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ON NI Γ-SEMIRINGS

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Abstract: In this paper, the notions of semi-reduced prime Γ-semiring, semi-reduced prime ideal and NI Γ-semiring are introduced. We will characterize the NI Γ-semiring by semi-reduced prime ideals.

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1. Introduction

The concept of an NI ring was first given by G. Marks in 2001 as follows: A ring R is called a NI ring if and only if $\mathcal{N}^*(R) = \mathcal{N}(R)$, where $\mathcal{N}^*(R)$ is the unique maximal nil ideal of the ring R and $\mathcal{N}(R)$ is the set of all nilpotent elements of R. We notice that C.Y. Hong et al.

Kwak in 2000 characterized a ring whose unique maximal nil ideal coincides with the set of all its nilpotent elements. Recently, T.K. Dutta and M.L. Das have further extended the notion of NI rings to NI semirings. Also, Singular

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ideals of ternary semirings have ben studied by T K Duta and S. Mandal in [11].

In this paper, the concept of a NI Γ-semiring with unity whose every ideal is a k-ideal is introduced. Some earlier works on Γ-semirings may be found in [6], [7], [8], [9] and [13]. We will concentrate in the study of the NI Γ-semirings. Some new results which are closely related to the semi-reduced prime ideals of an NI Γ-semirings will be given.

2. Preliminaries

We first give the definition of a Γ-semiring.

Definition 2.1. (see [4]) Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping from $S \times \Gamma \times S \longrightarrow S$ (the image to be denoted by $a\alpha b$, for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;

(iii)$a(\alpha + \beta)c = a\alpha c + a\beta c$;

(iv)$a\alpha(b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.2. (see [4]) Let S be a Γ-semiring. If there exists an element $0 \in S$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$ and for all $\alpha \in \Gamma$ then '0' is called the zero element or simply the zero of the Γ-semiring S. In this case we say S is a Γ-semiring with zero.

Throughout this paper, we assume that a Γ-semiring always contains a zero element and S^* denotes the set of all nonzero elements of S i.e. $S^* = S \setminus \{0\}$.

We first state the definition of the right ideals of a Γ-semiring S.

Definition 2.3. (see [4]) A nonempty subset I of a Γ-semiring S is called a right ideal of S if $I + I \subseteq I$ and $I\Gamma S \subseteq I$, where for subsets U, V of S and Δ of Γ, the set $U\Delta V$ is defined by

$$\{\sum_{i=1}^{n} u_i\gamma_i v_i : u_i \in U, v_i \in V, \gamma_i \in \Delta, \text{ where } n \text{ is a positive integer}\}.$$

Similarly, we can define the left ideal of a Γ-semiring. A nonempty subset I of a Γ-semiring S is an ideal of S if it is a left as well as a right ideal of S.

Definition 2.4. (see [4]) An ideal I of a Γ-semiring S is called a k-ideal if for $x, y \in S$, $x + y \in I$ and $y \in I$ implies that $x \in I$.

The following definitions are similar to the corresponding definitions in ring theory.

Definition 2.5. (see [2]) Let S be a Γ-semiring. A proper ideal P of a Γ-semiring S is called a *prime ideal* if for any two ideals A, B of S $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

Definition 2.6. (see [2]) A Γ-semiring S is said to be a prime Γ-semiring if $\{0\}$ is a prime ideal of S.

Definition 2.7. (see [3]) Let S be a Γ-semiring. A proper ideal P of a Γ-semiring S is called a *semiprime ideal* if for any ideal A of S $A\Gamma A \subseteq P$ implies that $A \subseteq P$.

Definition 2.8. (see [2]) A Γ-semiring S is said to be a semiprime Γ-semiring if $\{0\}$ is a semiprime ideal of S.

Definition 2.9. An ideal I of a Γ-semiring S is said to be *completely prime* if $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$ for $a, b \in S$.

Definition 2.10. An ideal I of a Γ-semiring S is said to be *completely semiprime* if $a\Gamma a \subseteq I$ implies $a \in I$ for $a \in S$.

Definition 2.11. (see [2]) Let S be a Γ-semiring. Then a nonempty subset H of Γ-semiring S is said to be an *m-system* of S if $c, d \in H$ implies there exist $s \in S$ and $\alpha, \beta \in \Gamma$ such that $c\alpha s\beta d \in H$.

