1$, which is a contradiction as we consider $k > -1$.
`Case 2.` Let $k < -1$. Making use of (2.9) and (2.16) in (3.5) we have

$$\begin{aligned}&g((k+1)\phi^2 X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)\phi^2 X\\&-\eta(Y)hX - \mu\eta(X)hY + \{2(n-1)+\mu\}(\nabla_X h)Y\\&+\{2(n-1)+n(2k-\mu)\}\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\} - (X\lambda)Y = 0.\end{aligned} \tag{3.9}$$

Using (2.3), (2.12) and (2.14) in the above equation gives

$$\begin{aligned}&g((k+1)\phi^2X,Y)\xi - g(hX,Y)\xi + (k+1)\eta(Y)\phi^2X\\&-\eta(Y)hX - \mu\eta(X)hY - \{2(n-1)+\mu\}\{(k+1)g(X,\phi Y)\xi + g(X,\phi hY)\xi\\&-(k+1)\eta(Y)\phi X + \eta(Y)\phi hX + \mu\eta(X)\phi hY\} + \{2(n-1)+n(2k-\mu)\}\\&\{g(X,\phi Y)\xi + g(\phi hX,Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX\} - (X\lambda)Y = 0.\end{aligned} \tag{3.10}$$

Contracting X in (3.10) we have

$$2n(k+1)\eta(Y) = Y\lambda. \tag{3.11}$$

Again substituting $Y = Z = \xi$ in (3.3) gives $\lambda = 2nk$, which is a constant. Using this in (3.11) we obtain $k = -1$, which is a contradiction as we consider $k < -1$. Combining the two cases our theorem follows. □

Next, we prove the following:

Theorem 3.2. *If a (k,μ)-paracontact metric manifold M^{2n+1} $(n > 1)$ admits a Ricci almost soliton for $k = -1$ whose potential vector field is the Reeb vector field ξ, then the Ricci almost soliton is expanding with $Q\xi = -2n\xi$.*

Proof: Replacing Y by ξ in (3.4) we get $Q\xi = \lambda\xi$. On the other hand from (2.11) and $k = -1$ we have $Q\xi = -2n\xi$. Thus we obtain $\lambda = -2n$. This shows that the Ricci almost soliton is expanding. □

Remark 3.3. *Since λ= constant, the Ricci almost soliton in a (k,μ)-paracontact metric manifold reduces to a Ricci soliton.*

4. Gradient Ricci Almost Solitons in (k,μ)-paracontact Metric Manifolds

This section is devoted to study gradient Ricci almost soliton in (k,μ)-paracontact manifolds. We prove the following:

Theorem 4.1. *Let (M,g) be a $(2n+1)$-dimensional $(n > 1)$ (k,μ)-paracontact metric manifold with $k \neq -1$. If g is a gradient Ricci almost soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or, M^{2n+1} is an Einstein manifold.*

Proof: Let (M,g) be a $(2n+1)$-dimensional (k,μ)-paracontact metric manifold and g a gradient Ricci almost soliton. Then (1.2) reduces to

$$\nabla_Y Df = -QY + \lambda Y, \tag{4.1}$$

for any $Y \in \chi(M)$, where D denotes the gradient operator of g. From (4.1) it follows that

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y - (Y\lambda)X + (X\lambda)Y. \tag{4.2}$$

Taking covariant differentiation of (2.9) along arbitrary vector field X and using (2.14) we have

$$\begin{aligned}(\nabla_X Q)Y &= \{2(n-1)+n(2k-\mu)\}[g(X,\phi Y)\xi + g(\phi hX,Y)\xi \\ &\quad -\eta(Y)\phi X + \eta(Y)\phi hX] + \{2(n-1)+\mu\}(\nabla_X h)Y. \end{aligned} \tag{4.3}$$

Applying (2.12) in (4.3) gives

$$\begin{aligned}(\nabla_X Q)Y - (\nabla_Y Q)X &= \{2(n-1)+\mu\}[-(k+1)\{2g(X,\phi Y)\xi + \eta(X)\phi Y \\ &\quad -\eta(Y)\phi X\} + (1-\mu)\{\eta(X)\phi hY - \eta(Y)\phi hX\}] \\ &\quad +\{2(n-1)+n(2k-\mu)\}[2g(X,\phi Y)\xi + \eta(X)\phi Y \\ &\quad -\eta(Y)\phi X + \eta(Y)\phi hX - \eta(X)\phi hY]. \end{aligned} \tag{4.4}$$

