ON PROJECTIVE-SYMMETRIC SPACES

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Introduction

This paper deals with a type of Riemannian space $V_n (n \geqq 2)$ for which the first covariant derivative of Weyl's projective curvature tensor

(1) $$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik})$$

is everywhere zero, that is,

(2) $$W^h_{ijk,l} = 0$$

where comma denotes covariant differentiation with respect to the metric tensor g_{ij} of V_n. Such a space has been called a projective-symmetric space by Gy. Soós [1]. We shall denote such an n-space by ψ_n. It will be proved in this paper that decomposable Projective-Symmetric spaces are symmetric in the sense of Cartan. In sections 3, 4 and 5 non-decomposable spaces of this kind will be considered in relation to other well-known classes of Riemannian spaces defined by curvature restrictions. In the last section the question of the existence of fields of concurrent directions in a ψ_n will be discussed.

1. Scalar curvature of a ψ_n

Gy. Soós [1] has proved that for every $\psi_n (n > 2)$

(1.1) $$R_{ij,k} - R_{ik,j} = 0.$$

From the identities of Bianchi we have

$$R_{ij,k} - R_{ik,j} + g^{hm} R_{mikj,h} = 0.$$

In virtue of (1.1) this reduces to

$$g^{hm} R_{mikj,h} = 0$$

or

$$\tfrac{1}{2} R_{,j} = 0.$$

Hence R is a constant.

For a ψ_2,

$$R_{hijk,m} = g_{hk}R_{ij,m} - g_{hj}R_{ik,m} \tag{1.2}$$
$$= \frac{R_{,m}}{2}(g_{hk}g_{ij} - g_{hj}g_{ik}).$$

From (1.2) it follows that in a ψ_2 the scalar curvature R is a constant if and only if $R_{hijk,m} = 0$.

It is known that for a V_2

$$R_{hijk} = -\frac{R}{2}(g_{hj}g_{ik} - g_{hk}g_{ij}).$$

Therefore, in a V_2

$$W_{hijk} = R_{hijk} - \frac{R}{2}(g_{hk}g_{ij} - g_{hj}g_{ik})$$
$$= 0.$$

This shows that every V_2 is a ψ_2.

We can therefore state the following theorem:

THEOREM 1. *Every V_2 is a ψ_2. The scalar curvature of a $\psi_n (n > 2)$ is a constant but that of a ψ_2 is, in general, not so. A ψ_2 is of constant scalar curvature if and only if it is symmetric in the sense of Cartan.*

2. Decomposable ψ_n

A Riemannian space V_n is said to be decomposable if it can be expressed as a product $V_r \times V_{n-r}$ for some r, i.e., if coordinates can be found so that its metric takes the form

$$ds^2 = \sum_{\alpha_1,\beta_1=1}^{r} g_{\alpha_1\beta_1}dx^{\alpha_1}dx^{\beta_1} + \sum_{\alpha_2,\beta_2=r+1}^{n} g_{\alpha_2\beta_2}dx^{\alpha_2}dx^{\beta_2} \tag{2.1}$$

where the $g_{\alpha_1\beta_1}$ are functions of $x^1, x^2, \cdots x^r$ only and the $g_{\alpha_2\beta_2}$ are functions of $x^{r+1}, x^{r+2} \cdots x^n$ only. Greek letters with subscript 1 are taken to have the range 1 to r and those with subscript 2 to have the range $r+1$ to n. The two parts of (2.1) are the metrics of V_r and V_{n-r} and are called decomposition spaces of V_n. We now suppose that a ψ_n which is not of constant non-vanishing curvature is a product space $V_{n-r} \times V_r$. The curvature restriction mentioned above is necessary, because, as proved by Ficken [2], a space of constant non-vanishing curvature cannot be decomposable. Now,

$$W_{\alpha_1\beta_2\gamma_1\delta_2} = R_{\alpha_1\beta_2\gamma_1\delta_2} - \frac{1}{n-1}(g_{\alpha_1\delta_2}R_{\beta_2\gamma_1} - g_{\alpha_1\gamma_1}R_{\beta_2\delta_2}) \tag{2.2}$$
$$= \frac{1}{n-1}g_{\alpha_1\gamma_1}R_{\beta_2\delta_2}$$

because, the components of the metric tensor, the curvature tensor and the Ricci tensor of V_n are zero unless all subscripts of the Greek letters are alike. Therefore

$$W_{\alpha_2\beta_2\gamma_2\delta_2,\lambda_2} = \frac{1}{n-1} g_{\alpha_2\gamma_2} R_{\beta_2\delta_2,\lambda_2}. \tag{2.3}$$