Definition 2.12. (see [4]) Let S be a Γ-semiring and F the free additive commutative semigroup generated by $S \times \Gamma$. Then the relation ρ on F defined by $\sum_{i=1}^{m}(x_i, \alpha_i)\rho\sum_{j=1}^{n}(y_j, \beta_j)$ if and only if $\sum_{i=1}^{m} x_i\alpha_i s = \sum_{j=1}^{n} y_j\beta_j s$ for all $s \in S(m, n \in \mathbb{Z}^+$, the set of all positive integers), is a congruence on F.

We denote the congruence class containing $\sum_{i=1}^{m}(x_i, \alpha_i)$ by $\sum_{i=1}^{m}[x_i, \alpha_i]$. Then F/ρ is an additive commutative semigroup. Now F/ρ forms a semiring under the multiplication defined by $(\sum_{i=1}^{m}[x_i, \alpha_i])(\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j]$. We denote this semiring by L and call it the left operator semiring of the Γ-semiring S.

Dually, we can define the right operator semiring R of the Γ-semiring S

where $R = \{\sum_{i=1}^{m}[\alpha_i, x_i] : \alpha_i \in \Gamma, x_i \in S, i = 1, 2, ...m; m \in \mathbb{Z}^+\}$ and the multiplication on R is defined as $(\sum_{i=1}^{m}[\alpha_i, x_i])(\sum_{j=1}^{n}[\beta_j, y_j]) = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$.

Definition 2.13. (see [4]) Let S be a Γ-semiring and L be the left operator semiring and R be the right operator semiring of S. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L$ (respectively $\sum_{j=1}^{n}[\nu_j, f_j] \in R$) such that $\sum_{i=1}^{m} e_i\delta_i a = a$(respectively $\sum_{j=1}^{n} a\nu_j f_j = a$) for all $a \in S$ then S is said to have an *left unity* $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively the *right unity*$\sum_{j=1}^{n}[\nu_j, f_j]$).

Definition 2.14. (see [5]) An element a of a Γ-semiring S is said to be *nilpotent* if for any $\gamma \in \Gamma$ there exists a positive integer $n = n(\gamma, a)$ such that $(a\gamma)^{n-1}a = 0$.

Definition 2.15. (see [5]) An element a of a Γ-semiring S is said to be *strongly nilpotent* if there exists a positive integer n such that $(a\Gamma)^{n-1}a = 0$.

Definition 2.16. (see [9]) A Γ-semiring S is said to be *SN* Γ*-semiring* if $\mathcal{N}(S) = \mathcal{N}_\Gamma(S)$, where $\mathcal{N}(S)$ is the set of all nilpotent elements of S and $\mathcal{N}_\Gamma(S)$ is the set of all strongly nilpotent elements of S.

Definition 2.17. (see [12]) A semiring D with identity is called a *division semiring* if and only if every non zero element has an inverse.

Definition 2.18. (see [4]) For an ideal A of a Γ-semiring S, the Γ-congruence on S, denoted by ρ_A and defined as $s\rho_A s'$ if and only if $s + a_1 = s' + a_2$, for some $a_1, a_2 \in A$, is called the Bourne Γ-congruence on S defined by the ideal A.

Now, we denote the Bourne Γ-congruence (ρ_A) class of an element r of S by r/ρ_A or simply by r/A and denote the set of all such Γ-congruences classes of the elements of the Γ-semiring S by S/ρ_A or simply by S/A.

Definition 2.19. (see [4]) For an ideal A of a Γ-semiring S if the Bourne Γ-congruence ρ_A, defined by A, is proper i.e. $0/A \neq S$ then we define on S/A the following operations: $s/A + s'/A = (s + s')/A$ and $(s/A)\alpha(s'/A) = (s\alpha s')/A$ for all $\alpha \in \Gamma$. Now S/A is a Γ-semiring with these operations. We call this Γ-semiring the Bourne factor Γ-semiring or simply the factor Γ-semiring of S

by A.

Definition 2.20. (see [9]) A Γ-semiring S is said to be right symmetric if for $a, b, c \in S$, $a\Gamma b\Gamma c = 0$ implies $a\Gamma c\Gamma b = 0$. An ideal I of a Γ-semiring S is said to be right symmetric if $a\Gamma b\Gamma c \subseteq I$ implies $a\Gamma c\Gamma b \subseteq I$ for $a, b, c \in S$.

Similarly we can define left symmetric Γ-semiring and left symmetric ideal.

