

# ON PRIMITIVE FOR THE GAP-INTEGRAL

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ABSTRACT. The concept of the GAP-integral was introduced by the authors. In this paper some results on primitive for the GAP-integral are presented.

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## 1. Introduction

The Approximately Continuous Perron integral was defined by Burkill [2] and its Riemann-type definition was given by Bullen [3]. Schwabik [6] presented a generalized version of the Perron integral leading to the new approach to a generalized ordinary differential equation. The authors developed a generalization of Approximately Continuous Perron (GAP)-integral and defined the primitive for the GAP-integral [1]. In the present paper attempt has been made to study some properties of primitive related to GAP-integral and also it is possible to give a characterization of the GAP-integral by its primitive. A fundamental type theorem for the GAP-integral has been established.

## 2. Preliminaries and definitions

**DEFINITION 2.1.** A collection  $\Delta$  of closed subintervals of  $[a, b]$  is called an approximate full cover (AFC) if for every  $x \in [a, b]$  there exists a measurable set  $D_x \subset [a, b]$  such that  $x \in D_x$  and  $D_x$  has density 1 at  $x$ , with  $[u, v] \in \Delta$  whenever  $u, v \in D_x$  and  $u \leq x \leq v$ .

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A division of  $[a, b]$  obtained by  $a = x_0 < x_1 < \dots < x_n = b$  and  $\{\xi_1, \xi_2, \dots, \xi_n\}$  is called a  $\Delta$ -division if  $\Delta$  is an approximate full cover with  $[x_{i-1}, x_i]$  coming from  $\Delta$  or more precisely, if we have  $x_{i-1} \leq \xi_i \leq x_i$  and  $x_{i-1}, x_i \in D_{\xi_i}$  for all  $i$ . We call  $\xi_i$  the associated point of  $[x_{i-1}, x_i]$  and  $x_i$  ( $i = 0, 1, \dots, n$ ) the division points.

A division of  $[a, b]$  given by  $a \leq y_1 < z_1 < y_2 < z_2 \dots < y_m < z_m \leq b$  and  $\{\zeta_1, \zeta_2, \dots, \zeta_m\}$  is called a  $\Delta$ -partial division if  $\bigcup_{i=1}^m [y_i, z_i] \subset [a, b]$  and  $\Delta$  is an approximate full cover with  $[y_i, z_i]$  coming from  $\Delta$  or more precisely,  $y_i \leq \zeta_i \leq z_i$  and  $y_i, z_i \in D_{\zeta_i}$  for all  $i$ .

In [1], the GAP-integral is defined as follows:

**DEFINITION 2.2.** A function  $U: [a, b] \times [a, b] \rightarrow \mathbb{R}$  is said to be Generalized Approximate Perron (GAP)-integrable to a real number  $A$  if for every  $\varepsilon > 0$  there is an AFC  $\Delta$  of  $[a, b]$  such that for every  $\Delta$ -division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$  we have

$$\left| (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - A \right| < \varepsilon$$

and we write  $A = (\text{GAP}) \int_a^b U$ .

The set of all functions  $U$  which are GAP-integrable on  $[a, b]$  is denoted by  $\text{GAP}[a, b]$ . We use the notation

$$S(U, D) = (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\}$$

for the Riemann-type sum corresponding to the function  $U$  and the  $\Delta$ -division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$ .

Note that the integral is uniquely determined.

With the notion of partial division we have proved the following theorem in [1].

