ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVIII, 2012, f.1
DOI: 10.2478/v10157-012-0001-3

ON PSEUDO RICCI SYMMETRIC MANIFOLDS

BY

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Abstract. The object of the present paper is to study pseudo Ricci symmetric manifolds. Among others we obtain a sufficient condition for a pseudo Ricci symmetric manifold to be a quasi Einstein manifold. We prove that in a pseudo Ricci symmetric quasi Einstein manifold the scalar curvature vanishes and pseudo Ricci symmetric quasi Einstein perfect fluid spacetime has also been considered. Also we construct two examples of pseudo Ricci symmetric manifolds to justify our theorems.

Mathematics Subject Classification 2000: 53C15, 53C40, 53C25.

Key words: quasi Einstein manifold, pseudo Ricci symmetric manifold, pseudo Ricci symmetric quasi Einstein perfect fluid spacetime.

1. Introduction

The Einstein equations (see [10]), imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied (see [2]) if the energy-momentum tensor is covariant-constant. Chaki and Ray (see [2]) had shown that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime. If however, $\nabla S \neq 0$, then such a spacetime may be called pseudo Ricci symmetric. We may say that the Ricci symmetric condition is only a special case of the pseudo Ricci symmetric condition. It is, therefore, meaningful to study the properties of pseudo Ricci symmetric spacetimes in general relativity.

In 1967, Sen and Chaki (see [15]) studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expressions of the covariant derivative of Ricci tensor:

$$R_{ij,l} = 2\lambda_l R_{ij} + \lambda_i R_{lj} + \lambda_j R_{il} \ , \tag{1.1}$$

where λ_i is a non-zero covariant vector and ',' denotes covariant differentiation with respect to the metric tensor g_{ij} .

Later in 1988, CHAKI (see [1]) called a non-flat Riemannian manifold a pseudo Ricci symmetric manifold if its Ricci tensor satisfies (1.1). In index free notation this can be stated as follows:

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0,2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2B(X)S(Y,Z) + B(Y)S(X,Z) + B(Z)S(X,Y), \tag{1.2}$$

where ∇ denotes the Levi-Civita connection and B is a non-zero 1-form such that

$$g(X,U) = B(X), \tag{1.3}$$

for all vector fields X; U being the vector field corresponding to the associated 1-form B. Pseudo Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh (see [16]). It is known (see [1]) that in a $(PRS)_n$ if the scalar curvature r is constant then $r = 0$. A concrete example of a $(PRS)_n$ is given by OZEN and ALTAY (see [11]). Also, SEN and CHAKI (see [15]) studied hypersurfaces of a conformally flat space of class-one and obtained in a natural way the notion of the pseudo Ricci symmetric manifold. On the other hand, CHAKI (see [1]) proved the existence of a pseudo Ricci symmetric manifold by considering a linear connection on a Riemannian manifold.

Also, RAY-GUHA (see [13]) proved that a conformally flat perfect fluid pseudo Ricci symmetric spacetime obeying Einstein equation without cosmological constant and having the basic vector field of pseudo Ricci symmetric spacetime as the velocity vector field of the fluid is infinitesimally spatially isotropic relative to the velocity vector field.

So pseudo Ricci symmetric manifolds have some importance in the general theory of relativity. Considering this aspect we are motivated to study such a manifold.

In 2000, CHAKI and MAITY (see [4]) introduced the notion of a quasi Einstein manifold.

A non-flat Riemannian manifold(M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.4}$$

where a, b are non-zero scalars, ρ is a unit vector and A is a non-zero 1-form such that

$$g(X,\rho) = A(X), \tag{1.5}$$

for all vector fields X. If $b = 0$ or, $A = 0$ or, both A and b vanishes, then the manifold reduces to an Einstein manifold. We shall call A the associated 1-form and ρ is the generator of the manifold and $a,\ b$ are associated scalars.