Definition 2.21. (see [9]) A one sided ideal I of a Γ-semiring S is said to have the *insertion of factors property* or simply IFP if for any $a, b \in S$, $a\Gamma b \subseteq I$ implies $a\Gamma S\Gamma b \subseteq I$.

Definition 2.22. (see [6]) Let S be a Γ-semiring and L (respectively R) be its left (respectively right) operator semiring. Then for $A \subseteq L, B \subseteq R, C \subseteq S$, $A^+ = \{x \in S : [x, \alpha] \in A \text{ for all } \alpha \in \Gamma\}$, $B^* = \{x \in S : [\alpha, x] \in B \text{ for all } \alpha \in \Gamma\}$, $C^{+'} = \{\sum_{i=1}^{n}[x_i, \alpha_i] \in L : \sum_{i=1}^{n} x_i\alpha_i s \in C \text{ for all } s \in S\}$ and $C^{*'} = \{\sum_{j=1}^{m}[\beta_j, y_j] \in R : \sum_{j=1}^{m} s\beta_j y_j \in C \text{ for all } s \in S\}$.

Proposition 2.23. *(see [6]) Let S be a Γ-semiring with unity and L its left operator semiring. Then the following statements hold:*

(i) if Q is a prime ideal of S then $Q^{+'}$ is a prime ideal of L;
(ii) if P is a prime ideal of L then P^+ is a prime ideal of S;

(iii) $(Q^{+'})^+ = Q$ and $(P^+)^{+'} = P$

Throughout this paper, we assume that a Γ-semiring S always contain a unity whose every ideal is a k-ideal.

3. Semi-Reduced Prime Γ-Semirings

We begin with the following examples:

Example 3.1. Let M be a Γ-ring with unity. Then M is a Γ-semiring with unity and every ideal of M is a k-ideal.

Example 3.2. Let R be a Γ-ring with unity, $S = \{r\omega : r \in \mathbb{R}_0^+\}$ and $\Gamma_1 = \{r\omega^2 : r \in \mathbb{R}_0^+\}$, where ω be a cube root of unity and $\mathbb{R}_0^+$ is the set of all non negetive real numbers. Then S is a Γ_1-semiring with usual addition and multiplication. Also $R \times S$ is a $\Gamma \times \Gamma_1-$ semiring with unity which is not a $\Gamma \times \Gamma_1-$ ring but every ideal of $R \times S$ is a k-ideal.

Example 3.3. Let L be a bounded distributive lattice with maximal element 1. Then L is a Γ-semiring with unity with respect to componentwise addition and multiplication, where $\Gamma = L$. Now L is not a Γ-ring. Also every ideal of S is a k-ideal.

The following definition is an important definition in this paper.

Definition 3.4. A Γ-semiring S is said to be a *semi-reduced prime* Γ*-semiring* if S is a prime Γ-semiring and it has no nonzero nil ideals.

Example 3.5. Let D be a division semiring. Consider two sets: $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \\ 0 & 0 \end{pmatrix} : d \in D \right\}$ and $\Gamma = \left\{ \begin{pmatrix} d_4 & d_5 & d_6 \\ 0 & d_7 & d_8 \end{pmatrix} : d_4, d_5, d_6, d_7, d_8 \in D \right\}$. Then S is a Γ-semiring with respect to usual matrix addition and matrix multiplication.

Here, $\mathcal{N}(S) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Hence, S is semi-reduced prime Γ-semiring.

Example 3.6. Let $S = \{r\omega : r \in \mathbb{Z}\}$ and $\Gamma = \{r\omega^2 : r \in \mathbb{Z}\}$, where ω is a cube root of unity and $\mathbb{Z}$ is the set of all integers. Then S is a Γ-semiring with usual addition and multiplication. Here S has no nonzero nil ideal and S is a prime Γ-semiring. Therefore, S is a semi-reduced prime Γ-semiring.

Example 3.7. Let M be a maximal ideal of the Γ-semiring S. Then S/M is a semi-reduced prime Γ-semiring.

Definition 3.8. A proper ideal P of a Γ-semiring S is said to be a *semi-reduced prime ideal* if and only if S/P is a semi-reduced prime Γ-semiring.

Example 3.9. Every maximal ideal of a Γ-semiring is a semi-reduced prime ideal.

