Phase induced transport of a Brownian particle in a periodic potential in the presence of an external noise: A semiclassical treatment

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We develop, invoking a suitable system-reservoir model, the Langevin equation with a state-dependent dissipation associated with a quantum Brownian particle submerged in a heat bath that offers a state-dependent friction to study the directed motion (by studying the phase-induced current) in the presence of an external noise. We study the phase induced current when both system and bath are subjected to external modulation by the noise and thereby expose the system to two cross-correlated noises. We also demonstrate the well-known fact that two noises remain mutually correlated if they share a common origin. We study the effects of correlation on the current in a periodic potential and envisage that the steady state current increases with increase in the extent of correlation, implying that exercising control on the degree of correlation can enhance the current in a properly designed experiment. To establish our model, we analyze numerically the effect of the external noise on system and bath separately as well as on composition of both. © *2011 American Institute of Physics*. [doi:10.1063/1.3614776]

I. INTRODUCTION

It is now well documented in the current literature that the motion of a Brownian particle in phase space, under the influence of a potential, is an important physicochemical and biological process which plays a central role in many practical dynamical problems[1–4] and is quite frequently used in the treatment of various phenomena in quantum optics,[5] quantum tunnelling and coherence effects in condensed matter physics,[6–8] activated processes in chemical physics,[9–11] and so on. The conventional means for describing this kind of problem are the Fokker-Planck equations, or master equations, or the Langevin equations for the coordinate and the conjugated momentum of the particle in the external potential. In the classical domain, the corresponding theoretical description is well developed. In many cases, particularly at low temperatures, a theory of the classical Brownian motion may be inadequate because it ignores quantum effects. Quantum noise arising from quantum fluctuations is also important in nano-scale and biological systems, tunnelling and transfer of electrons, and quasi-particles. Most recently, it has regained considerable attention for quantum information processing, where noise appears as an unavoidable but yet undesirable effect.[12] The decay of a zero voltage state in a biased Josephson junction, flux quantum transitions in a SQUID,[9] and the possible reversal by quantum tunnelling of the magnetization of a single domain ferromagnetic particle are also nice examples of the broad applicability of the Brownian motion model for metastable decay.

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All these considerations initiated the development of formalisms for quantum Brownian motion and nonequilibrium dynamics of open quantum systems.[3,9,13] Despite the tremendous methodological developments,[3,9,13,14] very little is known about understanding of quantum effects on the Brownian motion in an external potential. The quantum mechanical description of the directed transport is only partially elaborated because the transport depends strongly on the mutual interplay of pure quantum effects such as tunnelling and particle-wave interference with the dissipation processes, nonequilibrium fluctuations, and external driving.[15] The quantum domain, unlike the classical regime, is devoid of an evolution equation for the density distribution. The quantum regime is typically attested by the presence of long range retardation forces stemming out of the interplay of quantum fluctuations in the surrounding heat bath and the system dynamics. There exist, however, certain formally exact expressions of the time dependent reduced density matrices derived on the basis of path integrals, but their implementations have remained restricted only to certain very simple model systems.[3] These developments thus chiefly remained hinged to the adiabatic limit, and in this regime the external force reduces to a binary mode of a periodic 'up' and 'down' switch.[16–18] The last two explorations have been implemented in the case of infinite tight-binding lattices. This has the advantage that the exact path integral expression can be exploited to arrive at a retarded master equation that can be further reduced to a time-local equation of motion in the Markovian limit.[19]

The Smoluchowski limit[20] of strong or high friction, however, offers an enormous simplification to what we have said above.[21] Such a simplification owes a lot to the fact that the velocity component of the phase space distribution of the Brownian particle relaxes to equilibrium quite rapidly and thereby facilitates a description through the position distribution alone for times that are appreciably larger than the velocity relaxation time. Thus, under such circumstances, a viable Kramers' equation emerges, which can be further cast in a structurally much simpler looking Smoluchowski equation[4,6] in the classical regime. Different approaches have been proposed in the recent years that are seemingly not wholly consistent with each other. A few years ago, quantum Brownian motion in the strong friction limit had been studied by Ankerhold-Pechukas-Grabert[22,23] using the exact path integral formulation of dissipative systems. They had shown that quantum fluctuations might appear at relatively elevated temperatures and substantially influence the quantum dynamics. The quantum (or semiclassical) Smoluchowski equation provides a semiclassical description of the evolution of the diagonal matrix elements of the system density operators in the coordinate representation and allows one to study the influence of thermal, as well as quantum fluctuations. In the strong friction regime (*Smoluchowski limit*), the time scale of relaxation of position surpasses all other time scales that shape the dynamics of the system, and hence in this limit, one enjoys the possibility of exploring the formally exact path integral expression to arrive at a time-local evolution equation for the marginal position distribution. One may also envisage the so-called quantum Smoluchowski equation in the low temperature regime[22] where the thermal energy scale $k_B T$ is much smaller in comparison to the energy scale for the friction $\hbar\gamma$, with γ being the friction constant. Conceptually, this domain is far from being classical which requires just the opposite condition $\hbar\gamma \ll k_B T$ even though quantum fluctuations can be treated perturbatively. As shown recently by Coffey *et al.*, the phase-space formalism (master equation for the Wigner distribution function) can be used to derive a semiclassical Smoluchowski equation for the configuration-space probability distribution function for noninertial translational Brownian motion via perturbation theory in $\hbar^2$ ($\hbar$ is Planck's constant).[24,25] In general, these methods allow one to achieve a deep understanding of the dynamics of dissipative quantum systems. The range of validity of the semiclassical Smoluchowski equation derived by Coffey *et al.*[24] has been discussed by Tsekov.[26] In this context, we also mention the work due to Maier and Ankerhold.[27]

The quantum Smoluchowski equation[22,23,27,28] has found applications involving systems that are driven as well as to those devoid of any external drive, as in the case of the study of the quantum decay rates for driven potential barriers,[29] quantum phase diffusion, and charging effects in Josephson junction,[30] quantum diffusion in tilted periodic potentials,[31,32] and the quantum versions of the nonequilibrium fluctuation theorems.[33] However, the exact expression of quantum Smoluchowski equation of non-driven as well as driven systems is still the subject of discussions and a rigorous derivation for driven quantum system is still lacking.[24–26,31,34]

In one of the recent advancements, an attempt has been made to study the quantum Langevin equation in terms of the classical approach[35] that implements the idea of a coherent state representation of the noise operator and a canonical thermal Wigner distribution of the bath oscillators. This, along with its different variants,[36,37] have found applications in successful explanation of several facets of reaction rate theory in condensed phases in a quantum mechanical context. Motivated by the success of the above mentioned theory, we have put forth a system-reservoir nonlinear coupling model for a quantum system in situations where the associated heat bath is not in thermal equilibrium and is rather modulated by an external noise agency,[38] as well as for a system driven by an external noise.[39] Both these developments[38,39] are attuned to a microscopic approach to the quantum state-dependent diffusion and multiplicative noise in terms of a quantum Langevin equation and succeeds by a semiclassical realization of it. In the present work, we contemplate a situation in which the actual system is a part of a larger system and the whole is being modulated externally by a noise (random force). The subsequent dynamics of the subsystem is then monitored to explore the possibility of the generation of quantum current and the influence of the external modulation on the quantum transport. The sections to follow will reveal the fact that our system-reservoir model (real systems interact with surrounding reservoirs which typically contain a macroscopic number of degrees of freedom and thus constitute heat baths) leads to a quantum Langevin equation in a situation when the Brownian particle is driven by a pair of mutually correlated noises - one additive and the other multiplicative with state-dependent dissipation, on top of the thermal noise. We mention that the model considered here differs substantially from the expression derived by Ford and O'Connell,[40] but the reason is obvious. The theory of Ford and O'Connell describes the motion of a classical Brownian particle in a quantum environment. This is evident from the fact that in the case of removal of the thermal bath, the remaining equation of motion of their particle is the Newtonian one. The subject of our formalism is just the opposite: a quantum particle moving in a semiclassical environment. Hence, without the thermal bath, our particle is described by the Schrödinger equation.

The exploration of nonlinear dynamical systems modulated by cross-correlated noises has become an area of avid research in recent times,[41] since a vast variety of physical phenomena are successfully explained on the basis of such processes. These includes the understanding of the statistical properties associated with the single mode lasers,[42] bistable kinetics,[43] barrier crossing dynamics,[44,45] steady state entropy production,[46] and noise induced transport.[47] It is now well accepted that the effect of correlation between additive and multiplicative noises is indispensable in explaining phenomena such as phase transition transport in motor proteins, etc., and the presence of cross-correlated noises changes the dynamic of the systems.[48] The ubiquitous presence of the cross correlated noises and the effect of the correlation between the additive and multiplicative noises has been established beyond doubt in explaining crucial physical and biological phenomena such as phase transitions and the transport of motor proteins and how the cross correlated noises shape the dynamics of the system.[47] In all of the aforesaid developments, the corresponding Langevin equation has been explored only in the classical regime, and as far as our knowledge goes, no genuine quantum mechanical development has been put forth till date.

The paper is organized as follows: Sec. II describes the model we use in the present paper. In Sec. III, we outline the development of Fokker-Planck equation in the context of our present model. Section IV illustrates numerical results of our present development to illustrate its applicability. Finally, Sec. V summarizes the present development and puts forth the concluding remarks.

II. THE MODEL: STOCHASTIC DYNAMICS OF QUANTUM OPEN SYSTEM

As mentioned in Sec. I, although in general, the Hamiltonian for the system-reservoir model is simple, the dynamical evolution of the system after appropriate elimination of the reservoir degrees of freedom poses challenging tasks for an exact solution. Depending on the phenomena, various approximations have been introduced to extract out the nature of the underlying quantum stochastic processes that govern the dynamics of the system.