In a recent paper, (see [6]) De and Ghosh cited some examples of a quasi Einstein manifold.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds (see [9]). Also quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity (see [7]). So quasi Einstein manifolds have also some importance in the general theory of relativity.

The paper is organized as follows:

In section 2 it is shown that a conformally flat $(PRS)_n$ is a quasi Einstein manifold. In section 3 we have enquired under what condition a $(PRS)_n$ will be a quasi Einstein manifold. Then we construct an example to justify this Theorem. Section 4 of this paper deals with pseudo Ricci symmetric manifold which is also quasi Einstein. An example of such type of manifold is also given. Finally, we study pseudo Ricci symmetric quasi Einstein perfect fluid spacetime.

2. Conformally flat $(PRS)_n$

It is known (see [3]) that in a conformally flat $(PRS)_n$

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ T(X)T(Z)g(Y,W) - T(Y)T(Z)g(X,W) \\ &+ T(Y)T(W)g(X,Z) - T(X)T(W)g(Y,Z)], \end{aligned} \tag{2.1}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $T(X) = \frac{B(X)}{\sqrt{B(U)}}$.

Note that in a proper (non-flat) conformally flat $(PRS)_n$ the scalar curvature r can not be zero, for if $r = 0$, then from (2.2) it will follow that the manifold is flat. Now contracting (2.2) with respect to X and W we get

$$S(Y,Z) = \frac{r}{(n-1)}g(Y,Z) - \frac{r}{(n-1)}T(Y)T(Z). \tag{2.2}$$

Hence the manifold is a quasi Einstein manifold. Therefore, the following theorem holds.

Theorem 2.1. *A conformally flat* $(PRS)_n$ *is a quasi Einstein manifold.*

3. Sufficient condition for a pseudo Ricci symmetric manifold to be a quasi Einstein manifold

In a pseudo Ricci symmetric manifold the Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z) = 2B(X)S(Y,Z) + B(Y)S(X,Z) + B(Z)S(X,Y). \tag{3.1}$$

In a Riemannian manifold, a vector field $\bar{U}$ defined by $g(X,\bar{U}) = A(X)$ for all vector fields X is said to be a concircular vector field (see [14]) if

$$(\nabla_X A)(Y) = \alpha g(X,Y) + \omega(X)A(Y), \tag{3.2}$$

where α is a non-zero scalar and ω is a closed 1-form. If $\bar{U}$ is a unit one then the equation (3.2) can be written as

$$(\nabla_X A)(Y) = \alpha[g(X,Y) - A(X)A(Y)]. \tag{3.3}$$

We suppose that $(PRS)_n$ admits a unit concircular vector field defined by (3.3) where α is a non-zero constant. Applying Ricci-identity to (3.3) we obtain

$$A(R(X,Y)Z) = \alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)]. \tag{3.4}$$

Putting $Y = Z = e_i$ in (3.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \le i \le n$, we get

$$A(QX) = (n-1)\alpha^2 A(X),$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$, which implies that

$$S(X,\bar{U}) = (n-1)\alpha^2 A(X). \tag{3.5}$$

Now, $(\nabla_Y S))(X,\bar{U}) = \nabla_Y S(X,\bar{U}) - S(\nabla_Y X,\bar{U}) - S(X,\nabla_Y \bar{U})$. Applying (3.5) and (3.3), we get

$$(\nabla_Y S)(X,\bar{U}) = (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] - S(X,\nabla_Y \bar{U}). \tag{3.6}$$

Now, we have

$$\begin{aligned}(\nabla_Y A)(X) &= \nabla_Y A(X) - A(\nabla_Y X) = \nabla_Y g(X,\bar{U}) - g(\nabla_Y X,\bar{U}) \\ &= g(X,\nabla_Y \bar{U}), \quad \text{since } (\nabla_X g)(Y,\bar{U}) = 0.\end{aligned}$$