Definition 3.10. Let S be a Γ-semiring and α be any element of Γ. A nonempty subset M of S called an α*-semigroup* if $x\alpha y \in M$ for all $x, y \in M$.

Example 3.11. Every Γ-semiring is an α-semigroup for all $\alpha \in \Gamma$.

In the following propositions, we study the semi-reduced prime ideal of a Γ-semiring S.

Proposition 3.12. *Let S be a Γ-semiring and M be an α-semigroup in $S - \{0\}$, where $\alpha \in \Gamma$. Suppose that P is an ideal of S maximal with respect to the property $P \cap M = \phi$. Then P is a semi-reduced prime ideal of S.*

Proof. We first prove that P is a prime ideal of S. Let $A\Gamma B \subseteq P$, where A

and B are any two ideals of S. If possible, let $A \nsubseteq P$ and $B \nsubseteq P$. Then there exists $a \in A$ and $b \in B$ such that $a, b \notin P$. Therefore $(P+ < a >) \cap M \neq \phi$ and $(P+ < b >) \cap M \neq \phi$. Let $x_1 \in (P+ < a >) \cap M$ and $x_2 \in (P+ < b >) \cap M$. So $x_1 \alpha x_2 \in (P+ < a >)\Gamma(P+ < b >) \subseteq P+ < a > \Gamma < b > \subseteq P + P \subseteq P$. Also since M is an α-semigroup, $x_1 \alpha x_2 \in M$. Therefore, $P \cap M \neq \phi$, a contradiction. Hence $A\Gamma B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$. Thus P is a prime ideal of S.

To prove that P is a semi-reduced prime ideal of S, we first show that S/P contains no nonzero nil ideal. Suppose if possible, let I/P be a nonzero nil ideal of S/P. Then $P \subset I$ and I is an ideal of S. Therefore $I \cap M \neq \phi$. Let $x \in I \cap M$. So $x \in I$ and $x \in M$. Now $x/P \in I/P$ implies that, for any $\gamma \in \Gamma$ there exixts a positive integer n depending on x and γ such that$((x/P)\gamma)^{n-1}(x/P) = 0/P$ i.e. $((x\gamma)^{n-1}x)/P = 0/P$. Thus for all $\gamma \in \Gamma$ there exixts a positive integer n such that $(x\gamma)^{n-1}x \in P$ as P is a k-ideal of S. Also since M is an α-semigroup, in $S - \{0\}$, $x \in M$ implies $(x\alpha)^{m-1}x \in M$ for some positive integer m. Hence, $P \cap M \neq \phi$, a contradiction. This shows that P is a semi-reduced prime ideal of S. □

We now have the following corollary.

Corollary 3.13. *Let S be a Γ-semiring and x a nonzero element of S. Let $M = \{x, x\alpha x, (x\alpha)^2 x, (x\alpha)^3 x, ...\}$, where $\alpha \in \Gamma$ such that $(x\alpha)^{n-1}x \neq 0$ for all natural number n. Then there exists a semi-reduced prime ideal P of S such that $P \cap M = \phi$.*

Proof. Since x is a nonzero element of S and $(x\alpha)^{n-1}x \neq 0$ for all natural number n, M is an α-semigroup in $S - \{0\}$. Let $\mathcal{F} = \{I : I$ is an ideal of S such that $I \cap M = \phi\}$. Then it is clear that $\{0\} \in \mathcal{F}$. Thus $\mathcal{F}$ is a nonempty poset with respect to set inclusion. Suppose that $\mathcal{C} = \{K_i : i \in \Lambda\}$ be a chain in $\mathcal{F}$. Then $K = \bigcup_{i \in \Lambda} K_i$ is an upper bound of $\mathcal{C}$ in $\mathcal{F}$. Now, by Zorn's lemma, we can easily verify that $\mathcal{F}$ has a maximal element, say P. This leads to P is an ideal of S maximal with respect to the property $P \cap M = \phi$. Thus by **Proposition 3.12**, P is a semi-reduced prime ideal of S. □

Proposition 3.14. *Let S be a Γ-semiring and M be an α-semigroup in $S - \{0\}$, where $\alpha \in \Gamma$. Let I be an ideal of S such that $I \cap M = \phi$. Then there exists a semi-reduced prime ideal P of S such that $I \subseteq P$ and $P \cap M = \phi$.*