We consider a model characterized by a particle of unit mass coupled to a reservoir composed of N number of mass weighted harmonic oscillators (bosonic reservoir) with frequency $\{\omega_j\}$. The system is in thermal equilibrium with the reservoir at temperature T initially, (i.e., when $t = 0$). At

$t = 0_+$, the system as well as the heat bath is subjected to the action of an external Gaussian random force $\epsilon(t)$, which has an arbitrary decaying correlation function. The total Hamiltonian of such a composite system can be written as

$$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}) + \sum_{j=1}^{N} \left\{ \frac{\hat{p}_j^2}{2} + \frac{1}{2}\omega_j^2 (\hat{x}_j - c_j f(\hat{q}))^2 \right\} - \sum_j \kappa_j \hat{x}_j \epsilon(t) + \hat{q}\epsilon(t), \tag{1}$$

where, $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators of the system (Brownian particle) and $\{\hat{x}_j, \hat{p}_j\}$ is the set of normal coordinate (position) and momentum operators for the bath oscillators (environmental oscillators). For clarity, in this part, we shall put hats on the operators, to distinguish them from the corresponding scalars wherever necessary; viz., operator $\hat{A}$, scalar A. The system particle is coupled to the bath oscillators nonlinearly through the general coupling term $c_j \omega_j^2 f(\hat{q})$ where, c_j is the coupling constant. The interaction between the heat bath and the external noise is represented by the term $H_{int} = \sum_j \kappa_j \hat{x}_j \epsilon(t)$, where κ_j denotes the strength of the interaction. The potential $V(\hat{q})$ is due to external force field for the system particle. The last term is the direct driving term. We consider $\epsilon(t)$ to be a stationary Gaussian noise with zero mean and arbitrary decaying correlation function,

$$\left.\begin{aligned} \langle \epsilon(t) \rangle &= 0 \\ \langle \epsilon(t)\epsilon(t') \rangle &= 2D\psi(t - t') \end{aligned}\right\}, \tag{2}$$

where D is the strength of the external noise, $\psi(t)$ is some arbitrary decaying memory kernel and $\langle \cdots \rangle$ implies averaging over each realization of the external noise $\epsilon(t)$. The coordinate and momentum operators satisfy the standard commutation relations,

$$\left.\begin{aligned} [\hat{q}, \hat{p}] &= i\hbar \\ \left[\hat{x}_j, \hat{p}_k\right] &= i\hbar\delta_{jk} \end{aligned}\right\}. \tag{3}$$

Eliminating the bath degrees of freedom, we get the following Langevin equation for the system particle,

$$\left.\begin{aligned} \dot{\hat{q}} &= \hat{p} \\ \dot{\hat{p}} &= -V'(\hat{q}) - f'(\hat{q}) \textstyle\int_0^t dt' \gamma(t - t') f'(\hat{q}(t'))\hat{p}(t') + f'(\hat{q})\hat{\eta}(t) + f'(\hat{q})\pi(t)\hat{I} + \epsilon(t)\hat{I} \end{aligned}\right\}, \tag{4}$$

where the dot denotes the time derivative and the prime indicates the derivative with respect to the coordinate. Here, $\hat{I}$ is the unit operator and

$$\pi(t) = \int_0^t dt' \varphi(t - t')\epsilon(t'), \tag{5}$$

$$\gamma(t) = \sum_{j=1}^{N} c_j^2 \omega_j^2 \cos \omega_j t, \tag{6}$$

$$\varphi(t) = \sum_{j=1}^{N} c_j \omega_j \kappa_j \sin \omega_j t. \tag{7}$$

$\gamma(t)$ expresses the memory kernel and $\pi(t)$ is the fluctuating force generated due to the external stochastic forcing of the bath by $\epsilon(t)$. $\hat{\eta}(t)$ is the internal thermal noise operator and is given by

$$\hat{\eta}(t) = \sum_{j=1}^{N} c_j \omega_j^2 \left\{ \left[\hat{x}_j(0) - c_j f(\hat{q}(0))\right] \cos \omega_j t + \frac{\hat{p}_j(0)}{\omega_j} \sin \omega_j t \right\} \tag{8}$$

with the following statistical characteristics:

$$\langle \hat{\eta}(t) \rangle_{QS} = 0, \tag{9}$$

$$\frac{1}{2}\langle \hat{\eta}(t)\hat{\eta}(t') + \hat{\eta}(t')\hat{\eta}(t) \rangle_{QS} = \frac{1}{2} \sum_{j=1}^{N} c_j^2 \omega_j^2 \hbar\omega_j \coth\left(\frac{\hbar\omega_j}{2k_B T}\right) \cos \omega_j (t - t'). \tag{10}$$

Here, $\langle \ldots \rangle_{QS}$ implies quantum statistical average on the bath degrees of freedom and is defined for any bath operator $\hat{O}(\hat{x}_j, \hat{p}_j)$ as follows (k_B is the Boltzmann constant):

$$\langle \hat{O} \rangle_{QS} = \frac{\mathrm{Tr}\left[\hat{O} \exp(-\frac{\hat{H}_B}{k_B T})\right]}{\mathrm{Tr}\left[\exp(-\frac{\hat{H}_B}{k_B T})\right]}, \tag{11}$$

where

$$\hat{H}_B = \sum_{j=1}^{N} \left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\omega_j^2 (\hat{x}_j - c_j f(\hat{q}))^2 \right], \quad \text{at} \quad t = 0. \tag{12}$$

We now carry out a quantum mechanical averaging of the operator equation, Eq. (4) to get

$$\langle \dot{\hat{q}} \rangle_Q = \langle \hat{p} \rangle_Q, \tag{13}$$

$$\begin{aligned} \langle \dot{\hat{p}} \rangle_Q = & -\langle V'(\hat{q}) \rangle_Q - \langle f'(\hat{q}) \int_0^t dt' \gamma(t - t') f'(\hat{q}(t')) \hat{p}(t') \rangle_Q \\ & + \langle f'(\hat{q}) \hat{\eta}(t) \rangle_Q + \langle f'(\hat{q}) \rangle_Q \pi(t) + \epsilon(t), \end{aligned} \tag{14}$$

where the quantum mechanical average $\langle \cdots \rangle_Q$ is taken over the initial product separable quantum states of the particle and the jth-bath oscillators at $t = 0$, $|\phi\rangle\{|\alpha_j\rangle; j = 1, 2, \ldots, N\}$. Here $|\phi\rangle$ denotes any arbitrary initial state of the system and $\{|\alpha_j\rangle\}$ corresponds to the initial coherent state of the bath oscillators. Since $\hat{\eta}(t)$ contains operators at time $t = 0$, one may write $\langle f'(\hat{q})\hat{\eta}(t) \rangle_Q = \langle f'(\hat{q}) \rangle_Q \langle \hat{\eta}(t) \rangle_Q$. This factorization is strictly valid for Markovian cases and with this factorization we neglect the initial correlation, if any, between the system and the bath. Equation (14) can be written as

$$\begin{aligned} \langle \dot{\hat{p}} \rangle_Q = & -\langle V'(\hat{q}) \rangle_Q - \langle f'(\hat{q}) \int_0^t dt' \gamma(t - t') f'(\hat{q}(t')) \hat{p}(t') \rangle_Q \\ & + \langle f'(\hat{q}) \rangle_Q \langle \hat{\eta}(t) \rangle_Q + \langle f'(\hat{q}) \rangle_Q \pi(t) + \epsilon(t), \end{aligned} \tag{15}$$

$\langle \hat{\eta}(t) \rangle_Q$ is now a classical like noise term, which, because of quantum mechanical averaging, in general, is a non-zero number and is given by

$$\langle \hat{\eta}(t) \rangle_Q = \sum_{j=1}^{N} \left[c_j \omega_j^2 \{ [\langle \hat{x}_j(0) \rangle_Q - c_j \langle f(\hat{q}(0)) \rangle_Q] \cos \omega_j t + \frac{\langle \hat{p}_j(0) \rangle_Q}{\omega_j} \sin \omega_j t \} \right]. \tag{16}$$

To realize $\langle \hat{\eta}(t) \rangle_Q$ as an effective $c-$number noise, we now introduce the Ansatz that the momenta $\langle \hat{p}_j(0) \rangle_Q$ and the shifted coordinate $\left(\langle \hat{x}_j(0) \rangle_Q - c_j \langle f(\hat{q}(0)) \rangle_Q \right)$ of the bath oscillators are distributed according to the canonical thermal Wigner distribution (canonical distribution of Gaussian) of the form,[49,50]

$$P_j = \mathcal{N} \exp \left\{ - \frac{\langle \hat{p}_j(0) \rangle_Q^2 + \omega_j^2 [\langle \hat{x}_j(0) \rangle_Q - c_j \langle \hat{q}(0) \rangle_Q]^2}{2\hbar\omega_j \left(\bar{n}_j(\omega_j) + \frac{1}{2} \right)} \right\}, \tag{17}$$

so that, for any quantum mechanical mean value of operator $\langle \hat{O} \rangle_Q$ which is a function of bath variables, its statistical average is

$$\langle \langle \hat{O} \rangle_Q \rangle_S = \int \left[\langle \hat{O} \rangle_Q P_j d\{\omega_j^2 [\langle \hat{x}_j(0) \rangle_Q - c_j \langle f(\hat{q}(0)) \rangle_Q]\} d\langle \hat{p}_j(0) \rangle_Q \right]. \tag{18}$$

Here, $\mathcal{N}$ is the normalization constant and P_j is the exact solution of Wigner equation for harmonic oscillator[49] and forms the basis for description of the quantum noise characteristics of the bath kept in thermal equilibrium at temperature T. It is worth stressing here that the Wigner representation contains only features common to both quantum and classical statistical mechanics and formally represents quantum mechanics as a statistical theory on classical phase space. In Eq. (17), $\bar{n}_j(\omega_j)$ is the average thermal photon number of the jth bath oscillator at temperature T and is given by

$$\bar{n}_j(\omega_j) = \left[\exp\left(\frac{\hbar\omega_j}{k_B T}\right) - 1\right]^{-1}. \tag{19}$$

The distribution P_j, given by Eq. (17) and the definition of statistical average imply that the c-number noise $\langle\hat{\eta}(t)\rangle_Q$ given by Eq. (16) must satisfy

$$\langle\langle\hat{\eta}(t)\rangle_Q\rangle_S = 0, \tag{20}$$

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{1}{2}\sum_{j=1}^{N} c_j^2\omega_j^2\hbar\omega_j \coth\left(\frac{\hbar\omega_j}{2k_B T}\right)\cos\omega_j(t-t'). \tag{21}$$

