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Plane symmetric charged dust distribution

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An attempt is made to obtain the complete set of solutions of Einstein's equations for a nonstatic charged dust distribution with a comoving system of coordinates and the most general plane symmetric metric, i.e.,

$$ds^2 = \exp[2u(x,t)]dt^2 - \exp[2v(x,t)]dx^2 - \exp[2w(x,t)](dy^2 + dz^2).$$

The field equations have been reduced to a single ordinary differential equation which is integrated in a particular case.

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1. INTRODUCTION

If some matter distribution flattens in the form of a disk, the likeliest symmetry it will have is plane symmetry. Since in the universe there is no dearth of such flattened cosmological objects, study of matter distribution with plane symmetry has some relevance. Charge-carrying incoherent matter in the form of dust is our concern here. Since it has already been established[1] that plane-symmetric incoherent dust distribution in equilibrium is not possible in general relativity, we seek solutions of Einstein's equations for a nonstatic charge-carrying dust distribution with the most general plane-symmetric metric given by

$$ds^2 = e^{2u(x,t)}dt^2 - e^{2v(x,t)}dx^2 - e^{2w(x,t)}(dy^2 + dz^2) \tag{1}$$

and

$$(t,x,y,z) \equiv (x^0,x^1,x^2,x^3).$$

Some particular solutions of this type were previously reported by one of the present authors.[2,3] The present paper is an attempt to obtain the complete set of solutions of the above type for a comoving frame, i.e.,

$$\left.\begin{aligned} v^0 &= \frac{1}{(g_{00})^{1/2}} = e^{-u} \\ \text{and} \\ v^\mu &= 0 \quad \text{for } \mu = 1,2,3. \end{aligned}\right\} \tag{2}$$

2. FIELD EQUATIONS

Equations (1) and (2) reduce the Einstein field equations for a charged dust to

$$\begin{aligned} R^0_0 &= e^{-2u}[\ddot v + 2\ddot w + \dot v^2 + 2\dot w^2 - \dot u(\dot v + 2\dot w)] \\ &\quad - e^{-2v}[u'' + u'(u' - v' + 2w')] \\ &= -4\pi\rho + F^{01}F_{01}, \end{aligned} \tag{3a}$$

$$\begin{aligned} R^1_1 &= e^{-2u}[\ddot v + \dot v^2 - \dot u\dot v + 2\dot v\dot w] \\ &\quad - e^{-2v}[u'' + 2w'' + u'^2 + 2w'^2 - v'(u' + 2w')] \\ &= 4\pi\rho + F^{01}F_{01}, \end{aligned} \tag{3b}$$

$$\begin{aligned} R^2_2 = R^3_3 &= e^{-2u}[\ddot w + 2\dot w^2 + \dot w(\dot v - \dot u)] \\ &\quad - e^{-2v}[w'' + 2w'^2 + w'(u' - v')] \\ &= 4\pi\rho - F^{01}F_{01}, \end{aligned} \tag{3c}$$

$$R_{01} = 2[\dot w(w' - u') + (\dot w' - w'\dot v)] = 0, \tag{3d}$$

and the Maxwell equations for the same to

$$F^{01} = C(x)e^{-(u + v + 2w)}, \tag{4a}$$

$$4\pi\sigma = C'(x)e^{-(v + 2w)}, \tag{4b}$$

where ρ is the mass density, σ is the charge density and $F^{\mu\nu}$ is the electromagnetic field tensor whose only nonvanishing components are F^{01} and F^{10} due to symmetry of the problem. (A dot indicates differentiation with respect to t and a prime indicates differentiation with respect to x.)

Equations (3) and (4) provide the complete set of equations to be solved. However, to simplify the calculations we make direct use of conservation laws as follows.