Making use of (4.4) we have from (4.2)

$$\begin{aligned}R(X,Y)Df &= \{2(n-1)+\mu\}[(k+1)\{2g(X,\phi Y)\xi + \eta(X)\phi Y \\ &\quad -\eta(Y)\phi X\} + (\mu-1)\{\eta(X)\phi hY - \eta(Y)\phi hX\}] \\ &\quad -\{2(n-1)+n(2k-\mu)\}[2g(X,\phi Y)\xi + \eta(X)\phi Y \\ &\quad -\eta(Y)\phi X + \eta(Y)\phi hX - \eta(X)\phi hY] - (Y\lambda)X + (X\lambda)Y. \end{aligned} \tag{4.5}$$

Taking inner product of (4.5) with ξ we obtain

$$\begin{aligned}g(R(X,Y)Df,\xi) &= 2(\mu - 2k + \mu k + n\mu)g(X,\phi Y) \\ &\quad -(Y\lambda)\eta(X) + (X\lambda)\eta(Y). \end{aligned} \tag{4.6}$$

Substituting $X=\xi$ in (4.6) yields

$$g(R(\xi,Y)Df,\xi) = (\xi\lambda)\eta(Y) - (Y\lambda). \tag{4.7}$$

Also from (2.6) it follows that

$$R(\xi,Y)X = k\{g(X,Y)\xi - \eta(X)Y\} + \mu\{g(hX,Y)\xi - \eta(X)hY\}. \tag{4.8}$$

Taking inner product of (4.8) with ξ gives

$$g(R(\xi,Y)Df,\xi) = kg(Y, Df - (\xi f)\xi) + \mu g(hY, Df). \tag{4.9}$$

In view of (4.7) and (4.9) we have

$$kg(Y, Df - (\xi f)\xi) + \mu g(hY, Df) - (\xi\lambda)\eta(Y) + (Y\lambda) = 0, \tag{4.10}$$

from which we obtain

$$kDf - k(\xi f)\xi + \mu hDf + D\lambda - (\xi\lambda)\xi = 0. \tag{4.11}$$

Contracting X in (4.2) and noticing that the scalar curvature of the manifold is constant, we have $QDf = -2nD\lambda$. Applying this in (4.11) gives

$$2nkDf + 2n\mu hDf = QDf + 2n(k(\xi f) + (\xi\lambda))\xi. \tag{4.12}$$

Taking inner product of (4.12) with ξ and since $Q\xi = 2nk\xi$ it follows that $k(\xi f) + (\xi\lambda) = 0$. Using this in the above equation one gets

$$2nkDf + 2n\mu hDf = QDf. \tag{4.13}$$

Putting $X = \xi$ in (4.1) we obtain

$$\nabla_\xi Df = (\lambda - 2nk)\xi. \tag{4.14}$$

Replacing X by ϕX and Y by ϕY in (4.6) and (2.6), respectively, then comparing the right hand sides we have

$$(\mu - 2k + \mu k + n\mu)g(\phi X, Y) = 0. \tag{4.15}$$

Since $d\eta \neq 0$, it follows from the above equation

$$k = \frac{\mu(n+1)}{2-\mu}. \tag{4.16}$$

Differentiating (4.13) along ξ implies

$$2nk\nabla_\xi Df + 2n\mu(\nabla_\xi h)Df + 2n\mu h(\nabla_\xi Df) = (\nabla_\xi Q)Df + Q(\nabla_\xi Df). \tag{4.17}$$

Making use of (2.12), (4.3) and (4.14) in (4.17) we obtain

$$\mu\{\mu(2n-1) - 2(n-1)\}h\phi Df = 0. \tag{4.18}$$

Operating h on (4.18) and since $k \neq -1$ one obtains

$$\mu\{\mu(2n-1) - 2(n-1)\}\phi Df = 0. \tag{4.19}$$

Thus we consider the following cases:
`Case1.` If $\mu = 0$, then from (4.16) it follows that $k = 0$. Consequently (2.6) gives $R(X,Y)\xi = 0$. Therefore, using Lemma 2.2 we can state M^{2n+1} is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.
`Case2.` If $\phi Df = 0$. Applying ϕ on both sides we obtain

$$Df = (\xi f)\xi. \tag{4.20}$$

Taking differentiation of (4.20) along any arbitrary vector field X, we have $\nabla_X Df = X(\xi f)\xi + (\xi f)(-\phi X + \phi hX)$. Replacing X by ϕX and taking inner product with ϕY we have

$$g(\nabla_{\phi X} Df, \phi Y) = -(\xi f)\{g(X, \phi Y) + g(hX, \phi Y)\}. \tag{4.21}$$