Equation (21) expresses the quantum fluctuation-dissipation relation. To identify Eqs. (13) and (15) as generalized Langevin equation, some conditions must be imposed on the coupling coefficients c_j and κ_j, on bath frequencies $\{\omega_j\}$ and on the number of bath oscillators (N) that will ensure that $\gamma(t)$ is indeed dissipative. For $\gamma(t)$ to be dissipative, it must be positive-definite and should decrease monotonically with time. These conditions are achieved if $N \to \infty$, and if $c_j^2\omega_j^2$ and ω_j are sufficiently smooth functions of j.[51] As $N \to \infty$, one replaces the sum by an integral over ω, weighted by the density of states $\mathcal{D}(\omega)$. Thus, to obtain a finite result in the continuum limit, the coupling functions $c_j = c(\omega)$ and $\kappa_j = \kappa(\omega)$ are chosen as $c(\omega) = \frac{c_0}{\omega\sqrt{\tau_c}}$ and $\kappa(\omega) = \kappa_0\omega\sqrt{\tau_c}$ respectively. Consequently, $\gamma(t)$ and $\varphi(t)$ reduce to the following form:

$$\left.\begin{aligned} &\gamma(t) = \tfrac{c_0^2}{\tau_c}\int_0^\infty d\omega\mathcal{D}(\omega)\cos\omega t \\ &\text{and}\quad \varphi(t) = c_0\kappa_0\int_0^\infty d\omega\omega\mathcal{D}(\omega)\sin\omega t \end{aligned}\right\}, \tag{22}$$

where c_0 and κ_0 are constants and $\omega_c = \frac{1}{\tau_c}$ is the cut-off frequency of the bath oscillators. τ_c may be characterized as the correlation time of the bath and $\mathcal{D}(\omega)$ is the density of modes of the heat bath which is assumed to be Lorentzian,

$$\mathcal{D}(\omega) = \frac{2}{\pi}\frac{1}{\tau_c(\omega^2 + \tau_c^{-2})}. \tag{23}$$

With these forms of $\mathcal{D}(\omega)$, $c(\omega)$, and $\kappa(\omega)$; $\gamma(t)$ and $\varphi(t)$ take the following forms:

$$\gamma(t) = \frac{c_0^2}{\tau_c}\exp(-\frac{t}{\tau_c}) = \frac{\Gamma}{\tau_c}\exp(-\frac{t}{\tau_c}), \tag{24}$$

and

$$\varphi(t) = \frac{c_0\kappa_0}{\tau_c}\exp(-\frac{t}{\tau_c}), \tag{25}$$

with $\Gamma = c_0^2$. For $\tau_c \to 0$, Eqs. (24) and (25) reduce to $\gamma(t) = 2\Gamma\delta(t)$ and $\varphi(t) = 2c_0\kappa_0\delta(t)$, respectively. The noise correlation function, Eq. (21) then becomes

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{1}{2}\frac{\Gamma}{\tau_c}\int_0^\infty d\omega\hbar\omega\coth\left(\frac{\hbar\omega}{2k_B T}\right)\cos\omega(t-t')\mathcal{D}(\omega). \tag{26}$$

Equation (26) is an exact expression for two time correlation.

Now, writing $q = \langle\hat{q}\rangle_Q$ and $p = \langle\hat{p}\rangle_Q$, we can re-write Eqs. (13) and (14) as

$$\dot{q} = p, \tag{27}$$

and

$$\dot{p} = -\langle V'(\hat{q})\rangle_Q - \Gamma\langle\left[f'(\hat{q})\right]^2\hat{p}\rangle_Q + \langle f'(\hat{q})\rangle_Q\eta(t) + \langle f'(\hat{q})\rangle_Q\pi(t) + \epsilon(t). \tag{28}$$

We now add $V'(q)$, $\Gamma\left[f'(q)\right]^2 p$, and $f'(q)\xi(t)$ to both sides of Eq. (28) and rearrange it to obtain

$$\dot{q} = p, \tag{29}$$

$$\dot{p} = -V'(q) + Q_V - \Gamma\left[f'(q)\right]^2 p + Q_1 + f'(q)\xi(t) + Q_2 + \epsilon(t), \tag{30}$$

where, $\xi(t) = \eta(t) + \pi(t)$ is the effective multiplicative noise and

$$\left.\begin{aligned} Q_V &= V'(q) - \langle V'(\hat{q})\rangle_Q \\ Q_1 &= \Gamma\left\{[f'(q)]^2 p - \langle [f'(\hat{q})]^2 \hat{p}\rangle_Q\right\} \\ Q_2 &= \xi(t)[\langle f'(\hat{q})\rangle_Q - f'(q)] \end{aligned}\right\}. \tag{31}$$

We now write the system operators $\hat{q}$ and $\hat{p}$ as

$$\left.\begin{aligned} \hat{q} &= q + \delta\hat{q} \\ \hat{p} &= p + \delta\hat{p} \end{aligned}\right\}, \tag{32}$$

where $q(= \langle\hat{q}\rangle_Q)$ and $p(= \langle\hat{p}\rangle_Q)$ are the quantum mechanical mean values and $\delta\hat{q}$ and $\delta\hat{p}$ are the corresponding quantum fluctuations. By construction, $\langle\delta\hat{q}\rangle_Q = \langle\delta\hat{p}\rangle_Q = 0$ and they also follow the usual commutation relation $[\delta\hat{q}, \delta\hat{p}] = i\hbar$. Using Eq. (32) in $V'(\hat{q})$, $[f'(\hat{q})]^2\hat{p}$ and in $f'(\hat{q})$, a Taylor series expansion in $\delta\hat{q}$ around $\hat{q}$, Q_V, Q_1, and Q_2 can be obtained as

$$Q_V = -\sum_{n\geq 2}\frac{1}{n!}V^{(n+1)}(q)\langle\delta\hat{q}^n\rangle_Q, \tag{33}$$

$$Q_1 = -\Gamma\left[2pf'(q)Q_f + pQ_3 + 2f'(q)Q_4 + Q_5\right], \tag{34}$$

$$Q_2 = \xi(t)Q_f, \tag{35}$$

where

$$\left.\begin{aligned} Q_f &= \textstyle\sum_{n\geq 2}\frac{1}{n!}f^{(n+1)}(q)\langle\delta\hat{q}^n\rangle_Q \\ Q_3 &= \textstyle\sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m!}\frac{1}{n!}f^{(m+1)}(q)f^{(n+1)}(q)\langle\delta\hat{q}^m\delta\hat{q}^n\rangle_Q \\ Q_4 &= \textstyle\sum_{n\geq 1}\frac{1}{n!}f^{(n+1)}(q)\langle\delta\hat{q}^n\delta\hat{p}\rangle_Q \quad \text{and} \\ Q_5 &= \textstyle\sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m!}\frac{1}{n!}f^{(m+1)}(q)f^{(n+1)}(q)\langle\delta\hat{q}^m\delta\hat{q}^n\delta\hat{p}\rangle_Q \end{aligned}\right\}. \tag{36}$$

It is now clear that Q_V represents quantum correction due to the nonlinearity of the system potential, Q_1 and Q_2 describe the quantum corrections owing to the nonlinear nature of the system-bath coupling function. Using Eqs. (33), (34), and (35), we get the dynamical equations for the system variables from Eqs. (29) and (30) as

$$\dot{q} = p, \tag{37}$$

$$\begin{aligned} \dot{p} = &-V'(q) + Q_V - \Gamma\left[f'(q)\right]^2 p - 2\Gamma pf'(q)Q_f - \Gamma pQ_3 \\ &- 2\Gamma f'(q)Q_4 - \Gamma Q_5 + f'(q)\xi(t) + Q_f\xi(t) + \epsilon(t). \end{aligned} \tag{38}$$

Note that in Eq. (38), the classical force term, V', is associated with its correction term, Q_V. Not only that the Γ terms containing Γ are nonlinear dissipative in nature where Q_f, Q_3, Q_4, and Q_5 are due to associated quantum contribution in addition to classical nonlinear dissipative term, $\Gamma\left[f'(q)\right]^2 p$. The above equations contain a multiplicative noise term $Q_f\xi(t)$ in addition to the usual classical contribution $f'(q)\xi(t)$. Moreover, quantum dispersion due to nonlinearity of the potential and the coupling function in the Hamiltonian make their presence felt in Eqs. (37) and (38). Therefore, quantum dispersions are entangled with nonlinearity. For a classical mechanical system, all the quantum contributions, namely, Q_V, Q_f, Q_3, Q_4, and Q_5 would be zero. Consequently, in the absence of external noise $\epsilon(t)$, Eq. (38) reduces to

$$\dot{p} = -V'(q) - \Gamma \left[f'(q)\right]^2 p + f'(q)\eta(t). \tag{39}$$

Equation (39) was obtained earlier by Lindenberg and Seshadri.[52]

In the overdamped limit, the adiabatic elimination of the fast variable is usually done by simply putting $\dot{p} = 0$. This adiabatic elimination provides the correct equilibrium distribution only when the dissipation is state independent. But, for the state dependent dissipation, the conventional adiabatic elimination of the fast variable in the overdamped limit does not provide the correct result. To obtain a correct equilibrium distribution, an alternative approach proposed by Sancho *et al.*[53] is used here to obtain the dynamical equation of motion for the position coordinate in the case of state dependent dissipation. Based on the Langevin equation, Sancho *et al.* carried out a systematic expansion of the relevant variables in powers of Γ^{-1} neglecting the terms smaller than $O(\Gamma^{-1})$. We follow the same procedure in our present development. In this limit, the transient correction terms Q_4 and Q_5 do not affect the dynamics of the position, which varies over a much slower time scale in the overdamped limit. So, the equations governing the dynamics of the system variables are

$$\dot{q} = p, \tag{40}$$

$$\dot{p} = -V'(q) + Q_V - \Gamma h(q)p + g(q)\xi(t) + \epsilon(t), \tag{41}$$

where

$$\left.\begin{aligned} h(q) &= [f'(q)]^2 + 2f'(q)Q_f + Q_3 \\ g(q) &= f'(q) + Q_f \end{aligned}\right\}. \tag{42}$$