By (3.3) this implies $\alpha[g(X,Y) - A(X)A(Y)] = g(X,\nabla_Y\bar{U})$ i.e., $g(\alpha Y, X) - g(\alpha A(Y)\bar{U}, X) = g(\nabla_Y \bar{U}, X)$ which implies $\nabla_Y\bar{U}{=}\alpha Y - \alpha A(Y)\bar{U}{=}\alpha(Y - A(Y)\bar{U})$. Therefore $S(X,\nabla_Y\bar{U}) = S(X,\alpha Y) - S(X,\alpha A(Y)\bar{U})$. Hence

$$S(X,\nabla_Y\bar{U}) = \alpha[S(X,Y) - A(Y)S(X,\bar{U})]. \tag{3.7}$$

Applying (3.7) in (3.6) we get

$$\begin{aligned}(\nabla_Y S)(X,\bar{U}) &= (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] \\ &\quad -\alpha S(X,Y) + \alpha A(Y)S(X,\bar{U}).\end{aligned} \tag{3.8}$$

Applying (3.5) in (3.8) we get

$$(\nabla_Y S)(X,\bar{U}) = (n-1)\alpha^3 g(X,Y) - \alpha S(X,Y). \tag{3.9}$$

Putting $Z = \bar{U}$ and using (3.5) and (3.9) in (3.1) we get,

$$\begin{aligned}&(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y) \\ &= 2(n-1)\alpha^2 A(Y)B(X) + (n-1)\alpha^2 A(X)B(Y) + B(\bar{U})S(X,Y),\end{aligned}$$

which implies,

$$\begin{aligned}(\alpha + B(\bar{U}))S(X,Y) &= (n-1)\alpha^3 g(X,Y) - 2(n-1)\alpha^2 A(Y)B(X) \\ &\quad - (n-1)\alpha^2 A(X)B(Y).\end{aligned} \tag{3.10}$$

Putting $Y = \bar{U}$ in (3.10) and using (3.5) we get

$$\begin{aligned}(\alpha + B(\bar{U}))(n-1)\alpha^2 A(X) &= (n-1)\alpha^3 A(X) - 2(n-1)\alpha^2 B(X) \\ &\quad - (n-1)\alpha^2 A(X)B(\bar{U}),\end{aligned}$$

which implies,

$$B(X) = -B(\bar{U})A(X). \tag{3.11}$$

Let us suppose

$$\alpha + B(\bar{U}) \neq 0. \tag{3.12}$$

Putting (3.11) in (3.10) we have

$$S(X,Y)=\frac{(n-1)\alpha^3}{\alpha+B(\bar{U})}g(X,Y)+\frac{3(n-1)\alpha^2B(\bar{U})}{\alpha+B(\bar{U})}A(X)A(Y),$$

that is,

$$S(X,Y)=ag(X,Y)+bA(X)A(Y), \tag{3.13}$$

where $a=\frac{(n-1)\alpha^3}{\alpha+B(\bar{U})}$ and $b=\frac{3(n-1)\alpha^2B(\bar{U})}{\alpha+B(\bar{U})}\neq 0$, as $\alpha\neq 0$ and $B\neq 0$. Thus we have the following theorem:

Theorem 3.1. *If a $(PRS)_n$ admits a unit concircular vector field $\bar{U}$ whose associated scalar α is a non-zero constant and satisfies the condition* (3.12) *then the manifold reduces to a quasi Einstein manifold.*

Now Theorem 3.1 will be justified by the following example:

Example 3.1. Let us consider a Riemannian metric g on $\mathbb{R}^4$ given by

$$ds^2=g_{ij}dx^idx^j=(x^4)^{\frac{4}{3}}[(dx^1)^2+(dx^2)^2+(dx^3)^2]+(dx^4)^2, \tag{3.14}$$