Interchanging X and Y in the above equation yields

$$g(\nabla_{\phi Y} Df, \phi X) = -(\xi f)\{g(Y, \phi X) + g(hY, \phi X)\}. \tag{4.22}$$

Applying Poincaré's lemma: On a contractible manifold, all closed forms are exact. Therefore $d^2f(X,Y) = 0$, for all $X, Y \in \chi(M)$. From which we have

$$XY(f) - YX(f) - [X,Y]f = 0,$$

that is,

$$Xg(\text{grad}f, Y) - Yg(\text{grad}f, X) - g(\text{grad}f, [X,Y]) = 0.$$

This is equivalent to

$$\nabla_X g(\text{grad}f, Y) - g(\text{grad}f, \nabla_X Y) - \nabla_Y g(\text{grad}f, X) + g(\text{grad}f, \nabla_Y X) = 0.$$

Since $\nabla g = 0$, the above equation yields

$$g(\nabla_X \text{grad}f, Y) - g(\nabla_Y \text{grad}f, X) = 0,$$

that is, $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$. Replacing X by ϕX and Y by ϕY in the foregoing equation we obtain $g(\nabla_{\phi X} Df, \phi Y) = g(\nabla_{\phi Y} Df, \phi X)$. Applying this in (4.21) and (4.22) we have $(\xi f)g(X, \phi Y) = 0$, that is, $(\xi f)d\eta(X,Y) = 0$. Since $d\eta \neq 0$, it follows that $\xi f = 0$. Consequently from (4.20) we obtain $Df = 0$, this implies f is constant. Therefore from (4.1) we have

$$S(X,Y) = \lambda g(X,Y).$$

This shows the manifold is an Einstein manifold.
`Case3.` If $\mu(2n-1) - 2(n-1) = 0$, that is, $\mu = \frac{2(n-1)}{2n-1}$. Using (4.16) we get $k = n - \frac{1}{n}$. From (2.9) and (4.13) we obtain

$$(2(1-n) + n\mu - 2nk)(Df - (\xi f)\xi) + (2(n-1) + \mu - 2n\mu)hDf = 0. \tag{4.23}$$

Making use of $\mu = \frac{2(n-1)}{2n-1}$ and $k = n - \frac{1}{n}$ in the above equation and noticing $n > 1$ we have $Df = (\xi f)\xi$. Proceeding in the same way as in Case 2 we obtain the manifold is an Einstein manifold. This completes the proof of our theorem. □

5. Example of a 5-dimensional (k,μ)-paracontact Metric Manifold

In this section we give an example of a 5-dimensional (k,μ)-paracontact metric manifold such that $k = -2$ and $\mu = 2$. In [6], the authors construct an example of a 5-dimensional (k,μ)-paracontact metric manifold. With the help of that example we construct a new example as follows:

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G admits a basis $\{e_1, e_2, e_3, e_4, e_5\}$ such that [6]

$$\begin{aligned}
&[e_1,e_5] = e_1 + e_2,\ [e_2,e_5] = e_1 + e_2,\ [e_3,e_5] = -e_3 + e_4,\\
&[e_4,e_5] = e_3 - e_4,\ [e_1,e_2] = e_1 + e_2,\ [e_1,e_3] = e_2 + e_4 - 2e_5,\\
&[e_1,e_4] = e_2 + e_3,\ [e_2,e_3] = e_1 - e_4,\ [e_2,e_4] = e_1 - e_3 + 2e_5,\\
&[e_3,e_4] = -e_3 + e_4.
\end{aligned}$$

We consider the metric such that

$$g(e_1,e_1)=g(e_4,e_4)=g(e_5,e_5)=1,$$
$$g(e_2,e_2)=g(e_3,e_3)=-1 \text{ and } g(e_i,e_j)=0, \text{ for } i\neq j.$$

Setting $\xi=e_5$ and denote by η its dual 1-form. We define a tensor ϕ by $\phi e_1 = e_3$, $\phi e_2=e_4$, $\phi e_3=e_1$, $\phi e_4=e_2$, $\phi e_5=0$. Therefore we have $\phi^2X=X-\eta(X)\xi$ and $g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y)$. Thus (ϕ,ξ,η,g) makes G a paracontact metric manifold.