The functions $h(q)$ and $g(q)$ arise due to the nonlinearity of the system-bath coupling function $f(q)$. One can now easily identify Eq. (42)) as the c-number analogue of the quantum Langevin equation [Eq. (4)], valid for Ohmic bath. In passing, we note that $\pi(t)$ (and hence $\xi(t)$) and $\epsilon(t)$ are not statistically independent

$$\langle \pi(t)\epsilon(t')\rangle = \langle \epsilon(t)\pi(t')\rangle = \beta(t,t'), \tag{43}$$

which we shall calculate later for a particular $\psi(t)$. One can easily observe that in Eqs. (40) and (41), two mutually correlated noises appear in the dynamical equation of the open system. This is not surprising as both the system and the heat bath are exposed to the same random force $\epsilon(t)$. The appearance of cross-correlated noises has already been encountered while explaining various physical phenomena. In recent years, the literature regarding the correlation effects of additive and multiplicative noises is extensive.[54–56] In the most initial publications, multiplicative and additive noises, present simultaneously in a physical process, have usually been treated as uncorrelated random variables as it has been assumed that they have different origins. However, Fulinski and Telejko[54] demonstrated that the noises in some stochastic process may have a common origin. They analyzed the interference among the additive and multiplicative white noise in the kinetics of bistable systems. Madureira *et al.*[55] have pointed out the probability of cross correlated noise in the ballast resistor model showing bistable behavior of the system. They have also illustrated that the suppression of escape rate by an order of magnitude can be observed by controlling the correlation strength between additive and multiplicative noise sources. The idea of mutually correlated noises has also been generalized to the other subjects of stochastic system such as stochastic resonance.[54–57] The idea that the same microscopic source can be responsible of correlated additive and multiplicative noises has been discussed by Borromeo *et al*[58] for the classical case. It is now well accepted that the effect and interplay of correlation between additive and multiplicative noise is indispensable in explaining various phenomena such as phase transition and transport of motor proteins.

In this section, we wish to develop the quantum Langevin equation for a system which remains coupled to the noise source both directly and through bath, in an attempt to explore directed motion in such cases. As an off shoot, we demonstrate in the present work the well-known fact that if we choose to work with a pair of noises that share a common origin, the noises are seen to remain mutually correlated at all future times. Though the current development is quantum mechanical, it is valid in the classical domain as well retaining the full spirit.

A. Two-time correlation, Wigner function, and the semiclassical approximation

We now proceed to analyze the two-time correlation function given by Eq. (21). The continuum version of this equation is

$$F(t-t') = \langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_{\rm Q}\rangle_{\rm S} = \frac{\Gamma}{\tau_c}\int_0^\infty d\omega \frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\left[\omega(t-t')\right]\mathcal{D}(\omega). \tag{44}$$

If we substitute the explicit form of $\mathcal{D}(\omega)$ as given in Eq. (23), $F(t-t')$ can be expressed as

$$F(t-t') = \frac{\Gamma}{\pi\tau_c^2}\int_0^\infty d\omega \frac{\hbar\omega}{\omega^2+\tau_c^{-2}}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\left[\omega(t-t')\right]. \tag{45}$$

The integrals that appear in either Eq. (44) or Eq. (45) are not analytically tractable; this remains so even if we choose to work with a different physically acceptable form for the density of states $\mathcal{D}(\omega)$. Thus, we find ourselves in an analytically hopeless situation. In passing, we mention that at high temperature ($k_BT \gg \hbar\omega$), or in the classical limit ($\hbar \to 0$), $F(t-t')$ decays on a time scale of τ_c. At lower temperatures, however, the corresponding function decays over most of its domain on a time scale of $\hbar/k_BT$, rather than τ_c. The time scale of $\hbar/k_BT$ is very large at low temperatures and is seen to obey an inverse power-law decay,[59] where the bath can dissipate excitations whose energies lie in the range $(0, \hbar\omega_c)$, with $\omega_c = 1/\tau_c$. The spontaneous fluctuations occur only in the range $(0, k_BT)$ if $k_BT < \hbar\omega_c$. This correlation time of the fluctuations is, therefore, the larger of $\hbar/k_BT$ and τ_c. As the temperature is lowered, the correlations in the fluctuations become increasingly long-lived, even for an infinitely broad bath spectrum. Thus, the fluctuations and dissipation may enjoy widely apart time scales. In the classical regime ($\hbar \to 0$), or in the high temperature limit ($k_BT \gg \hbar\omega$), $\coth\left(\frac{\hbar\omega}{2k_BT}\right)$ can be approximated as $2k_BT/\hbar\omega$, and consequently, $F(t-t')$ can be written as

$$F(t-t') \approx \frac{2\Gamma k_BT}{\pi\tau_c^2}\int_0^\infty d\omega \frac{\cos\left[\omega(t-t')\right]}{\omega^2+\tau_c^{-2}} = \frac{\Gamma k_BT}{\tau_c}\exp\left(-\frac{|t-t'|}{\tau_c}\right). \tag{46}$$

In the limit of $\tau_c \to 0$, Eq. (46) reduces to a δ-correlated noise process,

$$F(t-t') = 2\Gamma k_BT\,\delta(t-t'). \tag{47}$$

This is the well-known Einsteins's fluctuation-dissipation relation in the Markovian limit. Eqs. (46) and (47) clearly demonstrate that in the classical limit, $F(t-t')$ is either proportional to some exponential or is proportional to a δ function in time, and both fluctuation and dissipation share a common origin with the same correlation. This is, however, not the case in the quantum domain, where the fluctuation and the dissipation possess characteristically different correlation times. Here, the correlation time for fluctuation is $\hbar/k_BT$, and is independent of the friction constant Γ and progressively becomes longer as the temperature decreases.

The above discussion is an indication of the ensuing formal complexity of the quantum case since $F(t-t')$ does not lend tractability to an analytically closed form with simple mathematical functions as achieved in the classical domain. Therefore, any quantum mechanical treatment warrants a situation that compels us to resort to some physically motivated approximation.

At this point we digress a little more on the issue that we have stated above. The reason for us being able to express $F(t-t')$ in terms of a δ function in time in the classical limit ($\hbar \to 0$) stems out from the fact that the term $\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$ in Eq. (44) in the classical limit becomes independent of ω, and yields k_BT, the average energy associated with an equilibrated bath at a temperature T. This is very much reminiscent of the existence of a principle of equipartition of energy (that arises out of a Boltzmann distribution) in the classical regime. However, the quantum case does not afford such a simplification. Nevertheless, we assume that despite having a frequency distribution, all the bath oscillators enjoy the same frequency Ω_0, so that, even in the quantum case, the mean energy of this quantum heat bath after attaining the thermodynamic equilibrium at temperature T is given by[60]

$$\langle E\rangle = \left(\frac{\hbar\Omega_0}{2}\right)\left[\frac{\exp\left(\frac{\hbar\Omega_0}{k_B T}\right)-1}{\exp\left(\frac{\hbar\Omega_0}{k_B T}\right)+1}\right] = \left(\frac{\hbar\Omega_0}{2}\right)\coth\left(\frac{\hbar\Omega_0}{2k_B T}\right). \tag{48}$$

Therefore, the integral in Eq. (44) affords an analytic handling, since now, the term $\frac{\hbar\Omega_0}{2}\coth\left(\frac{\hbar\Omega_0}{2k_B T}\right)$ no longer remains a part of the integrand. Therefore, Eq. (44), in the quantum case, with the above approximation in operation, takes the form

$$F(t-t') = \frac{\Gamma}{\tau_c}\frac{\hbar\Omega_0}{2}\coth\left(\frac{\hbar\Omega_0}{2k_B T}\right)\int_0^\infty d\omega \cos\left[\omega(t-t')\right]\mathcal{D}(\omega). \tag{49}$$

With a choice of $\mathcal{D}(\omega)$ as in Eq. (23), and with $\tau_c \to 0$, we obtain, for the quantum case

$$F(t-t') = \langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_{\mathrm{Q}}\rangle_{\mathrm{S}} = 2D_q\delta(t-t') \tag{50}$$

with

$$D_q = \frac{1}{2}\Gamma\hbar\Omega_0\coth\left(\frac{\hbar\Omega_0}{2k_B T}\right). \tag{51}$$

We would have arrived at the same results as in Eqs. (50) and (51) had we started with a canonical thermal Wigner distribution as

$$\mathcal{P}_j\left(\left[\langle\hat{x}_j(0)\rangle - c_j\langle\hat{q}(0)\rangle\right], \langle\hat{p}_j(0)\rangle\right) = \mathcal{N}\exp\left\{-\frac{\langle\hat{p}_j(0)\rangle^2 + \omega_j^2\left[\langle\hat{x}_j(0)\rangle - c_j\langle\hat{q}(0)\rangle\right]^2}{2\hbar\Omega_0\left[\overline{n}(\Omega_0)+\frac{1}{2}\right]}\right\} \tag{52}$$

instead of Eq. (17), Eq. (52) is a truly quantum mechanical expression with some approximation as stated above. Essentially, through the approximation that we have just made, we accept the action of some sort of equipartition principle in the quantum domain that yields the correct result of $k_B T$ in the high temperature classical domain. The existence of such a common frequency Ω_0 for all the bath modes has also been advocated, under the name of *average bath frequency*, by Ray and co-workers.[36,37] In their work,[36,37] they did not attribute any additional qualification to Ω_0. In the discussion to follow, we address this issue and assign a physically motivated significance to Ω_0.