$(i,j=1,2,3,4)$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma^1_{14}=\Gamma^2_{24}=\Gamma^3_{34}=\frac{2}{3x^4},\qquad \Gamma^4_{11}=\Gamma^4_{22}=\Gamma^4_{33}=-\frac{2}{3}(x^4)^{1/3},$$
$$R_{1441}=R_{2442}=R_{3443}=-\frac{2}{9(x^4)^{2/3}}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor and their covariant derivatives are:

$$R_{11}=-\frac{2}{9(x^4)^{2/3}},\ R_{22}=-\frac{2}{9(x^4)^{2/3}},\ R_{33}=-\frac{2}{9(x^4)^{2/3}},\ R_{44}=-\frac{2}{3(x^4)^2},$$
$$R_{11,4}=\frac{4}{9(x^4)^{5/3}},\ R_{22,4}=\frac{4}{9(x^4)^{5/3}},\ R_{33,4}=\frac{4}{9(x^4)^{5/3}},\ R_{44,4}=\frac{4}{3(x^4)^3}.$$

It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4,g)$ is $R=-\frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant. We shall now show that $\mathbb{R}^4$ is a $(PRS)_n$. Let us choose the associated 1-form as

$$B_i(x)=\begin{cases}-\frac{1}{x^4}, & \text{for i=4}\\ 0, & \text{otherwise,}\end{cases} \tag{3.15}$$

at any point $x \in \mathbb{R}^4$. Now, (1.2) reduces to the equations

$$R_{11,4} = 2B_4R_{11} + 2B_1R_{14}, \tag{3.16}$$

$$R_{22,4} = 2B_4R_{22} + 2B_2R_{24}, \tag{3.17}$$

$$R_{33,4} = 2B_4R_{33} + 2B_3R_{34}, \tag{3.18}$$

$$R_{44,4} = 2B_4R_{44} + 2B_4R_{44}, \tag{3.19}$$

since for the other cases (1.2) holds trivially. By (3.15) we get the following relation for the right hand side (r.h.s.) and the left hand side (l.h.s.) of (3.16)

$$\begin{aligned} \text{r.h.s. of (3.16)} &= 2B_4R_{11} + 2B_1R_{14} = \left(-\frac{2}{x^4}\right)\left(-\frac{2}{9(x^4)^{2/3}}\right) \\ &= \frac{4}{9(x^4)^{5/3}} = R_{11,4} = \text{l.h.s. of (3.16)}. \end{aligned}$$

By similar argument it can be shown that (3.17), (3.18) and (3.19) are true. So, $(\mathbb{R}^4, g)$ is a $(PRS)_n$ whose scalar curvature is non-zero and non-constant. It is to be noted that (1.2) can be satisfied by a number of 1-form B, namely, by those which fulfil (3.16), (3.17), (3.18) and (3.19). Thus we can state the following:

"Let $(\mathbb{R}^4, g)$ be a Riemannian manifold endowed with the metric given by $ds^2 = g_{ij}dx^idx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2$, $(i, j = 1, 2, 3, 4)$. Then $(\mathbb{R}^4, g)$ is a $(PRS)_4$ with non-zero and non-constant scalar curvature."

Now we shall show that this $(\mathbb{R}^4, g)$ admits a unit concircular vector field A^i whose associated scalar α is a non-zero constant and satisfies the condition (3.12) and hence is a quasi Einstein manifold.

For this let us chose A^i as follows:

$$A^i(x) = \begin{cases} \tan x^i, & \text{for i=4} \\ 0, & \text{otherwise,} \end{cases} \tag{3.20}$$

at any point $x \in \mathbb{R}^4$ and the associated scalar $\alpha = 1$. To prove that A^i is a unit concircular vector field we have to prove that

$$A_{i,j} = \alpha[g_{ij} + A_iA_j], \tag{3.21}$$

where ',' denotes the covariant differentiation with respect to the metric, and A_i is a non-zero 1–form defined by $A_i = g_{ij}A^j$. Since from the definition of the metric g it is clear that g^{i4} is non-zero only when $i = 4$ and $g^{44} = 1$, we get the 1-form A_i as follows:

$$A_i(x) = \begin{cases} \tan x^i, & \text{for i=4} \\ 0, & \text{otherwise.} \end{cases} \tag{3.22}$$