**THEOREM 2.1 (Saks-Henstock Lemma).** *Let  $U: [a, b] \times [a, b] \rightarrow \mathbb{R}$  be (GAP)-integrable over  $[a, b]$ . Then, given  $\varepsilon > 0$ , there is an approximate full cover  $\Delta$  of  $[a, b]$  such that for every  $\Delta$ -division  $D = \{([\alpha_{j-1}, \alpha_j], \tau_j) : j = 1, 2, \dots, q\}$  of  $[a, b]$  we have*

$$\left| \sum_{j=1}^q \{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (\text{GAP}) \int_a^b U \right| < \varepsilon.$$

*Then, if  $\{([\beta_j, \gamma_j], \zeta_j) : j = 1, 2, \dots, m\}$  represents a  $\Delta$ -partial division of  $[a, b]$ , we have*

$$\left| \sum_{j=1}^m \{U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)\} - (\text{GAP}) \int_{\beta_j}^{\gamma_j} U \right| < \varepsilon.$$

In [1], the indefinite GAP-integral is defined as follows:

**DEFINITION 2.3.** Let  $U \in \text{GAP}[a, b]$ . The function  $\phi: [a, b] \rightarrow \mathbb{R}$  defined by  $\phi(s) = (\text{GAP}) \int_a^s U$ ,  $a < s \leq b$ ,  $\phi(a) = 0$  is called the indefinite GAP-integral of  $U$ .

For  $[\alpha, \beta] \subset [a, b]$ , we put  $\phi(\alpha, \beta) = \phi(\beta) - \phi(\alpha) = (\text{GAP}) \int_\alpha^\beta U$ .

We take this function  $\phi$  as the primitive of  $U$ .

We need the following definitions to establish some results in the consequence:

**DEFINITION 2.4.** ([5]) If  $E$  is a measurable set, then  $E^d$  represents the set of all points  $x \in E$  such that  $x$  is a point of density of  $E$ .

**DEFINITION 2.5.** ([4]) A number  $A$  is said to be the approximate limit of a function  $f$  at  $x_0$  if for every  $\varepsilon > 0$  there exists a set  $D$  of density 1 at  $x_0$  such that

$$|f(x) - A| < \varepsilon \quad \text{for every } x \in D \setminus \{x_0\}.$$

We write  $\lim_{x \rightarrow x_0} \text{ap } f(x) = A$ .

Next we define a function  $f$  to be approximately continuous at  $x_0$  if

$$\lim_{x \rightarrow x_0} \text{ap } f(x) = f(x_0).$$

The approximate derivative of  $f$  at  $x_0$ , denoted by  $\text{AD } f(x_0)$ , is

$$\text{AD } f(x_0) = \lim_{x \rightarrow x_0} \text{ap } \frac{f(x) - f(x_0)}{x - x_0}.$$

We state Vitali's covering theorem without proof for further use:

**THEOREM 2.2.** ([4]) *If a family of closed intervals covers a set  $X$  in the Vitali sense, i.e., for every  $x \in X$  and  $\eta > 0$  there is a closed interval  $I$  in the family such that*

$$x \in I \quad \text{and} \quad |I| < \eta,$$

*then for every  $\varepsilon > 0$  there is a finite number of disjoint closed intervals  $I_1, I_2, \dots, I_n$  in the family such that*

$$|X| < \sum_{i=1}^n |I_i| + \varepsilon$$

where  $|X|$  denotes the outer measure of  $X$ .

We recall that the outer measure of  $X$  is defined to be

$$|X| = \inf \left\{ \sum_i |I_i| : \bigcup_i I_i \supset X \right\}$$

in which  $I_i$  denote intervals.

### 3. Some results on primitives

**THEOREM 3.1.** *If  $U \in \text{GAP}[a, b]$  and  $\phi$  is the primitive of  $U$ , then  $\phi$  is measurable if the function  $U(s, \cdot): [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  for  $s \in [a, b]$ .*

**Proof.** In order to prove that the function  $\phi$  is measurable, let  $\alpha$  be a real number.

Let  $E = \{s \in (a, b) : \phi(s) > \alpha\}$ . We shall show that the set  $E$  is measurable.

Let  $s \in E$  and choose  $\eta > 0$  such that  $3\eta = \phi(s) - \alpha$ .