Using the well known Koszul's folmula we have the following:

$$\nabla_{e_1}e_5=e_1-e_3, \nabla_{e_2}e_5=e_2-e_4, \nabla_{e_3}e_5=-e_1-e_3, \nabla_{e_4}e_5=-e_2-e_4,$$
$$\nabla_{e_5}e_1=-e_2-e_3, \nabla_{e_5}e_2=-e_1-e_4, \nabla_{e_5}e_3=-e_1-e_4, \nabla_{e_5}e_4=-e_2-e_3,$$
$$\nabla_{e_1}e_1=e_2-e_5, \nabla_{e_1}e_2=e_1, \nabla_{e_1}e_3=e_4-e_5, \nabla_{e_1}e_4=e_3,$$
$$\nabla_{e_2}e_1=-e_2, \nabla_{e_2}e_2=-e_1+e_5, \nabla_{e_2}e_3=-e_4, \nabla_{e_2}e_4=-e_3+e_5,$$
$$\nabla_{e_3}e_1=-e_2+e_5, \nabla_{e_3}e_2=-e_1, \nabla_{e_3}e_3=-e_4-e_5, \nabla_{e_3}e_4=-e_3,$$
$$\nabla_{e_4}e_1=-e_2, \nabla_{e_4}e_2=-e_1-e_5, \nabla_{e_4}e_3=-e_4, \nabla_{e_4}e_4=-e_3+e_5, \nabla_{e_5}e_5=0.$$

Comparing the above relations with (2.3) we get

$$he_1=e_3,\ he_2=e_4,\ he_3=-e_1,\ he_4=-e_2,\ he_5=0.$$

Using the formula $R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$, we may calculate the following:

$$R(e_1,e_2)e_1=e_2-e_4, R(e_1,e_2)e_2=e_1-e_3, R(e_1,e_2)e_3=e_2-e_4,$$
$$R(e_1,e_2)e_4=e_1-e_3, R(e_1,e_3)e_1=-4e_3, R(e_1,e_3)e_2=-2e_4,$$
$$R(e_1,e_3)e_3=-4e_1, R(e_1,e_3)e_4=-2e_2, R(e_1,e_4)e_1=e_2-e_4,$$
$$R(e_1,e_4)e_2=e_1+e_3, R(e_1,e_4)e_3=e_4-e_2, R(e_1,e_4)e_4=e_1+e_3,$$

$$R(e_1,e_5)e_1=2e_5, R(e_1,e_5)e_3=2e_5, R(e_1,e_5)e_5=2e_3-2e_1,$$
$$R(e_2,e_3)e_1=-e_2-e_4, R(e_2,e_3)e_2=e_3-e_1, R(e_2,e_3)e_3=-e_2-e_4,$$
$$R(e_2,e_3)e_4=e_1-e_3, R(e_2,e_4)e_1=2e_3, R(e_2,e_4)e_2=4e_4,$$
$$R(e_2,e_4)e_3=2e_1, R(e_2,e_4)e_4=4e_2, R(e_2,e_5)e_2=-2e_5,$$
$$R(e_2,e_5)e_4=-2e_5, R(e_2,e_5)e_5=2e_4-2e_2, R(e_3,e_4)e_1=e_2+e_4,$$
$$R(e_3,e_4)e_2=e_1+e_3, R(e_3,e_4)e_3=-e_2-e_4, R(e_3,e_4)e_4=-e_1-e_3,$$
$$R(e_3,e_5)e_1=2e_5, R(e_3,e_5)e_3=-2e_5, R(e_3,e_5)e_5=-2e_1-2e_3,$$
$$R(e_4,e_5)e_2=-2e_5, R(e_4,e_5)e_4=2e_5, R(e_4,e_5)e_5=-2e_2-2e_4.$$

With the help of the expressions of the curvature tensor we conclude that the manifold is a (k,μ)-paracontact metric manifold with $k=-2$ and $\mu=2$. Also from the above expressions we obtain the following:

$$S(e_1,e_1)=S(e_4,e_4)=-4,\ S(e_2,e_2)=S(e_3,e_3)=S(e_5,e_5)=0.$$

Using the above results it can be easily verified that such a manifold does not satisfy the Equation (3.1). Thus Theorem 3.1 is verified.