So (3.21) reduces to

$$A_{4,4} = \alpha[g_{44} + A_4 A_4], \tag{3.23}$$

since for the other cases (3.21) holds trivially. Again $A_{4,4} = \tan^2 x^4$. So with the help of (3.22) after straightforward calculation we see that (3.23) is true. Hence A^i is a concircular vector field with $\alpha = 1$. Again $\alpha + A^4 B^4 = 1 - \frac{\tan x^4}{x^4} \neq 0$, hence (3.12) is satisfied. Now to show that the manifold under consideration is a quasi Einstein manifold, let us choose the scalar functions a, b as follows:

$$a = -\frac{2}{9(x^4)^2}, \qquad b = -\frac{4}{9(x^4)^2 \tan^2 x^4} \tag{3.24}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equations

$$R_{11} = a g_{11} + b A_1 A_1, \tag{3.25}$$

$$R_{22} = a g_{22} + b A_2 A_2 \tag{3.26}$$

$$R_{33} = a g_{33} + b A_3 A_3, \tag{3.27}$$

and

$$R_{44} = a g_{44} + b A_4 A_4, \tag{3.28}$$

since, for the other cases (1.4) holds trivially. By (3.24) and (3.22) we get

$$\begin{aligned} \text{r.h.s. of (3.25)} &= a g_{11} + b A_1 A_1 = -\frac{2}{9(x^4)^2} \times (x^4)^{\frac{4}{3}} = -\frac{2}{9(x^4)^{2/3}} = R_{11} \\ &= \text{l.h.s. of (3.25)}. \end{aligned}$$

Again for (3.28) we have

$$\begin{aligned} \text{r.h.s. of (3.28)} &= a g_{44} + b A_4 A_4 = -\frac{2}{9(x^4)^2} - \frac{4}{9(x^4)^2 \tan^2 x^4} \times \{\tan x^4\}^2 \\ &= -\frac{2}{3(x^4)^2} = R_{44} = \text{l.h.s. of (3.28)}. \end{aligned}$$

By similar argument it can be shown that (3.26) and (3.27) are also true. So, $(\mathbb{R}^4, g)$ is a quasi Einstein manifold which justifies Theorem 3.1.

4. Pseudo Ricci symmetric quasi Einstein manifold

In the rest of the paper we consider that the associated scalars a and b of the quasi Einstein manifold defined by (1.4) are constant. Such a quasi Einstein manifold is called a special quasi Einstein manifold (see [8]). As a natural example we can mention that an η-Einstein Sasakian manifold is a special quasi Einstein manifold (see [17], p.285). The existence of such type of manifold was proved by the first author and De (see [5]).

Now taking covariant differentiation to the both sides of the equation (1.4) we get

$$(\nabla_Z S)(X,Y) = b(\nabla_Z A)(X)A(Y) + bA(X)(\nabla_Z A)(Y). \tag{4.1}$$

Again we consider that the manifold is also pseudo Ricci symmetric. Then from (1.2) and (4.1) we get

$$\begin{aligned}&2B(Z)S(X,Y) + B(X)S(Y,Z) + B(Y)S(X,Z)\\ &= b(\nabla_Z A)(X)A(Y) + bA(X)(\nabla_Z A)(Y).\end{aligned} \tag{4.2}$$

Now, contracting X and Y in (4.2) we have

$$rB(Z) + S(U,Z) = b(\nabla_Z A)(\rho), \tag{4.3}$$

where U is defined by (1.3). We know that in a $(PRS)_n$ (see [1]), $S(U,Z) = 0$ for all Z and $(\nabla_Z A)(\rho) = 0$, since ρ is a unit vector. So, from (4.3) we have

$$rB(Z) = 0. \tag{4.4}$$

But, since, $B \neq 0$, so $r = 0$. Hence we have the following theorem:

Theorem 4.1. *In a pseudo Ricci symmetric quasi Einstein manifold the scalar curvature vanishes.*

Now we justify Theorem 4.1 by the following example:

Example 4.1. On coordinate space $\mathbb{R}^n$ (with coordinates $x^1, x^2, ..., x^n$) we define a Riemannian space V_n. We calculate the components of the curvature tensor, the Ricci tensor and the components of its covariant derivatives and then we verify that this space is a pseudo Ricci symmetric space. Then we show that this space is a quasi Einstein space whose associated scalars are constant and hence is a space of vanishing scalar curvature which justifies Theorem4.1.

