

ON THE NATURE OF θ IN THE MEAN-VALUE
THEOREM OF THE DIFFERENTIAL
CALCULUS*

BY GANESH PRASAD

1. *Introduction.* If $f(x)$ is a single-valued function which is finite and continuous in an interval (a, b) , the ends being included, than the relation

$$(M) \quad f(x+h) = f(x) + hf'(x+\theta h), \quad 0 < \theta < 1,$$

holds for every value of x and h for which the interval $(x, x+h)$ is in the interval (a, b) ; provided that *either* $f'(x)$ exists at every point inside the interval (a, b) *or* a certain less restrictive condition † is satisfied. In recent years the nature of θ has been studied by a number of writers ‡ who start with the assumption that $f''(x)$ exists everywhere in the interval (a, b) . The two theorems, which it is the object of this paper to formulate and prove, are believed to be new and hold even if $f''(x)$ does not exist everywhere. For the sake of clarity and fixity of ideas, I consider θ only as a function of h , assuming x to be a constant, say 0, in the theorem (M).

2. THEOREM I. *If $\theta(h)$ is single-valued and continuous, it is not necessarily differentiable for every value of h .*

PROOF. Take $f(x)$ to be the indefinite integral of a monotone, increasing and continuous function which has a differential coefficient everywhere in the interval (a, b) , excepting the points

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† The condition of W. H. Young and G. C. Young, *Quarterly Journal of Mathematics*, vol. 40 (1909), p. 1; Hobson's *Theory of Functions of a Real Variable*, vol. 1, 3d edition, 1927, p. 384; or the still less restrictive condition of A. N. Singh, *Bulletin of the Calcutta Mathematical Society*, vol. 19 (1928), p. 43.

‡ R. Rothe (*Mathematische Zeitschrift*, vol. 9 (1921), p. 300; *Tôhoku Mathematical Journal*, vol. 29 (1928), p. 145); T. Hayashi (*Science Reports of the Tôhoku Imperial University*, (1), vol. 13 (1925), p. 385); O. Szász (*Mathematische Zeitschrift*, vol. 25 (1926), p. 116).

of an everywhere dense set. Such a function is that given by T. Broden.* Denoting Broden's function by w , let

$$f(x) = \int_0^x w(t)dt,$$

and let the everywhere dense set be denoted by S ; also let ξ stand for $h\theta$. Then it is easily seen that ξ is a single-valued and continuous function of h , and that, corresponding to each value of ξ , there is a value of h and only one value. Now (M) gives

$$f(h) = hf'(\xi) = hw(\xi),$$

whatever h may be.

Therefore, as $f'(h)$ exists,

$$\frac{d}{dh}\{hw(\xi)\}, \text{ that is, } w(\xi) + h\frac{dw}{dh},$$

must exist for every value of h . Thus, at any point $h = h'$ which corresponds to a point $\xi = \xi'$ of S , $d\xi/dh$ and, consequently, $d\theta/dh$ must be non-existent; otherwise $w'(\xi')$ will exist which is impossible.

Therefore it is proved that, for every value of h corresponding to which ξ is a point of S , $d\theta/dh$ is non-existent.

3. THEOREM II. *If $\theta(h)$ is single-valued, it is necessarily continuous for every value of h .*

PROOF. Assume, if possible, that \bar{h} is a point of discontinuity of $\theta(h)$. Then, denoting the corresponding values of ξ and θ by $\bar{\xi}$ and $\bar{\theta}$ respectively, we have by (M)

$$f(\bar{h}) = \bar{h}f'(\bar{\xi}).$$

Now two possibilities arise: the discontinuity may be of the first kind or of the second kind.

(a) If the discontinuity is of the first kind, then there must be a sequence $\{h_n\}$, tending to \bar{h} , for which the corresponding sequence $\{\xi_n\}$ does not tend to $\bar{\xi}$ but to $\bar{\xi}'$ different from $\bar{\xi}$. Thus

$$f'(\bar{\xi}) = f'(\bar{\xi}').$$

* Journal für Mathematik, vol. 118, p. 27; Hobson's *Theory of Functions of a Real Variable*, vol. 1, 1927, p. 389.

So, for the same value of h , namely, \bar{h} , there are two values of θ , namely, $\bar{\theta}$ and $\bar{\theta}'$, which is absurd, since θ is single-valued.

(b) If the discontinuity is of the second kind, then there must be a sequence $\{h_n\}$, tending to \bar{h} , for which the corresponding sequence $\{\xi_n\}$ does not tend to any limit. Therefore two values k_1 and k_2 of h can always be found as near as we please to \bar{h} such that the corresponding values η_1 and η_2 of ξ differ from each other by a quantity greater than a suitably prescribed positive quantity δ . But, from (M), $f(h)/h$ and, consequently, $f'(\xi)$ are continuous functions of h at \bar{h} . Therefore ξ must be multiple-valued at \bar{h} , which is absurd, since θ is single-valued.

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A NUMERICAL FUNCTION APPLIED TO CYCLOTOMY

BY EMMA T. LEHMER

A function $\phi_2(n)$ giving the number of pairs of consecutive integers each less than n and prime to n , was considered first by Schemmel.* In applying this function to the enumeration of magic squares, D. N. Lehmer† has shown that if one replaces consecutive pairs by pairs of integers having a fixed difference λ prime to $n = \prod_{i=1}^t p_i^{\alpha_i}$, then the number of such pairs (mod n) whose elements are both prime to n is also given by

$$\phi_2(n) = \prod_{i=1}^t p_i^{\alpha_i-1} (p_i - 2).$$

As is the case for Euler's totient function $\phi(n)$, the function $\phi_2(n)$ obviously enjoys the multiplicative property $\phi_2(m)\phi_2(n) = \phi_2(mn)$, $(m, n) = 1$, $\phi_2(1) = 1$. In what follows we call an integer simple if it contains no square factor > 1 . For a simple number n we have the following analog of Gauss' theorem:

$$(1) \quad \sum_{\delta|n} \phi_2(\delta) = \phi(n),$$

* Journal für Mathematik, vol. 70 (1869), pp. 191-2.

† Transactions of this Society, vol. 31 (1929), pp. 538-9.