Since  $U \in \text{GAP}[a, b]$ , given  $\eta > 0$ , there is an approximate full cover  $\Delta$  of  $[a, b]$  such that for every  $\Delta$ -division  $D$  of  $[a, b]$  we have

$$|S(U, D) - \phi(a, b)| < \eta.$$

Again, since the function  $U(s, \cdot)$  is continuous on  $[a, b]$ , for  $\eta > 0$  there exists a  $\delta_s > 0$  such that

$$|U(s, t) - U(s, s)| < \eta \quad \text{whenever} \quad |t - s| < \delta_s.$$

Let  $\{D_s : s \in [a, b]\}$  be the collection of sets generated by  $\Delta$ . We will assume that each point of  $D_s$  is a point of density of  $D_s$ .

Suppose that  $t \in (s, s + \delta_s) \cap D_s$ . Let  $D$  be a  $\Delta$ -division of  $[a, s]$  and let  $D' = D \cup ([s, t], s)$ . Then  $D'$  is a  $\Delta$ -division of  $[a, t]$ .

Hence

$$\begin{aligned} \phi(t) &= \phi(a, t) > S(U, D') - \eta \\ &= S(U, D) + \{U(s, t) - U(s, s)\} - \eta \\ &> \phi(a, s) - 3\eta = \phi(s) - 3\eta = \alpha. \end{aligned}$$

In a similar way, we can show that  $\phi(t) > \alpha$  for all  $t \in (s - \delta_s, s) \cap D_s$ .

For each  $s$ , let  $A_s = (s - \delta_s, s + \delta_s) \cap D_s$ .

Then  $A_s = A_s^d$  and  $A_s \subseteq E$ .

Hence the set  $E = \bigcup_{s \in E} A_s$  is measurable, ([5]).

Therefore,  $\phi$  is a measurable function. □

**THEOREM 3.2.** *If  $U \in \text{GAP}[a, b]$ , then its primitive  $\phi$  is approximately differentiable almost everywhere and the approximate derivative*

$$\text{AD } \phi(\tau) = \lim_{t \rightarrow \tau} \text{ap} \frac{U(\tau, t) - U(\tau, \tau)}{t - \tau}$$

for almost all  $\tau \in [a, b]$ , if the limit exists.

ON PRIMITIVE FOR THE GAP-INTEGRAL

Proof. Since  $U \in \text{GAP}[a, b]$ , given  $\varepsilon > 0$ , there exists an approximate full cover  $\Delta$  of  $[a, b]$  such that for every  $\Delta$ -division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$  we have

$$\sum |\{\phi(\beta) - \phi(\alpha)\} - \{U(\tau, \beta) - U(\tau, \alpha)\}| < \varepsilon.$$

Let

$$X = \left\{ \tau \in [a, b] : \text{either AD } \phi(\tau) \text{ does not exist,} \right. \\ \left. \text{or, if it does, it is not equal to } \lim_{t \rightarrow \tau} \text{ap } \frac{U(\tau, t) - U(\tau, \tau)}{t - \tau} \right\}.$$

We shall show that  $X$  is of measure zero.

From the definition of  $X$  we see that for each  $\tau \in [a, b] - X$ , there exists a measurable set  $D_\tau \subset [a, b]$  of density 1 at  $\tau$  and

$$\text{AD } \phi(\tau) = \lim_{\substack{\beta \rightarrow \tau \\ \beta \in D_\tau}} \frac{\phi(\beta) - \phi(\tau)}{\beta - \tau}.$$

Then for each  $\tau \in X$  there is a  $\eta(\tau) > 0$  such that for every approximate neighbourhood  $D_\tau$  of  $\tau$  either there is a point  $\alpha \in D_\tau$  with  $0 < \tau - \alpha < \delta$  ( $\delta > 0$ ) and

$$\left| \frac{\phi(\tau) - \phi(\alpha)}{\tau - \alpha} - \left\{ \frac{U(\tau, \tau) - U(\tau, \alpha)}{\tau - \alpha} \right\} \right| > \eta(\tau)$$

i.e.