Wigner[49] showed that quantum mechanics can be reformulated in terms of a phase space quasi-probability distribution,

$$W(q,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dy\mathcal{D}\left(q+\frac{1}{2}y, q-\frac{1}{2}y\right)\exp\left(-\frac{i}{\hbar}py\right), \tag{53}$$

where $\mathcal{D}(q,q') = \langle q|\hat{\mathcal{D}}|q\rangle$ is the density matrix. The Wigner function $W(q,p,t)$ exhibits most of the properties of a classical phase space distribution. The expectation value of a quantum mechanical operator $\hat{O}$ may then be calculated using the Wigner function from the corresponding classical variable $O(q,p)$ as

$$\langle O(t)\rangle = \int W(q,p,t)O(q,p)dqdp. \tag{54}$$

For closed quantum systems, the time behavior of the Wigner function is governed by an evolution equation equivalent to the Schrödinger equation. The Wigner formalism may also be applied to an open quantum system.[61] In particular, it provides a useful tool for introducing quantum correction to classical models of dissipation such as Brownian motion (that is the fate of a heavy particle immersed in a fluid of lighter particles).[24,25,61–64] In this context we consider a dynamical system characterized by a potential $V(q)$ coupled bilinearly to the environment. The latter is modelled as a set of harmonic oscillators with frequecies $\{\omega_j\}$. The oscillators represent the normal modes of the bath. The quantum dynamics of the bare system with unit mass ($m = 1$) is governed by the Wigner equation[49] as given by

$$\left(\frac{\partial W}{\partial t}\right)_{\text{dynamical}} = \hat{M}_W W, \tag{55}$$

where

$$\hat{M}_W W = -v\frac{\partial W}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial W}{\partial v} + \sum_{n\geq 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial q^{2n+1}}\frac{\partial^{2n+1}W}{\partial v^{2n+1}}. \tag{56}$$

The above c-number description given by Eq. (56) is equivalent to the full Schrödinger equation. The quantum correction to the classical Louville motion is contained in the $\hbar$ containing terms within the summation. The dissipative time evolution of the Wigner distribution function $W(q, v, t)$, on the other hand, for arbitrarily strong damping can be described by[61]

$$\left(\frac{\partial W}{\partial t}\right)_{\text{dissipative}} = \gamma\frac{\partial}{\partial v}(vW) + D\frac{\partial^2 W}{\partial v^2}, \tag{57}$$

where γ and D represent the dissipation constant and the diffusion coefficient, respectively. The first term in Eq. (57) is a direct consequence of the γ dependent term in the imaginary part of the exponent in the expression of the propagator for the density operator in the Feynman-Vernon theory[65] and has been shown to be responsible for the appearance of a damping force in the classical equation of motion for the Brownian particle to ensure quantum-classical correspondence. γ and D are related by the relation,

$$D = \frac{\gamma}{2}\hbar\omega_0 \coth\left(\frac{\hbar\omega_0}{2k_B T}\right). \tag{58}$$

(In the classical domain or in the high temperature limit, $D \to \gamma k_B T$ and $W(q, v, t)$ reduces to the classical phase-space distribution function.) In Eq. (58), ω_0 is the renormalized linear frequency of the nonlinear system.[61] We emphasize here that while Eq. (57) represents the semiclassical Brownian dynamics due to the coupling of an Ohmic environment to the system, Eq. (56) incorporates the full quantum effects of the system. The overall dynamics is then given by[66–68]

$$\frac{\partial W}{\partial t} = \left(\frac{\partial W}{\partial t}\right)_{\text{dynamical}} + \left(\frac{\partial W}{\partial t}\right)_{\text{dissipative}}. \tag{59}$$

The above Wigner-Leggett-Caldeira equation, Eq. (59), may have been the starting point of further analysis as in Ref. 66, however, we note that γ and D terms in Eq. (57) are strictly valid in Eq. (59), provided the system operators pertain to a harmonic oscillator. When the system is nonlinear, instead of Eq. (59), we assume the following generic form for the quantum master equation in the Wigner representation,

$$\frac{\partial W}{\partial t} = -v\frac{\partial W}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial W}{\partial v} + \gamma\frac{\partial}{\partial v}(vW) + \frac{\partial}{\partial v}D_{pp}\frac{\partial W}{\partial v} + \sum_{n\geq 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial q^{2n+1}}\frac{\partial^{2n+1}W}{\partial v^{2n+1}}, \tag{60}$$

where D_{pp} represents the coordinate and momentum dependent parameters to be determined keeping in mind that

1. in the classical limit ($\hbar \to 0$) or in the high temperature limit ($\hbar\omega/k_B T \ll 1$), $D_{pp} = \gamma k_B T$, and
2. when the system potential is harmonic, $D_{pp} = D$, as given in Eq. (58).

In passing, we note that we have allowed D_{pp} to include the nonlinearity of the system potential entangled with the quantum correction terms.

At this point we mention that some authors[24,25,64] have heuristically introduced, in Eq. (60), an additional mixed diffusion coefficient term D_{qp} in the (q, p) space depending on the color of the quantum noise. But, as this term is absent in the original formulation,[61] we do not include the mixed diffusion term in our equation.

In order to determine the explicit form of D_{pp}, we first recall Wigner's result[49] for the unnormalized equilibrium distribution $W_0(q, p)$, a stationary solution of

$$\frac{\partial W_0}{\partial t} = \hat{M}_W W_0(q, p),$$

and hence may be developed in a power series in $\hbar^2$[49] as

$$W_0(q, p) = e^{-\beta\epsilon(q,p)} + \hbar^2 W_2(q, p) + \hbar^4 W_4(q, p) + \ldots, \tag{61}$$

where $\beta = 1/k_B T$ and $\epsilon = \frac{v^2}{2} + V(q)$. Following Wigner,[49] one can easily show that

$$\begin{aligned} W_0(q, p) = e^{-\beta\epsilon(q,p)} \Bigg\{ 1 + \Lambda\left(\beta v^2 V'' - 3V'' + \beta V'^2\right) + 3\Lambda^2 \Bigg[v^4 \left(\frac{\left(\beta V''\right)^2}{6} - \frac{\beta V^{(4)}}{10} \right) \\ + v^2 \left(V^{(4)} - \frac{2\beta V''' V'}{5} + \frac{V''\left(\beta V'\right)^2}{3} - \frac{9\beta V''^2}{5} \right) + \frac{\beta^2 V'^4}{6} + \frac{5V''^2}{2} \\ - \frac{9\beta V'' V'^2}{5} + 2V''' V' - \frac{3V^{(4)}}{2\beta} \Bigg] \Bigg\} + \ldots, \end{aligned} \tag{62}$$

where

$$\Lambda = \hbar^2 \beta^2 / 24 \tag{63}$$

is the characteristic quantum parameter. Now, following Coffey and co-workers,[24] we make the Ansatz that $W_0(q, p)$ will also be the equilibrium solution of the generic master equation, Eq. (60). This implies that, $W_0(q, p)$ must satisfy

$$\frac{\partial}{\partial v} D_{pp} \frac{\partial W_0}{\partial v} + \gamma \frac{\partial}{\partial v} (v W_0) = 0. \tag{64}$$

By seeking a solution for D_{pp} of the form

$$D_{pp}(q, p) = \frac{\gamma}{\beta} + \hbar^2 d_2^{pp}(q, p) + \hbar^4 d_4^{pp}(q, p) + \ldots \tag{65}$$

and by substituting Eq. (62) into Eq. (60), one then finds that W_0 satisfies Eq. (64), if D_{pp} has the form,

$$D_{pp}(q, p) = \frac{\gamma}{\beta} \left\{ 1 + 2\Lambda V''(q) - \frac{2\Lambda^2}{5} \left[6V'''(q) V'(q) + 2V''(q)^2 + 3V^{(4)}(q) \left(v^2 - \frac{5}{\beta} \right) \right] + \ldots \right\}. \tag{66}$$

Thus, the explicit form of the master equation, Eq. (60), containing terms up to $O\left(\hbar^4\right)$ is

$$\begin{aligned} \frac{\partial W}{\partial t} + v\frac{\partial W}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial W}{\partial v} + \frac{\hbar^2}{24}\frac{\partial^3 W}{\partial v^3} - \frac{\hbar^4}{1920}\frac{\partial^5 V}{\partial q^5}\frac{\partial^5 W}{\partial v^5} + \ldots = \gamma \frac{\partial}{\partial v} \\ \times \left[vW + \frac{1}{\beta} \left\{ 1 + \frac{\hbar^2 \beta^2}{12} V'' - \frac{\hbar^4 \beta^4}{1440} \left[6V''' V' + 2V''^2 + 3V^{(4)} \left(v^2 - \frac{5}{\beta} \right) \right] \right\} \frac{\partial W}{\partial v} \right]. \end{aligned} \tag{67}$$

The above master equation, Eq. (67), can be exploited for any arbitrarily general potential $V(q)$. For a harmonic potential $V(q) = \omega_0^2 q^2 / 2$, from Eq. (66) we see that

$$D_{pp}(q, p) = \frac{\gamma}{\beta} \left[1 + \frac{\hbar^2 \beta^2 \omega_0^2}{12} - \frac{\hbar^4 \beta^4 \omega_0^4}{720} + \ldots \right] = \gamma \frac{\hbar\omega_0}{2} \coth\left(\frac{\beta\hbar\omega_0}{2} \right). \tag{68}$$

Thus, for the harmonic oscillator, $D_{pp} = D$, as given by the Caldeira-Leggett formulation,[61] and the master equation, Eq. (67), reads for the harmonic oscillator as

$$\frac{\partial W}{\partial t} + v\frac{\partial W}{\partial q} - \omega_0^2 q \frac{\partial W}{\partial v} = \gamma \frac{\partial}{\partial v} \left[vW + \frac{\hbar\omega_0}{2} \coth\left(\frac{\beta\hbar\omega_0}{2} \right) \frac{\partial W}{\partial v} \right], \tag{69}$$

which is the Leggett-Caldeira equation as obtained in Ref. 61. It is worth stressing that except the first, the other terms of D_{pp} are nonlinear with respect to the Planck constant $\hbar$ and measure typical quantum fluctuations.