Let each Latin index runs over $1, 2, ..., n$ and each Greek index over $2, 3, ..., (n-1)$. We define a Riemannian metric on the $\mathbb{R}^n (n \geq 3)$ by the formula

$$ds^2 = \phi(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n \ , \tag{4.5}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant and ϕ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n. In the metric considered, the only non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are (see [12])

$$\begin{gathered}\Gamma^\beta_{11} = -\frac{1}{2}K^{\alpha\beta}\phi_{.\alpha} \ , \quad \Gamma^n_{11} = \frac{1}{2}\phi_{.1} \ , \quad \Gamma^n_{1\alpha} = \frac{1}{2}\phi_{.\alpha} \ , \\ R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta} \ , \quad R_{11} = \frac{1}{2}K^{\alpha\beta}\phi_{.\alpha\beta} \ ,\end{gathered} \tag{4.6}$$

where '.' denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$.

Here we consider $K_{\alpha\beta}$ as Kronecker symbol $\delta_{\alpha\beta}$ and $\phi = \delta_{\alpha\beta}x^\alpha x^\beta e^{(x^1)^2}$. In this case ϕ reduces to $\phi = \sum_{\alpha=2}^{n-1} x^\alpha x^\alpha e^{(x^1)^2}$. Thus we have the following relations $\phi_{.\alpha\beta} = 2\delta_{\alpha\beta}e^{(x^1)^2}$, $\delta_{\alpha\beta}\delta^{\alpha\beta} = n-2$. Therefore, $\delta^{\alpha\beta}\phi_{.\alpha\beta} = 2\delta^{\alpha\beta}\delta_{\alpha\beta}e^{(x^1)^2} = 2(n-2)e^{(x^1)^2}$. Since $\phi_{.\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components for R_{hijk} and R_{ij} and $R_{ij,k}$ in virtue of (4.6) are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\phi_{.\alpha\alpha} = e^{(x^1)^2},$$

$$R_{11} = \frac{1}{2}\phi_{.\alpha\beta}\delta^{\alpha\beta} = (n-2)e^{(x^1)^2} \tag{4.7}$$

and

$$R_{11,1} = 2(n-2)x^1 e^{(x^1)^2}, \tag{4.8}$$

respectively. So neither the Ricci tensor nor its covariant derivative vanish.

We claim that $\mathbb{R}^n (n \geq 3)$ with the given metric g is a pseudo Ricci symmetric space. To verify the relation (1.2) it is sufficient to check the followings:

$$R_{11,1} = 2B_1R_{11} + B_1R_{11} + B_1R_{11} = 4B_1R_{11}, \tag{4.9}$$

$$R_{11,k} = 2B_kR_{11} + B_1R_{1k} + B_1R_{1k}, \quad k \neq 1, \tag{4.10}$$

since for the other cases (1.2) holds trivially. In virtue of (4.7) and (4.8), (4.9) holds if and only if

$$B_1 = \frac{x^1}{2}, \tag{4.11}$$

and (4.10) holds if and only if

$$B_k = 0, \quad k \neq 1. \tag{4.12}$$

These mean that (1.2) can be satisfied by the 1-form defined by (4.11) and (4.12). Thus $(\mathbb{R}^n, g)$, defined by (4.5), is a pseudo Ricci symmetric space.