In virtue of (2) it follows from (2.3) that

$$R_{\beta_2\delta_2,\lambda_2} = 0.$$

Similarly we have

$$R_{\alpha_1\gamma_1,\lambda_1} = 0.$$

Therefore

$$R_{\alpha_2\beta_2\gamma_2\delta_2,\lambda_2} = 0 \quad \text{and} \quad R_{\alpha_1\beta_1\gamma_1\delta_1,\lambda_1} = 0.$$

So the decomposition spaces are symmetric in the sense of Cartan and therefore their product is so. Hence we have the following theorem.

THEOREM 2. *A decomposable projective-symmetric space is symmetric in the sense of Cartan.*

Henceforth by a ψ_n we shall mean a non-decomposable ψ_n.

3. Three-dimensional projective-symmetric spaces

For a ψ_3 (1.1) holds and R is constant. Therefore

$$\begin{aligned} R_{ijk} &= R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik}R_{,j} - g_{ij}R_{,k}) \\ &= 0. \end{aligned}$$

Hence a ψ_3 is conformally flat.

For a V_3 the curvature tensor has the form

$$R_{hijk} = g_{hj}H_{ik} - g_{hk}H_{ij} + g_{ik}H_{hj} - g_{ij}H_{hk},$$

where

$$H_{ij} = -\left(R_{ij} - \frac{R}{4}g_{ij}\right).$$

Hence for a ψ_3

$$\begin{aligned} R_{hijk,l} &= g_{hj}H_{ik,l} - g_{hk}H_{ij,l} + g_{ik}H_{hj,l} - g_{ij}H_{hk,l} \\ &= \tfrac{1}{2}(g_{hk}R_{ij,l} - g_{hj}R_{ik,l}) \end{aligned} \tag{3.1}$$

Since in a ψ_3, R is constant

$$H_{ij,l} = -R_{ij,l}.$$

Therefore from (3.1) we have

$$-(g_{hj}R_{ik,l}-g_{hk}R_{ij,l}+g_{ik}R_{hj,l}-g_{ij}R_{hk,l}) = \tfrac{1}{2}(g_{hk}R_{ij,l}-g_{hj}R_{ik,l}).$$

Multiplying both sides by g^{ik} and summing for i and k we get

$$\tfrac{1}{2}(R_{hj,l}-g_{hj}R_{,l}) = -R_{hj,l}$$

whence

$$R_{hj,l} = 0.$$

Therefore from (3.1) it follows that the space is symmetric in the sense of Cartan. We can therefore state the following theorem.

THEOREM 3. *Every ψ_3 is a conformally flat symmetric space.*

4. Conformally-flat ψ_n ($n \geqq 4$)

We now consider a $\psi_n (n \geqq 4)$ and suppose that it is conformally flat. Then

$$\begin{aligned} R_{hijk,l} &= g_{hj}H_{ik,l}-g_{hk}H_{ij,l}+g_{ik}H_{hj,l}-g_{ij}H_{hk,l} \\ &= \frac{1}{n-1}(g_{hk}R_{ij,l}-g_{hj}R_{ik,l}) \end{aligned} \tag{4.1}$$

where

$$H_{ij} = -\frac{1}{n-2}\left[R_{ij}-\frac{R}{2(n-1)}g_{ij}\right]. \tag{4.2}$$

Since R is constant,

$$H_{ij,l} = -\frac{1}{n-2}R_{ij,l}.$$

Hence from (4.1) we have

$$\begin{aligned} -\frac{1}{n-2}&(g_{hj}R_{ik,l}-g_{hk}R_{ij,l}+g_{ik}R_{hj,l}-g_{ij}R_{hk,l}) \\ &= \frac{1}{n-1}(g_{hk}R_{ij,l}-g_{hj}R_{ik,l}). \end{aligned} \tag{4.3}$$

Multiplying both sides of (4.3) by g^{ik} and summing for i and k we have

$$\frac{n}{n-1}R_{hj,l} = 0$$

whence

$$R_{hj,l} = 0.$$

Therefore from (4.1) it follows that the space is symmetric in the sense of Cartan.