$$|\{\phi(\tau) - \phi(\alpha)\} - \{U(\tau, \tau) - U(\tau, \alpha)\}| > \eta(\tau)|\tau - \alpha|.$$

or there is a point  $\beta \in D_\tau$  with  $0 < \beta - \tau < \delta$  and

$$|\{\phi(\beta) - \phi(\tau)\} - \{U(\tau, \beta) - U(\tau, \tau)\}| > \eta(\tau)|\beta - \tau|.$$

Let  $X_n = \{\tau \in X : \eta(\tau) \geq \frac{1}{n}\}$ ,  $n = 1, 2, \dots$  and so  $X = \bigcup_{n=1}^{\infty} X_n$ .

Then the above family of closed intervals  $[\alpha, \tau]$  or  $[\tau, \beta]$  covers  $X_n$  in the Vitali sense.

By the Vitali's covering theorem, for every  $\varepsilon > 0$ , we can find a finite number of disjoint closed intervals  $[\alpha_k, \beta_k]$  for  $k = 1, 2, \dots, m$  with  $\alpha_k = \tau_k$  or  $\beta_k = \tau_k$  such that

$$|X_n| < \sum_{k=1}^m |\beta_k - \alpha_k| + \varepsilon \\ < \sum_{k=1}^m \frac{|\{\phi(\beta_k) - \phi(\alpha_k)\} - \{U(\tau_k, \beta_k) - U(\tau_k, \alpha_k)\}|}{\eta(\tau_k)} + \varepsilon \\ < n\varepsilon + \varepsilon = \varepsilon(n + 1)$$

Since  $\varepsilon > 0$  is arbitrary, the outer measure of  $X_n$  is 0 and so is  $X$ . □

**THEOREM 3.3.** *Let  $U : [a, b] \times [a, b] \rightarrow \mathbb{R}$  be a function such that  $U(\tau, \cdot) : [a, b] \rightarrow \mathbb{R}$  is a continuous function on  $[a, b]$  for  $\tau \in [a, b]$ . Also let  $\phi : [a, b] \rightarrow \mathbb{R}$  be a function which is approximately continuous on  $[a, b]$  and approximately differentiable nearly everywhere i.e. everywhere except perhaps for a countable number of points in  $[a, b]$  such that*

$$\text{AD } \phi(\tau) = \lim_{t \rightarrow \tau} \text{ap} \frac{U(\tau, t) - U(\tau, \tau)}{t - \tau} \quad \text{for } \tau \in [a, b].$$

Then  $U \in \text{GAP}[a, b]$  and  $(\text{GAP}) \int_a^b U = \phi(b) - \phi(a)$ .

**PROOF.** Let  $X = \{\tau_n : n \in \mathbb{Z}^+\}$  be the set of all points  $\tau \in [a, b]$  for which  $\text{AD } \phi(\tau)$  does not exist.

For each  $\tau \in [a, b] - X$ , there exists a measurable set  $D_\tau \subseteq [a, b]$  of density 1 at  $\tau$  and

$$\text{AD } \phi(\tau) = \lim_{\substack{\beta \rightarrow \tau \\ \beta \in D_\tau}} \frac{\phi(\beta) - \phi(\tau)}{\beta - \tau}.$$

Again, by the given condition, for given  $\varepsilon > 0$  we have

$$\left| \text{AD } \phi(\tau) - \frac{U(\tau, \beta) - U(\tau, \tau)}{\beta - \tau} \right| < \varepsilon \quad \text{whenever } \beta \in D_\tau.$$

For each  $n$ , there exists a measurable set  $D_{\tau_n} \subseteq [a, b]$  of density 1 at  $\tau_n$  and

$$\phi(\tau_n) = \lim_{\substack{\tau \rightarrow \tau_n \\ \tau \in D_{\tau_n}}} \phi(\tau).$$