Proof. Let $\mathcal{F} = \{J : J$ be an ideal of S containing I and $J \cap M = \phi\}$. Since $I \in \mathcal{F}$, $\mathcal{F}$ is a nonempty set. Now $\mathcal{F}$ is a poset with respect to the set inclusion. Let $\mathcal{C} = \{K_i : i \in \Lambda\}$ be a chain in $\mathcal{F}$. Then $K = \bigcup_{i \in \Lambda} K_i$ is an upper bound of

$\mathcal{C}$ in $\mathcal{F}$. Therefore by Zorn's lemma, $\mathcal{F}$ has a maximal element, say P. Then $I \subseteq P$. Hence P is maximal with respect to the property $P \cap M = \phi$. Now as in Proposition 3.12, we can show that P is a semi-reduced prime ideal of S. Thus the result follows. □

Corollary 3.15. *Let S be a Γ-semiring without nonzero nilpotent elements and I a completely semiprime ideal of S such that $x \notin I$. Then there exists a semi-reduced prime ideal P of S containing I such that $x \notin P$.*

Proof. Since x is not a nonzero nilpotent element of S, there exists an element $\alpha \in \Gamma$ such that $(x\alpha)^{n-1}x \neq 0$ for all natural number n. Let $M = \{x, x\alpha x, (x\alpha)^2 x, (x\alpha)^3 x, ...\}$ Then M is a α-semigroup in $S - \{0\}$. Also we see that $(x\Gamma)^{n-1}x \nsubseteq I$ for each positive integer n, as $x \notin I$ and I is a completely semiprime ideal of S. Thus $I \cap M = \phi$. Hence by Proposition 3.14, there exists a semi-reduced prime ideal P of S such that $I \subseteq P$ and $P \cap M = \phi$ i.e. $x \notin P$. □

Proposition 3.16. *Let S be a Γ-semiring without nonzero nilpotent elements. Then every completely semiprime ideal I of S is the intersection of all semi-reduced prime ideals of S containing I.*

Proof. Let $H = \bigcap\{P : P$ is a semi-reduced prime ideal of S containing $I\}$. Obviously $I \subseteq H$. If possible, let $I \neq H$. Then there exists an element $a \in H$ such that $a \notin I$. Then by **Corollary 3.15**, there exists a semi-reduced prime ideal P of S containing I such that $a \notin P$. Therefore $a \notin H$, a contradiction. Hence $I = H$ i.e. $I = \bigcap\{P : P$ is a semi-reduced prime ideal of S such that $I \subseteq P\}$. □

In the following Propositions, we describe the properties of semi-reduced prime ideals of the Γ-semiring.

Proposition 3.17. *Let S be a Γ-semiring. Then $\mathcal{N}^*(S) = \bigcap\{P : P$ is a semi-reduced prime ideal of $S\}$, where $\mathcal{N}^*(S)$ denotes the unique maximal nil ideal of S.*

Proof. If possible, we let $\mathcal{N}^*(S) \nsubseteq P$ for some semi-reduced prime ideal P of S. Then $(\mathcal{N}^*(S)+P)/P$ is a nonzero nil ideal of S/P, a contradiction. Hence, $\mathcal{N}^*(S) \subseteq P$ for each semi-reduced prime ideal P of S. Thus $\mathcal{N}^*(S) \subseteq \bigcap\{P : P$ is a semi-reduced prime ideal of $S\}$.

Conversely, let $a \in \bigcap\{P : P$ is a semi-reduced prime ideal of $S\}$. If possible, let $a \notin \mathcal{N}^*(S)$. Then a is not a nilpotent element of S. Therefore by **Corollary**

3.13, there exists a semi-reduced prime ideal P of S such that $a \notin P$, a contradiction. Therefore $a \in \mathcal{N}^*(S)$. So $\bigcap\{P : P$ is a semi-reduced prime ideal of $S\} \subseteq \mathcal{N}^*(S)$. Hence $\mathcal{N}^*(S) = \bigcap\{P : P$ is a semi-reduced prime ideal of $S\}$ □

4. Some Special Subsets of a Γ-Semiring

We now define some special subsets in a Γ-semiring S.

Definition 4.1. For a semi-reduced prime ideal P of a Γ-semiring S, we define the following subsets in S:

(i) $N^*(P) = \{x \in S : x\Gamma S\Gamma y \subseteq \mathcal{N}^*(S)$ for some $y \in S \setminus P\}$;

(ii) $N^*_P = \{x \in S : x\Gamma y \subseteq \mathcal{N}^*(S)$ for some $y \in S \setminus P\}$;

(ii) $\overline{N^*_P} = \{x \in S : (x\Gamma)^{n-1}x \subseteq N^*_P$, for some positive integer $n\}$.