From (1) and (2) $T^{\mu\nu}{}_{;\nu} = 0$ gives

$$F^{01} = -(\rho/\sigma)u'e^{-(u + 2v)} \tag{5}$$

and conservation of mass gives

$$e^{v + 2w}\rho = \text{const.} \tag{6}$$

From (4b) and (6), (ρ/σ) must be a function of x and hence

$$C^2(x) = (\rho/\sigma)^2 A^2(x), \tag{7}$$

where $A(x)$ is some function of x. From (4a), (5), and (7),

$$u'e^{(2w - v)} = -A(x). \tag{8}$$

From (3d) and (8),

$$(\dot w' + \dot u') + (w' + u')(\dot w - \dot v) = 0,$$

which on integration gives

$$w' + u' = B(x)e^{(v - w)}, \tag{9}$$

where $B(x)$ is some function. From (8) and (9)

$$w' = B(x)e^{(v - w)} + A(x)e^{(v - 2w)}. \tag{10}$$

Using (4b), Eq. (3a) − Eq. (2b) + 2×Eq. (3c) gives

$$\begin{aligned} &e^{-2u}(4\ddot w + 6\dot w^2 - 4\dot u\dot w) - e^{-2v}(2w'^2 + 4u'w') \\ &\quad = 2C^2e^{-2w}. \end{aligned} \tag{11}$$

Putting w' and u' from (8) and (10) into (11) and integrating,

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$$w = \pm e^{u}\{[A^{2}(x) - C^{2}(x)]e^{-4w} + D(x)e^{-3w} + B^{2}(x)e^{-2w}\}^{1/2} \tag{12}$$

where $D(x)$ is some function. Similarly, using (4b), (6), and (12), Eq. (3a) — Eq. (3b) — $2\times$Eq. (3c) gives

$$\dot{v} = \pm \frac{\{[A'(x) \pm C'(x)C(x)/A(x)]e^{-(v+2w)} + B'(x)e^{-(v+2w)} + [C^{2}(x) - A^{2}(x)]e^{-4w} - [D(x)/2]e^{-3w}\}}{e^{-u}\{[A^{2}(x) - C^{2}(x)]e^{-4w} + D(x)e^{-3w} + B^{2}(x)e^{-2w}\}^{1/2}}.$$

From (10), (12), and $\partial w/\partial x = \partial w'/\partial t$ one sees that the negative sign must be taken in $\pm C'(x)C(x)/A(x)$ in the above expression for $\dot{v}$, i.e.,

$$\dot{v} = \pm \frac{\{[A'(x) - C'(x)C(x)/A(x)]e^{-(v+2w)} + B'(x)e^{-(v+2w)} + [C^{2}(x) - A^{2}(x)]e^{-4w} - [D(x)/2]e^{-3w}\}}{e^{-u}\{[A^{2}(x) - C^{2}(x)]e^{-4w} + D(x)e^{-3w} + B^{2}(x)e^{-2w}\}^{1/2}} \tag{13}$$

and further

$$2B(x)[A(x)A'(x) - C(x)C'(x)] = A(x)[D'(x) - 2A(x)B'(x)]. \tag{14}$$

From Eq. (12) and (13)

$$\left.\frac{de^{v}}{dw}\right|_{x=\text{const}} = P(x,w)e^{v} + Q(x,w),$$

which is equivalent to

$$e^{v} = \exp\left[\int P(x,w)\,dw\right]\left\{\int Q(x,w)\exp\left[-\int P(x,w)\,dw\right]dw + R(x)\right\}, \tag{15}$$

where $R(x)$ is a function of x, and

$$P(x,w) = \frac{[C^{2}(x) - A^{2}(x)] - \frac{1}{2}D(x)e^{w}}{[A^{2}(x) - C^{2}(x)] + D(x)e^{w} + B^{2}(x)e^{2w}},$$

$$Q(x,w) = \frac{[A'(x) - C(x)C'(x)/A(x)] + B'(x)e^{w}}{[A^{2}(x) - C^{2}(x)]e^{-2w} + D(x)e^{-w} + B^{2}(x)}.$$