For each  $\tau \in [a, b] - X$ , there exists  $\delta_\tau > 0$  such that  $\beta \in D_\tau \cap (\tau - \delta_\tau, \tau + \delta_\tau) = S_\tau$  implies

$$|\{\phi(\beta) - \phi(\tau)\} - \text{AD } \phi(\tau)(\beta - \tau)| \leq \varepsilon |\beta - \tau|$$

and

$$|\text{AD } \phi(\tau)(\beta - \tau) - \{U(\tau, \beta) - U(\tau, \tau)\}| < \varepsilon |\beta - \tau|$$

Therefore,

$$\begin{aligned} & |\{\phi(\beta) - \phi(\tau)\} - \{U(\tau, \beta) - U(\tau, \tau)\}| \\ & \leq |\{\phi(\beta) - \phi(\tau)\} - \text{AD } \phi(\tau)(\beta - \tau)| + |\text{AD } \phi(\tau)(\beta - \tau) - \{U(\tau, \beta) - U(\tau, \tau)\}| \\ & < \varepsilon |\beta - \tau| + \varepsilon |\beta - \tau| < 2\varepsilon |\beta - \tau|. \end{aligned}$$

The set  $S_\tau$  is measurable and  $\tau \in S_\tau^d$ .

For each  $n$ , there exists  $\delta_n > 0$  such that  $\alpha, \beta \in D_{\tau_n} \cap (\tau_n - \delta_n, \tau_n + \delta_n) = S_{\tau_n}$  implies

$$|\phi(\beta) - \phi(\alpha)| < \varepsilon 2^{-n}.$$

Again, since  $U(\tau, \cdot)$  is continuous on  $[a, b]$ , we have

$$|U(\tau_n, \beta) - U(\tau_n, \alpha)| < \varepsilon 2^{-n} \quad \text{for } \alpha, \beta \in S_{\tau_n}.$$

ON PRIMITIVE FOR THE GAP-INTEGRAL

The set  $S_{\tau_n}$  is measurable and  $\tau_n \in S_{\tau_n}^d$ .

The sets  $\{S_\tau : \tau \in [a, b] - X\}$  and  $\{S_{\tau_n} : n \in Z^+\}$  define an approximate full cover  $\Delta$  of  $[a, b]$ .

Let  $D = ([\alpha, \beta], \tau)$  be any  $\Delta$ -division of  $[a, b]$ .

We split up the sum  $\sum$  over the  $\Delta$ -division into two partial sums  $\sum_1$  and  $\sum_2$  in which  $\tau \in [a, b] - X$  and  $\tau \in X$  respectively.

Then we obtain

$$\begin{aligned} & \left| \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - \phi(a, b) \right| \\ & \leq \sum_1 |\{U(\tau, \beta) - U(\tau, \alpha)\} - \phi(\alpha, \beta)| + \sum_2 |\phi(\alpha, \beta)| + \sum_2 |U(\tau, \beta) - U(\tau, \alpha)| \\ & < \sum_1 [|\{U(\tau, \beta) - U(\tau, \tau)\} - \{\phi(\beta) - \phi(\tau)\}| \\ & \quad + |\{U(\tau, \tau) - U(\tau, \alpha)\} - \{\phi(\tau) - \phi(\alpha)\}|] + 2 \sum_2 \varepsilon 2^{-n} \\ & < \sum_1 \{2\varepsilon(\beta - \tau) + 2\varepsilon(\tau - \alpha)\} + 2\varepsilon \\ & < 2\varepsilon(b - a) + 2\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, it follows that

$$U \in \text{GAP}[a, b] \quad \text{and} \quad (\text{GAP}) \int_a^b U = \phi(b) - \phi(a).$$