We now show that $\mathbb{R}^n$ is a quasi Einstein space. In order to verify the relation (1.4) it is sufficient to check

$$R_{11} = ag_{11} + bA_1A_1. \tag{4.13}$$

Let us consider the 1-form A_i as follows:

$$A_i(x) = \begin{cases} \sqrt{\sum_{\alpha=2}^{n-1} x^\alpha x^\alpha - 1}\ e^{\frac{(x^1)^2}{2}}\ , & \text{for i=1} \\ 0\ , & \text{otherwise,} \end{cases} \tag{4.14}$$

at any point $x \in \mathbb{R}^n$ and the real numbers

$$a = n-2, \quad b = 2-n. \tag{4.15}$$

Now from (4.14), (4.15), (4.7) and by the definition of ϕ we get the following relation for the right hand side (r.h.s.) and the left hand side (l.h.s.) of (4.13)

$$\begin{aligned} r.h.s.\ of\ (4.13) &= ag_{11} + bA_1A_1 = a\phi + b(A_1)^2 = (n-2)\sum_{\alpha=2}^{n-1} x^\alpha x^\alpha e^{(x^1)^2} \\ &\quad -(n-2)\{(\sqrt{\sum_{\alpha=2}^{n-1} x^\alpha x^\alpha - 1})\ e^{\frac{(x^1)^2}{2}}\}^2 \\ &= (n-2)e^{(x^1)^2} = R_{11} = l.h.s.\ of\ (4.13). \end{aligned}$$

Therefore, the space is also a quasi Einstein space whose associated scalars are constant.

Now from (4.5) we obtain $g_{ni} = g_{in} = 0$ for $i \neq 1$ which implies $g^{11} = 0$. Hence the scalar curvature R of this space is

$$R = g^{ij} R_{ij} = g^{11} R_{11} = 0. \tag{4.16}$$

Thus the space under consideration is a pseudo Ricci symmetric quasi Einstein space whose scalar curvature vanishes. This justifies Theorem 4.1.

5. Pseudo Ricci symmetric quasi Einstein perfect fluid spacetime

This section is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-, +, +, +)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a special type of spacetime which is called pseudo Ricci symmetric quasi Einstein spacetime. A semi-Riemannian four-dimensional pseudo Ricci symmetric quasi Einstein manifold may similarly be defined by taking a Lorentz metric g with signature $(-, +, +, +)$.

In this section we consider a pseudo Ricci symmetric quasi Einstein perfect fluid spacetime having the basic vector field of the quasi Einstein manifold as the timelike velocity vector field ρ of the fluid, that is, $g(\rho, \rho) = -1$. So, Theorem 4.1 will also hold in such a spacetime.

For a perfect fluid spacetime, we have the Einstein's equation without cosmological constant as

$$S(X, Y) - \frac{r}{2} g(X, Y) = kT(X, Y), \tag{5.1}$$

where k is the gravitational constant, T is the energy momentum tensor of type $(0, 2)$ given by

$$T(X, Y) = (\sigma + p)A(X)A(Y) + pg(X, Y), \tag{5.2}$$

with σ *and* p as the energy density and isotropic pressure of the fluid respectively. Using (5.2) we can express (5.1) as

$$S(X, Y) - \frac{r}{2} g(X, Y) = k[(\sigma + p)A(X)A(Y) + pg(X, Y)]. \tag{5.3}$$

Taking a frame field and contracting (5.3) over X and Y we get

$$r = k(\sigma - 3p). \tag{5.4}$$

Now from Theorem 4.1 we have $r = 0$. So (5.4) gives us either $k = 0$ or, $\sigma = 3p$. Since, $k \neq 0$, so $p = \frac{1}{3}\sigma$. But $p = \frac{1}{3}\sigma$ corresponds to equation of state in the radiation era in the evolution of our universe. The radiation is the era before the present matter dominated era. This leads to the following:

Theorem 5.1. *A pseudo Ricci symmetric quasi Einstein perfect fluid spacetime represents the equation of state in the radiation era in the evolution of our universe.*

Acknowledgement. The authors are thankful to the referee for his valuable suggestions towards the improvement of this paper.

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Received: 23.IX.2009
Revised: 11.XI.2009
Revised: 9.XII.2009

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