The properties of the subsets $N^*(P)$ and $\overline{N^*_P}$ are given in the following propositions.

Proposition 4.2. *Let S be a Γ-semiring. Then for any semi-reduced prime ideal P of S, $N^*(P) \subseteq P$ and $N^*(P) \subseteq N^*_P \subseteq \overline{N^*_P}$.*

Proof. Let $x \in N^*(P)$. Then $x\Gamma S\Gamma y \subseteq \mathcal{N}^*(S)$ for some $y \in S \setminus P$. By **Proposition 3.17**, $x\Gamma S\Gamma y \subseteq P$ for any semi-reduced prime ideal P of S. Since P is prime and $y \in S \setminus P$, $x \in P$ (cf. [2]). Hence $N^*(P) \subseteq P$.

If $x \in N^*(P)$, then $x\Gamma S\Gamma y \subseteq \mathcal{N}^*(S)$, for some $y \in S \setminus P$. Since S is a Γ-semiring with unity, $x\Gamma y \subseteq \mathcal{N}^*(S)$, where $y \in S \setminus P$. Hence $x \in N^*_P$, and whence $N^*(P) \subseteq N^*_P$. It is obvious that $N^*_P \subseteq \overline{N^*_P}$. □

Proposition 4.3. *Let S be a Γ-semiring and P a semi-reduced prime ideal of S. Then $N^*(P) = \{x \in S : x\Gamma S\Gamma < y > \subseteq \mathcal{N}^*(S)$ for some $y \in S \setminus P\}$, where $< y >$ denotes the ideal of S generated by y.*

Proof. Let $A = \{x \in S : x\Gamma S\Gamma < y > \subseteq \mathcal{N}^*(S)$ for some $y \in S \setminus P\}$. Since $y \in < y >$, $A \subseteq N^*(P)$. Let $x \in N^*(P)$. Then $x\Gamma S\Gamma y \subseteq \mathcal{N}^*(S)$ for some $y \in S \setminus P$. Now any element of $< y >$ is of the form $ny + \sum_{i=1}^{m} y\alpha_i x_i + \sum_{j=1}^{t} z_j\beta_j y + \sum_{k=1}^{s} u_k\lambda_k y\mu_k v_k$, where $x_i, z_j, u_k, v_k \in S$, $\alpha_i, \beta_j, \lambda_k, \mu_k \in \Gamma$ and n, m, t, s are non negative integers. Hence $x\Gamma S\Gamma < y > \subseteq \mathcal{N}^*(S)$ as $\mathcal{N}^*(S)$ is an ideal of S. Therefore $x \in A$. Consequently $N^*(P) = A$. □

Proposition 4.4. *Let S be a Γ-semiring and P any semi-reduced prime ideal of S. Then $N^*(P)$ is a two sided ideal of S.*

Proof. Since $0 \in N^*(P)$, $N^*(P)$ is a non empty subset of S. Let $x_1, x_2 \in N^*(P)$. Then there exist $y_1, y_2 \in S \setminus P$ such that $x_1\Gamma S\Gamma < y_1 > \subseteq \mathcal{N}^*(S)$ and $x_2\Gamma S\Gamma < y_2 > \subseteq \mathcal{N}^*(S)$. Since P is a semi-reduced prime ideal i.e. a prime ideal of S, $S \setminus P$ is an m-system. So there exist $s \in S, \alpha, \beta \in \Gamma$ such that $y_1\alpha s\beta y_2 \in S \setminus P$. Now $< y_1\alpha s\beta y_2 > \subseteq < y_1 >$ and $< y_1\alpha s\beta y_2 > \subseteq < y_2 >$. Therefore, $(x_1 + x_2)\Gamma S\Gamma < y_1\alpha s\beta y_2 > \subseteq x_1\Gamma S\Gamma < y_1\alpha s\beta y_2 > + x_2\Gamma S\Gamma < y_1\alpha s\beta y_2 > \subseteq x_1\Gamma S\Gamma < y_1 > + x_2\Gamma S\Gamma < y_2 > \subseteq \mathcal{N}^*(S)$ as $\mathcal{N}^*(S)$ is an ideal of S. Thus $x_1 + x_2 \in N^*(P)$. Let $x \in N^*(P)$. Then there exists $y \in S - P$ such that $x\Gamma S\Gamma < y > \subseteq \mathcal{N}^*(S)$. Therefore $S\Gamma x\Gamma S\Gamma < y > \subseteq \mathcal{N}^*(S)$ and $x\Gamma S\Gamma S\Gamma < y > \subseteq x\Gamma S\Gamma < y > \subseteq \mathcal{N}^*(S)$. Thus $S\Gamma x, x\Gamma S \subseteq N^*(P)$ for all $x \in N^*(P)$. Hence $N^*(P)$ is a two sided ideal of S □