Acknowledgments

The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

References

1. C. L. Bejan and M. Crasmareanu, *Second order parallel tensors and Ricci solitons in* 3-*dimensional normal paracontact geometry,* Ann. Global Anal. Geom., 46, 117-127, (2014).
2. C. Calin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds,* Bull. Malays. Math. Sci. Soc. **33**, 361-368, (2010).
3. G. Calvaruso, *Homogeneous paracontact metric three-manifolds,* Illinois J. Math., 55, 697-718, (2011).
4. G. Calvaruso and A. Perrone, *Ricci solitons in three-dimensional paracontact geometry,* J. Geom. Phys. 98, 1-12, (2015).
5. G. Calvaruso and D. Perrone, *Geometry of H-paracontact metric manifolds,* Publ. Math. Debrecen 86, 325-346, (2015).
6. B. Cappelletti-Montano, I. Küpeli Erken and C. Murathan, *Nullity conditions in paracontact geometry,* Diff. Geom. Appl. 30, 665-693, (2012).
7. B. Cappelletti-Montano and L. Di Terlizzi, *Geometric structures associated to a contact metric (k,μ)-space,* Pacific J. Math. 246, 257-292, (2010).
8. J. T. Cho, *Notes on contact Ricci solitons,* Proc. Edinb. Math. Soc. 54, 47-53, (2011).
9. J. T. Cho, *Ricci solitons in almost contact geometry,* Proc. Seventeenth Int. Workshop on Diff. Geom. 17, 85-95, (2013).
10. B. Chow and D. Knopf, *The Ricci flow: An introduction, Mathematical surveys and Monographs,* Amer. Math. Soc. **110**(2004).
11. U. C. De and Y. Matsuyama, *Ricci solitons and gradient Ricci solitons in a Kenmotsu manifold,* Southeast Asian Bull. Math. 37, 691-697, (2013).
12. S. Deshmukh, *Jacobi-type vector fields on Ricci solitons,* Bull. Math. Soc. Math. Roumanie Tome 55(103), 41-50, (2012).
13. A. Ghosh, *Certain contact metrics as Ricci almost solitons,* Results Math. 65, 81-94, (2014).
14. R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity (Santa Cruz, CA, 1986),* Contemp. Math. **71**, Amer. Math. Soc., 237-262, (1988).
15. S. Kaneyuki and F. L. Williams, *Almost paracontact and parahodge structures on manifolds,* Nagoya Math. J. 99, 173-187, (1985).
16. V. Martin-Molina, *Paracontact metric manifolds without a contact metric counterpart,* Taiwanese J. Math. 19, 175-191, (2015).
17. V. Martin-Molina, *Local classification and examples of an important class of paracontact metric manifolds,* Filomat 29, 507-515, (2015).
18. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications,* Preprint, http://arxiv.org/abs/math.DG/02111159.
19. S. Pigola, M. Rigoli, M. Rimoldi and A. Setti, *Ricci almost solitons,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. X, 757-799, (2011).
20. R. Sharma, *Almost Ricci solitons and K-contact geometry,* Monatsh Math. 175, 621-628, (2014).
21. M. Turan, U. C. De and A. Yildiz, *Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds,* Filomat 26, 363-370, (2012).

22. Y. Wang, *A generalization of the Goldberg conjecture for coKähler manifolds,* Mediterr. J. Math., DOI: 10.1007/s00009-015-0646-8.

23. Y. Wang and X. Liu, *Ricci solitons on three-dimensional η-Einstein almost Kenmotsu manifolds,* Taiwanese J. Math. 19, 91-100, (2015).

24. A. Yildiz, U. C. De and M. Turan, *On* 3*-dimensional f-Kenmotsu manifolds and Ricci solitons,* Ukrainian Math. J. **65**, 684-693, (2013).

25. S. Zamkovoy, *Canonical connections on paracontact manifolds,* Ann. Global Anal. Geom. 36, 37-60, (2009).

26. S. Zamkovoy and V. Tzanov, *Non-existence of flat paracontact metric structures in dimension greater than or equal to five,* Annuaire Univ. Sofia Fac. Math. Inform. 100, 27-34, (2011).

U. C. De,
Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road, Kol-19,
West Bengal, India.
E-mail address: uc_de@yahoo.com

and

Krishanu Mandal,
Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road, Kol-19,
West Bengal, India.
E-mail address: krishanu.mandal013@gmail.com