Now it can be checked through direct substitution that all the equations of (3) and (4) are satisfied by Eqs. (10), (12), (14), and (15) combined. Of these equations, Eq. (15) is already in the integrated form, (14) can be regarded as a definition of $D(x)$ for preassigned $A(x)$, $B(x)$, and $C(x)$, and (12) can be regarded as a definition of u. Thus we are left with Eq. (10), where v is given by (15) and $D(x)$ by (14) for arbitrarily assigned $A(x)$, $B(x)$, and $C(x)$. However, solving (10) where e^{v} is given by (14) and (15), seems to be an impossible task, but the special case of $B(x)/A(x) = \text{const}$ can be solved as follows

3. SOLUTION FOR A SPECIAL CASE

Take

$$B(x) = kA(x), \tag{16}$$

where k is a constant. Here, from (8) and (9),

$$\left.\frac{dw}{du}\right|_{t=\text{const}} + 1 + ke^{w} = 0,$$

which on integration gives

$$e^{-w} + k = e^{u}, \tag{17}$$

where the arbitrary function of t arising in the integration has been absorbed in e^{u}. This is possible due to Eq. (1). Equation (17) reduces Eq. (12) to

$$t = \int_{x=\text{const}} \frac{e^{w}\,dw}{(ke^{w} + 1)\{[A^{2}(x) - C^{2}(x)]e^{-4w} + D(x)e^{-3w} + B^{2}(x)e^{-2w}\}^{1/2}} + L(x)\cdots. \tag{18}$$

Using (16) one can integrate (15) to get

$$k^{2}[A^{2}(x) - C^{2}(x)] = D(x) - 2kA^{2}(x) + l, \tag{19}$$

where l is an arbitrary constant.

We shall now show that if w is defined by (18) and u by (17), $A(x)$, $C(x)$, and $D(x)$ are connected by (19), and v is defined through (10), then that provides the complete set of solutions for the special case $B(x)/A(x) = \text{const}$ which is under consideration here. That Eqs. (10), (17)–(19) are necessary for the present case has been shown already. That they are sufficient to provide a solution can be seen as follows.

Equations (17) and (18) together ensures (12). Equations (10), (12), and (19) together with the consistency condition $\partial w/\partial x = \partial w'/\partial t$ give (13). For the present case, (19) is equivalent to (14); (12) and (13) together ensure (15). Thus when (16) holds, Eqs. (10) and (17)–(19) together are sufficient to ensure (10), (12), (14), and (15), which in turn are sufficient to ensure that all the field equations are satisfied.

4. CONCLUSION

Therefore, in summary, the coupled Einstein–Maxwell equations for a nonstatic charge-carrying dust distribution in a comoving frame with the most general plane symmetric metric (1) reduces to Eq. (10), where u and v are given, respectively, by (12) and (15), and further $A(x)$, $B(x)$, $C(x)$, and $D(x)$

are connected by (14). The complete solution has been possible only when $B(x)/A(x) = \text{const}$, in which case the solution is given as follows, w is given by (18), u by (17), v by (10), and $A(x)$, $B(x)$, $C(x)$, and $D(x)$ satisfy (16) and (19). A special case of this solution was previously reported by one of the authors in the case $B(x) = 0$.

Although the equations have been completely integrated for $B/A = \text{const}$, the resulting equations still appear to be too involved for physical conclusions to be drawn. However, the fact that a class of exact solutions of Einstein's equations for the metric (1) has been obtained seems to be significant, because, although an extensive literature exists for the solutions for Einstein's equations for the plane symmetic metric, $ds^2 = e^{2u(x,t)}(dt^2 - dx^2) - e^{2w(x,t)}(dy^2 + dz^2)$, which is a special case of (1) for $u = v$, very few solutions are known for the metric (1). The present paper may open up some possibilities in that direction.

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