Now, for the harmonic oscillator, $V(q) = \omega_0^2 q^2/2$, $Q_V = 0$, and in the absence of external noise $\epsilon(t)$, Eqs. (40) and (41) reads

$$\left.\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\omega_0^2 q - \Gamma p + \eta(t) \end{aligned}\right\}. \tag{70}$$

If we assume

$$\langle \eta(t)\eta(t')\rangle = \Gamma\hbar\omega_0 \coth\left(\frac{\hbar\omega_0}{2k_B T}\right)\delta(t-t')$$

the Fokker-Planck equation associated with Eq. (70) resembles Eq. (69) with $\gamma = \Gamma$. Thus, at least for the harmonic oscillator, if the noise correlation function is taken as Eq. (50), with Ω_0 being the frequency of the oscillator, one is able to recover the semiclassical result exactly. Now, Eq. (66) reveals that the nonlinearity in the system potential includes other terms in the diffusion coefficient D_{pp} through various higher derivatives of $V(q)$, and these are the terms that are responsible for the departure of D_{pp} from D_q. Thus, if one takes Eq. (50) as the two time correlation of the noise $\eta(t)$ with Ω_0 being the linear frequency of the anharmonic system, then with the Langevin equation [Eq. (70)], and the Wigner distribution [Eq. (52)], we obtain a result which is exact for the harmonic oscillator and includes the leading order quantum corrections, beyond the harmonic approximation, for any anharmonic system. As an off-shoot of our present analysis, we propose an equipartition theorem in the semiclassical domain through Eq. (52).

Returning back to the Langevin equation in the presence of an external noise, the main results are summarized as follows. The semiclassical Langevin equation for the system, when the system and the associated bath are exposed to an external noise $\epsilon(t)$ is given by [Eqs. (40) and (41)] (for Ohmic dissipation)

$$\left.\begin{aligned} \dot{q} &= p \\ \dot{p} &= -V'(q) + Q_V - \Gamma h(q)p + g(q)\left[\eta(t) + \pi(t)\right] + \epsilon(t) \end{aligned}\right\}, \tag{71}$$

where $\langle \eta(t)\rangle_S = 0$, $\langle \epsilon(t)\rangle_S = 0$, and

$$\langle \eta(t)\eta(t')\rangle_S = 2D_q\delta(t-t'), \tag{72a}$$

$$D_q = \frac{1}{2}\Gamma\hbar\Omega_0 \coth\left(\frac{\hbar\Omega_0}{2k_B T}\right) \tag{72b}$$

Ω_0 being the linear frequency of the system. The definition of averaging in terms of the Wigner distribution has been given earlier. A relevant pertinent point that needs attention in this context is that the original development by Ray and co-workers[36,37] and its subsequent implementations by Ray Chaudhuri and co-workers[38,39] defined Ω_0 as the average bath frequency.

In terms of an effective classical noise, $\xi(t) = \eta(t) + \pi(t)$, the Langevin equation [Eq. (71)] can be written as

$$\ddot{q}(t) = -V'(q) + Q_V - \Gamma h(q)\dot{q}(t) + g(q)\xi(t) + \epsilon(t). \tag{73}$$

The effective noise $\xi(t)$ thus has the statistical properties

$$\left.\begin{aligned} \langle \xi(t)\rangle_S &= 0 \\ \langle \xi(t)\xi(t')\rangle_S &= \tfrac{D_R}{\tau_R}\exp\left(-\tfrac{|t-t'|}{\tau_R}\right) \end{aligned}\right\}, \tag{74}$$

where, $D_R = D_q + \Gamma D_e \kappa_0^2$, and $\tau_R = \frac{D_e}{D_R}\Gamma\kappa_0^2\tau_e$, with D_R and τ_R being the strength and the correlation time of the effective noise $\xi(t)$, respectively. In passing, we mention that although the reservoir is being driven by a colored noise $\epsilon(t)$, with the noise intensity D_e and the correlation time τ_e, the dynamics of the system of interest is governed by the scaled colored noise $\xi(t)$ with a noise strength of D_R.

III. FOKKER-PLANCK DESCRIPTION AND PHASE-INDUCED CURRENT

Before examining the noise induced transport in a periodic potential, it is instructive to consider the particular noise properties. In the Langevin equation [Eq. (41)], the multiplicative noise term $g(q)\xi(t)$ actually consists of two terms, namely, $g(q)\eta(t)$ and $g(q)\pi(t)$. $\Gamma h(q)$ is the state-dependent damping. $\eta(t)$ is the thermal noise which arises due to the system-bath interaction and the dressed noise $\pi(t)$ arises due to bath modulation by the external noise $\epsilon(t)$. The statistical properties of $\pi(t)$ are given by

$$\left.\begin{aligned}\langle\pi(t)\rangle &= 0\\ \langle\pi(t)\pi(t')\rangle &= 2D\int_0^t dt''\int_0^{t'} dt'''\varphi(t-t'')\varphi(t'-t''')\psi(t''-t''')\end{aligned}\right\}, \tag{75}$$

where $\langle\cdots\rangle$ denotes the ensemble average over each realization of $\epsilon(t)$. At this point, we assume that the external noise $\epsilon(t)$ is δ-correlated,

$$\langle\epsilon(t)\epsilon(t')\rangle = 2D_e\delta(t-t'). \tag{76}$$

Then the correlation function of $\pi(t)$ becomes in the limit $\tau_c \to 0$,

$$\langle\pi(t)\pi(t')\rangle = 2\Gamma\kappa_0^2 D_e\delta(t-t'), \tag{77}$$

and consequently, the effective noise $\xi(t)[=\eta(t)+\pi(t)]$ has a correlation

$$\langle\xi(t)\xi(t')\rangle = 2\Gamma D_0\delta(t-t'), \tag{78}$$

where

$$D_0 = \hbar\Omega_0\left[\bar{n}(\Omega_0)+\frac{1}{2}\right] + D_e\kappa_0^2. \tag{79}$$

It has been pointed earlier that $\pi(t)$ and $\epsilon(t)$ are not uncorrelated. With the above noise properties of $\epsilon(t)$, one can easily find that

$$\langle\pi(t)\epsilon(t')\rangle = \langle\epsilon(t)\pi(t')\rangle = 2c_0\kappa_0 D_e\delta(t-t'). \tag{80}$$

Now, in terms of an auxiliary function $G(q)$ and Gaussian noise $R(t)$, the Langevin equations [Eqs. (40) and (41)] can be written as

$$\dot{q} = p, \tag{81}$$

$$\dot{p} = -V'(q) + Q_V + \Gamma h(q)p + G(q)R(t), \tag{82}$$

where

$$\left.\begin{aligned}\langle\langle R(t)\rangle\rangle &= 0\\ \langle\langle R(t)R(t')\rangle\rangle &= 2\delta(t-t')\end{aligned}\right\}, \tag{83}$$

where $\langle\langle\ldots\rangle\rangle$ implies average over the noise process $R(t)$ (this averaging over $R(t)$ consists of two independent averaging, averaging over thermal noise $\eta(t)$ and over external noise $\epsilon(t)$. In Eq. (82),

$$G(q) = \left[\Gamma D_0 g^2(q) + 2c_0\kappa_0 D_e g(q) + D_e\right]^{1/2}. \tag{84}$$

Now, using van Kampen's lemma[69] and Novikov theorem,[70] one may easily obtain the Fokker-Planck-Smoluchowski equation corresponding to Eqs. (81) and (82) supplemented by Eqs. (83) and (84) as

$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\left[\frac{V'(q)-Q_V}{\Gamma h(q)}\right]P(q,t) + \frac{\partial}{\partial q}\left[\frac{G(q)G'(q)}{\Gamma h^2(q)}\right]P(q,t) + \frac{\partial}{\partial q}\left[\frac{G(q)}{\Gamma h(q)}\frac{\partial}{\partial q}\frac{G(q)}{\Gamma h(q)}\right]P(q,t). \tag{85}$$

According to Stratonovich prescription,[71] the Langevin equation corresponding to Fokker-Planck equation is

$$\dot{q} = -\frac{V'(q) - Q_V}{\Gamma h(q)} - D_0 \frac{G(q)G'(q)}{\Gamma h^2(q)} + \frac{G(q)}{\Gamma h(q)} R(t). \tag{86}$$

Equation (86) is the c-number quantum Langevin equation for multiplicative noise with state-dependent dissipation in the overdamped limit (refers to large Γ rather than $\Gamma[f'(q)]^2$), when the associated heat bath and the system are driven by an external noise $\epsilon(t)$. Since we are dealing with high temperature overdamped situation, our analysis is valid in the regime $\Gamma\hbar\Omega_0 \ll 1$.

The Fokker-Planck equation, Eq. (85), can be written in a more compact form as

$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\frac{1}{\Gamma h(q)}\left[V'(q) - Q_V + \frac{D_0}{\Gamma}\frac{\partial}{\partial q}\frac{G^2(q)}{h(q)}\right] P(q,t) \tag{87}$$

for which the stationary current can be obtained as

$$J = -\frac{1}{\Gamma h(q)}\left[\frac{D_0}{\Gamma}\frac{\partial}{\partial q}\frac{G^2(q)}{h(q)} + V'(q) - Q_V\right] P_{st}(q). \tag{88}$$

Integrating Eq. (88) we obtain the expression of stationary probability density in terms of stationary current as

$$P_{st}(q) = \frac{e^{-\phi(q)}h(q)}{G^2(q)}\left[\frac{G^2(0)}{h(0)}P_{st}(0) - J\frac{\Gamma^2}{D_0}\int_0^q h(q')e^{\phi(q')}dq'\right], \tag{89}$$

where

$$\phi(q) = \frac{\Gamma}{D_0}\int_0^q \frac{\left\{V'(q') - Q_V\right\} h(q')}{G^2(q')} dq' \tag{90}$$

is the effective potential. Any tilt of the effective potential $\phi(q)$ makes the transition between the left to right and right to left of the Brownian particle unequal and thereby creates a directed mass motion. If $\frac{V'(q)h(q)}{G^2(q)}$ and $\frac{Q_V h(q)}{G^2(q)}$ are periodic with the same period, there will be no symmetry breaking of the initial potential and $\phi(q)$ will remain symmetric periodic with the same period. Nonintegrability of $\phi(q)$ will create spatial symmetry breaking and consequently, there will be a net mass flow in a particular direction.