Let us now suppose that the rank of the matrix $((H_{ij}))$ is n where H_{ij} is given by (4.2).

Then there are uniquely determined quantities H^{ij} such that

$$H^{hj}H_{hk} = \delta^j_k, \quad H^{hj}H_{kj} = \delta^h_k.$$

Suppose that there exists a non-zero vector λ_l such that

$$\lambda_l R_{hijk} + \lambda_j R_{hikl} + \lambda_k R_{hilj} = 0. \tag{4.4}$$

Then

$$\begin{aligned} &\lambda_l(g_{hj}H_{ik} - g_{hk}H_{ij} + g_{ik}H_{hj} - g_{ij}H_{hk}) \\ &+\lambda_j(g_{hk}H_{il} - g_{hl}H_{ik} + g_{il}H_{hk} - g_{ik}H_{hl}) \\ &+\lambda_k(g_{hl}H_{ij} - g_{hj}H_{il} + g_{ij}H_{hl} - g_{il}H_{hj}) = 0. \end{aligned} \tag{4.5}$$

Multiplying both sides of (4.5) by $H^{ij}H^{hk}$ and summing for i, j, h, k we get

$$\lambda_l g_{hk}H^{hk} = \lambda_k g_{hl}H^{hk}. \tag{4.6}$$

Again multiplying (4.5) by H^{hj} and summing for h and j we get in virtue of

$$(n-3)(g_{il}\lambda_k - g_{ik}\lambda_l) = 0$$

whence

$$g_{il}\lambda_k = g_{ik}\lambda_l. \tag{4.7}$$

From (4.7) it follows that

$$(n-1)\lambda_l = 0$$

whence

$$\lambda_l = 0.$$

Thus there exists no non-zero vector λ_l such that (4.4) holds. The ψ_n therefore satisfies the following conditions

i) $R_{hijk,l} = 0$,

and

ii) $\lambda_l R_{hijk} + \lambda_j R_{hikl} + \lambda_k R_{hilj} \neq 0$

for a non-zero vector λ_l.

Hence it is a symmetric space of the first kind according to Hlávaty [3]. Therefore we have the following theorem.

THEOREM 4. *A conformally flat $\psi_n (n \geqq 4)$ is symmetric in the sense of Cartan. If further, the rank of the matrix $((H_{ij}))$ where H_{ij} is given by* (4.2), *be n then the ψ_n is a symmetric space of the first kind.*

5. Recurrent and Ricci-recurrent ψ_n $(n \geqq 4)$

Let a ψ_n be a recurrent space i.e. a non-flat space in which the Riemann curvature tensor satisfies the relation

$$R^h_{ijk,m} = \lambda_m R^h_{ijk} \tag{5.1}$$

for a non-zero vector λ_m.
Then

$$\begin{aligned} W^h_{ijk,m} &= R^h_{ijk,m} - \frac{1}{n-1} (\delta^h_k R_{ij,m} - \delta^h_j R_{ik,m}) \\ &= \lambda_m \left[R^h_{ijk} - \frac{1}{n-1} (\delta^h_k R_{ij} - \delta^h_j R_{ik}) \right] \\ &= \lambda_m W^h_{ijk} \end{aligned}$$

or

$$\lambda_m W^h_{ijk} = 0. \tag{5.2}$$

Since $\lambda_m \neq 0$ it follows from (5.2) that

$$W^h_{ijk} = 0. \tag{5.3}$$

As the space under consideration is not flat, (5.3) leads to a contradiction since it would require ψ_n to be a space of constant Riemannian curvature. Hence a ψ_n cannot be a recurrent space.

Next we suppose that a ψ_n is a Ricci-recurrent space, i.e. a space in which the Ricci tensor $R_{ij} (\not\equiv 0)$ satisfies the relation

$$R_{ij,m} = \lambda_m R_{ij} \tag{5.4}$$

for a non-zero vector λ_m.
In virtue of (2) and (5.4) we get

$$R_{hijk,m} = \lambda_m (R_{hijk} - W_{hijk}). \tag{5.5}$$

Multiplying both sides of (5.5) by g^{hk} and summing for h and k we have

$$R_{ij,m} = \lambda_m R_{ij}.$$

We can therefore state the following theorems:

THEOREM 5. *A non-flat $\psi_n (n \geqq 4)$ cannot be a recurrent space.*

THEOREM 6. *A necessary and sufficient condition that a $\psi_n (n \geqq 4)$ be a Ricci-recurrent space specified by a non-zero vector λ_m is that* (5.5) *holds.*