□

**THEOREM 3.4.** *A function  $U : [a, b] \times [a, b] \rightarrow \mathbb{R}$  is GAP-integrable on  $[a, b]$  with primitive  $\phi$  if and only if for every  $\varepsilon > 0$  there exists an approximate full cover  $\Delta$  of  $[a, b]$  such that*

$$(D) \sum |\phi(\alpha, \beta)| < \varepsilon \quad \text{and} \quad (D) \sum |U(\tau, \beta) - U(\tau, \alpha)| < \varepsilon$$

whenever  $D = ([\alpha, \beta], \tau)$  is a partial division in  $\Delta \cap \Gamma_\varepsilon$  where  $\Gamma_\varepsilon$  is defined as

$$\Gamma_\varepsilon = \left\{ ([\alpha, \beta], \tau) : |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \geq \varepsilon|\beta - \alpha| \right\}.$$

**Proof.** Let there be an approximate full cover  $\Delta$  of  $[a, b]$  such that the two inequalities are satisfied.

Then for any  $\Delta$ -division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$  we have

$$\begin{aligned} & (D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \\ & < (D \cap \Gamma_\varepsilon) \sum |\phi(\alpha, \beta)| + (D \cap \Gamma_\varepsilon) \sum |U(\tau, \beta) - U(\tau, \alpha)| \\ & \quad + (D \setminus \Gamma_\varepsilon) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \\ & < \varepsilon + \varepsilon + \sum \varepsilon |\beta - \alpha| \\ & < 2\varepsilon + \varepsilon(b - a) = \varepsilon(2 + b - a). \end{aligned}$$

Therefore,  $U \in \text{GAP}[a, b]$  with primitive  $\phi$ .

Conversely, suppose that  $U \in \text{GAP}[a, b]$  with primitive  $\phi$ .

Then for every  $\varepsilon \in (0, 1)$  and for every  $k \in \mathbb{N}$  there exists an approximate full cover  $\Delta_k$  of  $[a, b]$  such that for any  $\Delta_k$ -division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$  we have

$$(D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| < \frac{\varepsilon^2}{k2^{k+1}}$$

Let

$$E_k = \left\{ ([\alpha, \beta], \tau) : k - 1 \leq \left| \frac{U(\tau, \beta) - U(\tau, \alpha)}{\beta - \alpha} \right| < k, \tau \in [a, b] \right\}, \quad k = 1, 2, \dots$$

Consider

$$\Delta = \bigcup_{k=1}^{\infty} \{([\alpha, \beta], \tau) : ([\alpha, \beta], \tau) \in \Delta_k \cap E_k\}.$$

Then  $\Delta$  is an approximate full cover of  $[a, b]$ .

Then for any partial division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$  in  $\Delta \cap \Gamma_\varepsilon$  we have

$$\begin{aligned} & (D) \sum |U(\tau, \beta) - U(\tau, \alpha)| \\ & < \sum k|\beta - \alpha| \\ & < \sum_{k=1}^{\infty} \frac{k}{\varepsilon} (D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \\ & < \sum_{k=1}^{\infty} \frac{k}{\varepsilon} \frac{\varepsilon^2}{k2^{k+1}} \\ & < \varepsilon/2 < \varepsilon. \end{aligned}$$

Again, since  $U \in \text{GAP}[a, b]$  with primitive  $\phi$ , by Saks-Henstock Lemma, given  $\varepsilon > 0$ , there exists an approximate full cover  $\Delta$  of  $[a, b]$  such that for any partial division  $D = ([\alpha, \beta], \tau)$  of  $[a, b]$  in  $\Delta \cap \Gamma_\varepsilon$  we have

$$(D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| < \varepsilon.$$



ON PRIMITIVE FOR THE GAP-INTEGRAL

Hence

$$\begin{aligned} & (D) \sum |\phi(\alpha, \beta)| \\ & \leq (D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| + (D) \sum |U(\tau, \beta) - U(\tau, \alpha)| \\ & < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

□

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