We now consider a symmetric periodic potential with periodicity 2π and periodic derivative of the coupling function with the same periodicity, i.e., $V(q+2\pi) = V(q)$ and $f'(q+2\pi) = f'(q)$. Consequently, Q_V, $h(q)$ and $G(q)$ are also periodic functions of q with period 2π. Now, applying the periodic boundary condition on P_{st}, i.e., $P_{st}(q+2\pi) = P_{st}$, we have from Eq. (89)

$$\frac{G^2(0)}{h(0)}P_{st}(0) = \frac{\Gamma^2}{D_0}J\left[\frac{1}{1-e^{\phi(2\pi)}}\right]\int_0^{2\pi} h(q)e^{-\phi(q)}dq. \tag{91}$$

By applying the normalization condition on the stationary probability distribution given by $\int_0^{2\pi} P_{st}(q)dq = 1$, we get from Eq. (89),

$$\int_0^{2\pi}\frac{e^{-\phi(q)}h(q)}{G^2(q)}\left[\frac{G^2(0)}{h(0)}P_{st}(0) - J\frac{\Gamma^2}{D_0}\int_0^q dq' h(q')e^{\phi(q')}\right]dq = 1. \tag{92}$$

Now, eliminating $\frac{D_0}{\Gamma}\frac{G^2(0)}{h(0)}P_{st}(0)$ from Eqs. (89) and (92) one obtains the expression for stationary current as

$$J = \frac{D_0}{\Gamma^2}\left(1 - e^{\phi(2\pi)}\right)\left\{\int_0^{2\pi}\frac{h(q)}{G^2(q)}e^{-\phi(q)}dq\int_0^{2\pi} h(q')e^{\phi(q')}dq' - \left[1 - e^{\phi(2\pi)}\right]\int_0^{2\pi}\left(\frac{h(q)}{G^2(q)}e^{-\phi(q)}dq\int_0^q h(q')e^{\phi(q')}dq'\right)dq\right\}^{-1}. \tag{93}$$

From the condition of periodicity of potential and different quantum correction terms it is apparent that for the periodic potential and the periodic derivative of coupling function with same periodicity,

$\phi(2\pi) = 0$. Therefore, the numerator of the expression of current Eq. (93) reduces to zero. We thus conclude that there is no occurrence of current for a periodic potential and periodic derivative of coupling function with same periodicity as there will be no symmetry breaking mechanism.

To this end, we now indicate the relationship of our present formulation with other methods, such as Ankerhold *et al.*,[22] Coffey and co-workers,[24,25] and Lutz *et al.*[28] bearing kinship with ours. For linear system-reservoir coupling ($f(q) = q$), and in absence of external noise ($D_e = 0$), Eq. (87) will reduce to the following form:

$$\frac{\partial P(q,t)}{\partial t} = \frac{1}{\Gamma}\frac{\partial}{\partial q}\left[\left\{V'(q) + \sum_{n\geq 2}\frac{1}{n!}V^{(n+1)}(q)\langle\delta\hat{q}^n\rangle_Q\right\} + \frac{\partial}{\partial q}\left(\frac{D_q}{\Gamma}\right)\right]P(q,t), \tag{94}$$

where we have provided the explicit mathematical form for Q_V. Now, for harmonic oscillator, $V(q) = \Omega_0^2 q^2/2$, the term Q_V vanishes, and upon retaining terms correct up to $\hbar^2$, we obtain

$$\frac{\partial P}{\partial t} = \frac{1}{\Gamma}\frac{\partial}{\partial q}\left[V'(q) + \frac{1}{\beta}\frac{\partial}{\partial q}D_{\text{eff}}\right]P(q,t), \tag{95}$$

where $D_{\text{eff}} = 1 + 2V''(q)\Lambda$. Eq. (95) has the same form as obtained by Ankerhold *et al.*[22,23] and Lutz *et al.*[28]

We now illustrate the close resemblance of the present development with that of Coffey and co-workers.[25] To proceed in this direction, we now recall Eqs. (3) and (4) of Ref. 25,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial q}\left\{\frac{P}{\zeta}\frac{\partial V}{\partial q} + \frac{\partial}{\partial q}(DP)\right\}, \tag{96}$$

where $D(q)$ is the diffusion coefficient given by Coffey and co-workers[25] as

$$D(q) = \frac{1}{\beta\zeta}\left(1 + 2\Lambda V''(q) - \frac{4\Lambda^2}{5}\left\{\left[V''(q)\right]^2 + 3V'(q)V^{(3)}(q) - 3\beta^{-1}V^{(4)}(q)\right\} + \ldots\right). \tag{97}$$

Here $\Lambda = (\hbar\beta)^2/(24m)$ and ζ used by Coffey and co-workers[25] is same as Γ used by us in the present development. If we retain terms correct up to $\hbar^2$ in accordance with the above discussion, Eq. (97) assumes the form,

$$D(q) = \frac{1}{\beta\zeta}\left[1 + 2\Lambda V''(q)\right], \tag{98}$$

which is identical to our expression given by Eq. (95). It is enlightening to point out at this juncture that the quantum Smoluchowski equation deduced initially by Coffey *et al.*[24] is very similar, but not identical to Eqs. (96) and (97) [that is, Eqs. (3) and (4) of Ref. 25].

IV. RESULTS AND DISCUSSIONS

In this section, we put forth the issues pertaining to the implementation of the formulation proposed above. At the very outset, we emphasize that the expression for the phase induced current J [Eq. (93)] reveals certain key issues. First, we envisage that the development quantizes both – the system, as well as the bath. While the former quantization is borne out by the term $G(q)$ in Eq. (93), the quantization of the bath is being surfaced by the term D_0. Second, as have been amply stressed in the development, the system-bath combination is subject to a pair of external noises that originate from a common source, and by virtue of this, we could explicitly demonstrate that these two noises remain mutually correlated. Lastly, we point out that the present development bears a strikingly close structural kinship with two of our earlier developments,[38,39] however, the present work explores the dual role of the interplay of an additive noise and a multiplicative noise, and in this sense, the terms appearing in Eq. (93) bear distinctively different physical significance. This structural resemblance is a consequence of the fact that in all the cases (the present, as well as the ones developed earlier[38,39]), the Langevin equation bears a state-dependent dissipation and a multiplicative noise.

As we have pointed out earlier, the quantization of the system is embedded in the term Q_V [Eq. (33)], and in order to incorporate the quantum corrections to leading order we follow the route of Barik and Ray,[37] and evaluate the quantum dispersion term $\langle\delta\hat{q}^2\rangle_Q$ in the lowest order,

$$\langle\delta\hat{q}^2\rangle_Q = \Delta_q \left[V'(q)\right]^2, \tag{99}$$

where

$$\Delta_q = \frac{\langle\delta\hat{q}^2\rangle_Q^0}{[V'(q)]^2} \tag{100}$$

and q^0 is a quantum mechanical mean position at which $\langle\delta\hat{q}^2\rangle_Q$ becomes minimum: $\langle\delta\hat{q}^2\rangle_Q^0 = \frac{\hbar}{2\Omega_0}$, Ω_0 being defined earlier. With this value of $\langle\delta\hat{q}^2\rangle_Q$, one may easily obtain the various quantum corrections, for example, Q_V, Q_f and Q_3 in lowest order.

For numerical implementation of our results, we consider a sinusoidal periodic and symmetric potential

$$V(q) = V_0[1 + \cos(q + \theta)], \tag{101}$$

where, V_0 is the barrier height and θ is the phase factor which can be controlled externally. The coupling function $f(q)$ is chosen as $f(q) = (q + \alpha\sin q)$ so that the derivative of the coupling function becomes $f'(q) = 1 + \alpha\cos q$, where α is the modulation parameter. Consequently, the second order quantum correction in the over damped limit becomes

$$\langle\delta\hat{q}^2\rangle_Q = -\Delta_q V_0^2 \sin^2(q + \theta),$$

and the correction to the potential in the leading order is given by

$$Q_V = -\frac{1}{2}\Delta_q V_0^3 \sin^3(q + \theta). \tag{102}$$

The quantum corrections Q_f and Q_3 in the same order can be estimated as

$$Q_f = -\frac{1}{2}\Delta_q \alpha V_0^2 \cos q \sin^2(q + \theta), \tag{103}$$

$$Q_3 = \Delta_q \alpha^2 V_0^2 \sin^2 q \sin^2(q + \theta), \tag{104}$$

and the functions $h(q)$ and $g(q)$ are given by

$$h(q) = (1 + \alpha\cos q)^2 - \Delta_q \alpha V_0^2 \cos q \sin^2(q + \theta) \tag{105}$$

$$\times(1 + \alpha\cos q) + \Delta_q \alpha^2 V_0^2 \sin^2 q \sin^2(q + \theta),$$

$$g(q) = 1 + \alpha\cos q - \frac{1}{2}\Delta_q \alpha V_0^2 \cos q \sin^2(q + \theta). \tag{106}$$

In the unit of $\hbar = k_B = 1$, we set the parameters $\langle\delta\hat{q}^2\rangle_Q^0 = \frac{1}{2}$, the minimum uncertainty value, $\Delta_q = 0.5, V_0 = 1.0, \alpha = 1.0, T = 1.0, \Gamma = 1.0$.

In Figure 1, we plot the variation of effective potential $\phi(q)$ for a set of D_e values to observe the effect of D_e. The figure clearly reveals that a tilt to the effective potential has been generated. The unidirectional motion that appears is a consequence of this tilt in the effective potential $\phi(q)$ since this asymmetry makes the transition from left to right and right to left unequal. Additionally, we observe that the net slope of the tilt increases with increasing D_e. This is clearly seen from the dependence of the term $G(q)$ on D_e as given in Eq. (84).

The variation of current as a function of phase difference θ is shown in Figure 2 for different values of D_e. We observe that for $\theta = n\pi$, $n = 0, 1, 2, \ldots$ there is no current. This is so because for $\theta = n\pi$, $n = 0, 1, 2, \ldots$, the right hand side of Eq. (90) becomes integrable, and $\phi(2\pi) = 0$. Figure 2 also shows that the current increases for increasing D_e since, as we have just pointed out, the gradient in the tilt of the effective potential $\phi(q)$ increases with D_e. In addition to this, Eq. (79) shows that D_e is one of the measures of the effective temperature of the bath, and as expected, any increase in D_e amounts to an enhanced particle diffusion through the potential.