Let us now suppose that a $\psi_n (n \geqq 4)$ is a Ricci-recurrent space with λ_i as its vector of recurrence. Then from (1.1) we have

$$\lambda_k R_{ij} = \lambda_j R_{ik}$$

Hence

$$R_{ij} = s\lambda_i \lambda_j \quad (s \neq 0) \tag{5.6}$$

where s is a scalar factor of proportionality.
Therefore

$$R = g^{ij} R_{ij} = sg^{ij} \lambda_i \lambda_j . \tag{5.7}$$

It is known that in an irreducible Ricci-recurrent space the scalar curvature is zero. Hence from (5.7) we have

$$sg^{ij} \lambda_i \lambda_j = 0$$

whence

$$g^{ij} \lambda_i \lambda_j = 0 \qquad \text{because } s \neq 0.$$

The vector of recurrence is therefore a null vector. Again from (5.4)

$$\begin{aligned} R_{ij,ml} &= \lambda_m R_{ij,l} + \lambda_{m,l} R_{ij} \\ &= \lambda_l \lambda_m R_{ij} + \lambda_{m,l} R_{ij} . \end{aligned}$$

Therefore

$$R_{ij,ml} - R_{ij,lm} = R_{ij}(\lambda_{m,l} - \lambda_{l,m}). \tag{5.8}$$

It has been proved by Gy. Soós [1] that in a ψ_n

$$R_{ij,ml} - R_{ij,lm} = 0.$$

Hence from (5.8) we have

$$R_{ij}(\lambda_{m,l} - \lambda_{l,m}) = 0.$$

Since $R_{ij} \not\equiv 0$ we get

$$\lambda_{m,l} - \lambda_{l,m} = 0.$$

Thus we have the following theorem:

THEOREM 7. *In a Ricci-recurrent* $\psi_n (n \geqq 4)$, *the rank of the Ricci-tensor is* 1 *and the vector of recurrence is a null vector and the gradient of a scalar.*

6. Existence of fields of concurrent directions in a ψ_n $(n > 2)$

The question of the existence of fields of concurrent directions in a Riemannian space was discussed by Shirokov [4]. He proved that if in a

Riemannian space with metric tensor g_{ij} there exists a field of concurrent directions then the directions are determined by the equation

$$v_{\ ,i} = g_{ij}. \tag{6.1}$$

Let us now suppose that in a $\psi_n (n > 2)$ a vector v_i determines a field of concurrent directions. Then (6.1) will hold. From (6.1) we have

$$R_{tijk} v^k = 0. \tag{6.2}$$

Since

$$W_{tijk} = R_{tijk} - \frac{1}{n-1}(g_{tk}R_{ij} - g_{tj}R_{ik}) \tag{6.3}$$

$$\begin{aligned} W_{tijk}v^k &= R_{tijk}v^k - \frac{1}{n-1}(g_{tk}R_{ij}v^k - g_{tj}R_{ik}v^k) \\ &= -\frac{1}{n-1}g_{tk}R_{ij}v^k. \end{aligned} \tag{6.4}$$

Differentiating both sides of (6.4) covariantly we get

$$W_{tijk,l}v^k + W_{tijk}v^k_{,l} = -\frac{1}{n-1}g_{tk}(R_{ij,l}v^k + R_{ij}v^k_{,l}). \tag{6.5}$$

In virtue of (2) and (6.1) it follows from (6.5) that

$$W_{tijl} = -\frac{1}{n-1}g_{tk}R_{ij,l}v^k - \frac{1}{n-1}R_{ij}g_{tl}. \tag{6.6}$$

Making use of (6.3) we get from (6.6)

$$R_{tijl} + \frac{1}{n-1}g_{tj}R_{il} = -\frac{1}{n-1}g_{tk}R_{ij,l}v^k. \tag{6.7}$$

Multiplying both sides of (6.7) by g^{ij} and summing for i and j we have

$$R_{tl} + \frac{1}{n-1}R_{tl} = 0 \qquad \text{because } R \text{ is constant.}$$

Hence $R_{tl} = 0$.
Therefore from (6.6) and (6.3) we have

$$R_{tijk} = 0.$$

We can therefore state the following theorem:

THEOREM 8. *In a non-flat $\psi_n (n > 2)$ there cannot exist a field of concurrent directions.*

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