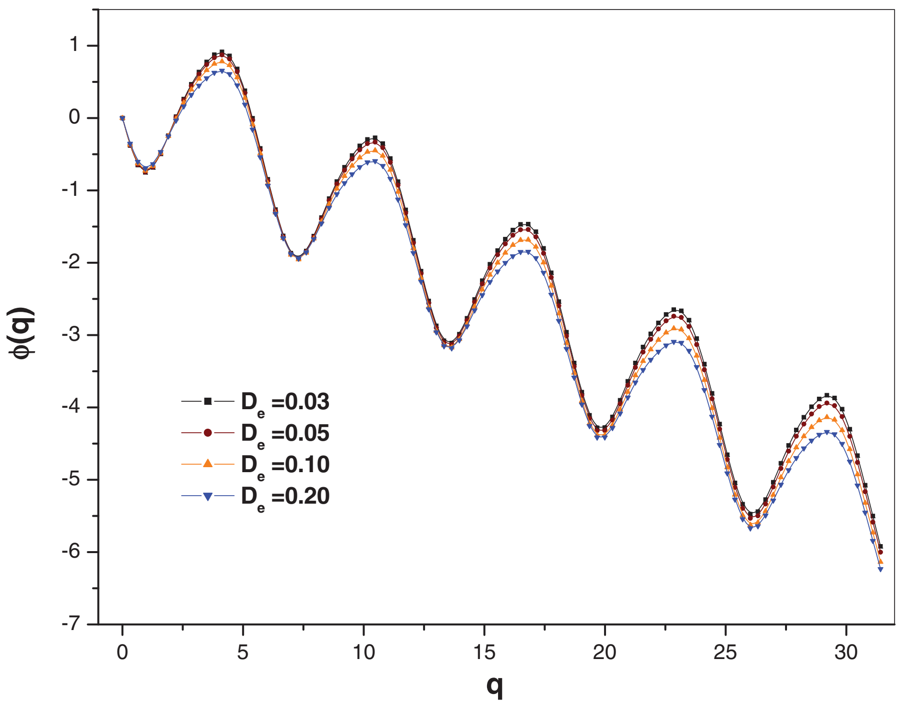

FIG. 1. (Color online) Variation of the effective potential $\phi(q)$ with q for different D_e values.

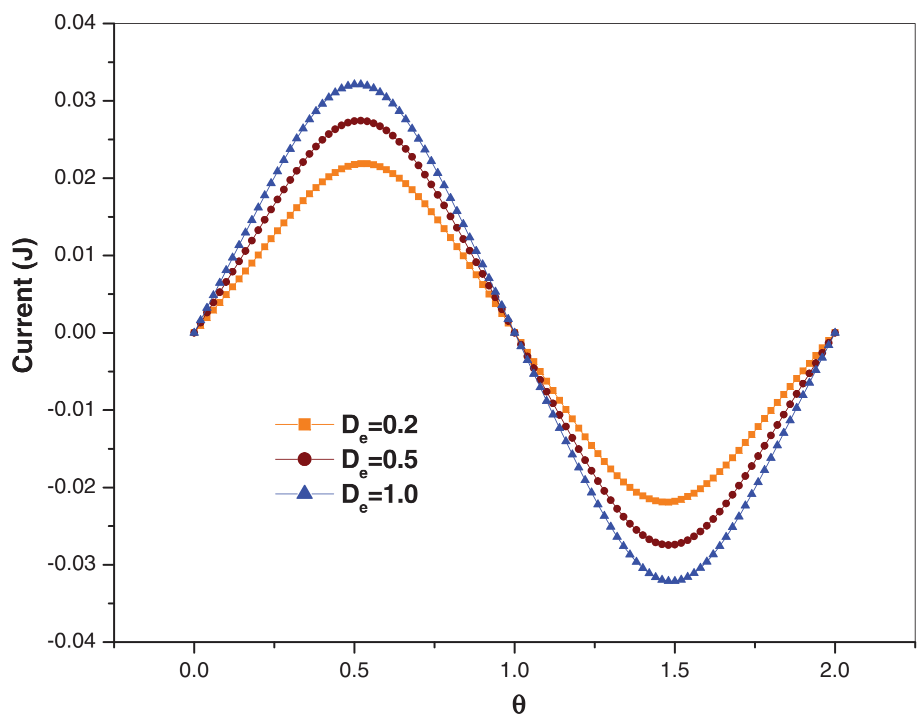

FIG. 2. (Color online) Variation of the current J as a function of phase difference θ for different D_e values.

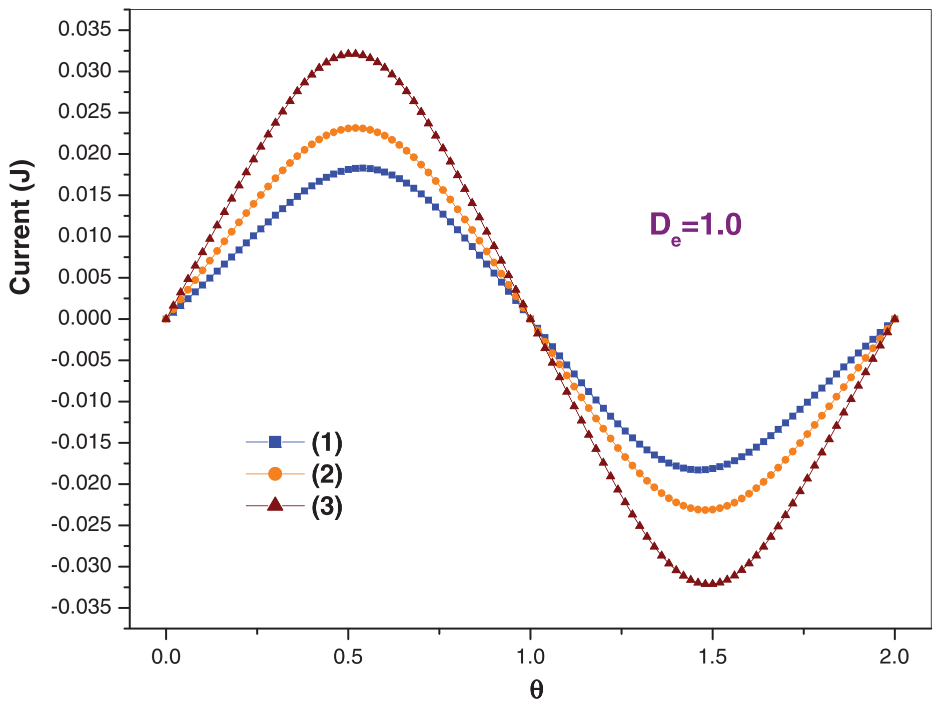

FIG. 3. (Color online) Variation of the current J (as a function of phase difference θ) with the nature of external driving for $D_e = 1.0$.

Figure 3 is a representation of the effect of the dependence of current on the nature of external driving for a fixed D_e. In Figure 3, the curves labelled (1), (2), and (3) represent, respectively, the cases where the bath, the system, and the system-bath combination has been driven. As it should be, the most pronounced current is envisaged in the situation where the system-bath combination is being driven. This is a consequence of the mutual correlation of the additive and the multiplicative noises. This aspect brings on to fore certain important generalizations pertaining to this development. Since the present work essentially deals with a situation that yields phase induced current as a consequence of a mutual interplay of an additive noise with a multiplicative one, such a theory can be equally applied to study the cases pertaining to say, a chemical reaction. Clearly enough, Figure 3 is an indication of the fact that driving the system alone, the bath alone, or the system-bath combination results in distinctly different situations in terms of the current yield. Thus, in the parlance of the theory of reaction rates, one may find a suitable control by modulating the external noises, and thereby control the rates effectively. Despite the fact that in this work we have put forth the quantum case, this observation applies to the classical domain as well.

V. CONCLUSIONS

In two of our earlier works,[38,39] we have studied the generation of phase induced current in a periodic potential for a quantum Brownian system where, either the bath[38] or the system[39] had been driven externally by a colored noise. However, from the standpoint of an experiment, possibly, it is more appealing to drive the system and the bath simultaneously, since this gives a better insight regarding the interplay of the noise processes and the effects stemming out therefrom. The present work is essentially such a development, where we have studied the phase induced current when both the system and the bath are being modulated by external noises and thereby exposed the system to the net effect of two cross-correlated noises. By starting out with a suitable system-reservoir model for a quantum system, we have developed the corresponding Langevin equation with a state-dependent dissipation and a multiplicative noise. We have also found out the expression for the effective potential that the system remains subjected to and the corresponding phase induced current. Our present formalism is a partial differential equation in phase space akin to the FokkerPlanck-Smoluchowski equation, and so operators are not involved. Note that at strong friction, the leading quantum corrections vanish for (biased) free quantum-Brownian motion, i.e., for potentials with $V(q)$=constant; or with linear $V(q) = \alpha q$; in these cases, no normalizable stationary state occurs. In principle, quantum effects on transport properties can be estimated for arbitrary potentials using our present model. We have also put the developed formulation to numerical calculations and analyzed the results. Our present studies illustrate the fact that the steady current is enhanced by an increase in the extent of correlation in a periodic potential. This aspect can be fruitfully implemented to intensify current in an appropriately designed experiment just by exercising a suitable control on the degree of correlation. In view of this, our model, although semiclassical, is of enormous utility in understanding and evaluating the influence of a medium on dynamical processes. We also indicate the relationship of the present method with other methods bearing close kinship with ours. We have illustrated in this paper that our quantum Smoluchowski equation (with constant dissipation) is identical to the quantum Smoluchowski equation originally obtained by Ankerhold *et al.*,[22] Coffey and co-workers,[25] and Lutz[28] for the harmonic oscillator case when terms up to $\hbar^2$ are being retained. Finally, we want to apply our present model to asymmetric (or ratchet) potentials and explore the possibility to obtain noise rectification from one fluctuation source[72] in near future.

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