Proposition 4.5. *Let S be a Γ-semiring and P be a semi-reduced prime ideal of S such that N_P^* is an ideal of S:*

(i) If N_P^* has the IFP, then $\overline{N_P^*}$ is an ideal of S;

(ii) N_P^* is a completely semiprime ideal of S if and only if $N_P^* = \overline{N_P^*}$.

Proof. (i) Since N_P^* is an ideal of S and $N_P^* \subseteq \overline{N_P^*}$, $\overline{N_P^*}$ is a nonempty subset of S. Let $x, y \in \overline{N_P^*}$. Then $(x\Gamma)^{n-1}x, (y\Gamma)^{m-1}y \subseteq N_P^*$, for some positive integers n, m. Since N_P^* has the IFP, the elements of the form

$$x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x ... x\alpha_{k-1}s_{k-1}\beta_{k-1}x$$

$(k \geq n)$ and $y\gamma_1 s_1\delta_1 y\gamma_2 s_2\delta_2 y ... y\gamma_{r-1}s_{r-1}\delta_{r-1}y$ $(r \geq m)$ belong to N_P^* i.e. an expression containing at least n x's or m y's must belong to N_P^*. Since N_P^* is an ideal having IFP, each term of $((x + y)\Gamma)^{m+n}(x + y)$ is contained in N_P^*. Also since N_P^* is an ideal, $((x + y)\Gamma)^{m+n}(x + y) \subseteq N_P^*$. So $x + y \in \overline{N_P^*}$. Since N_P^* is an ideal having IFP, $((S\Gamma x)\Gamma)^{n-1}(S\Gamma x), ((x\Gamma S)\Gamma)^{n-1}(x\Gamma S) \subseteq N_P^*$ i.e. $S\Gamma x, x\Gamma S \subseteq \overline{N_P^*}$. Therefore $\overline{N_P^*}$ is an ideal of S.

(ii) Suppose that N_P^* is a completely semiprime ideal of S. Clearly, $N_P^* \subseteq \overline{N_P^*}$. Let $a \in \overline{N_P^*}$. Then $(a\Gamma)^{n-1}a \subseteq N_P^*$, for some positive integer n. As N_P^* is completely semiprime ideal of S, $(a\Gamma)^{n-1}a \subseteq N_P^*$ implies $a \in N_P^*$. Therefore $N_P^* = \overline{N_P^*}$. The converse part is obvious. □

Definition 4.6. For a semi-reduced prime ideal P of a Γ-semiring S, we define:

(*i*) $O^*(P) = \{x \in S : x\Gamma S\Gamma y = 0 \text{ for some } y \in S \setminus P\}$;

(ii) $O^*_P = \{x \in S : x\Gamma y = 0$ for some $y \in S \setminus P\}$;

(iii) $\overline{O^*_P} = \{x \in S : (x\Gamma)^{n-1}x \subseteq O^*_P$, for some positive integer $n\}$.

In the following lemmas, we consider the semi-reduced prime ideals of Γ-semirings and SN Γ-semirings.

Lemma 4.7. *Let S be a Γ-semiring. Then for any semi-reduced prime ideal P of S, $O^*(P) \subseteq P$ and $O^*(P) \subseteq O^*_P \subseteq \overline{O^*_P}$.*

Proof. The proof is Similar to Proposition 4.2. □

Lemma 4.8. *Let S be a Γ-semiring and P a semi-reduced prime ideal of S. Then $O^*(P) = \{x \in S : x\Gamma S\Gamma < y >= 0$ for some $y \in S \setminus P\}$, where $< y >$ is the ideal of S generated by y.*

Proof. The proof is similar to Proposition 4.3. □

Lemma 4.9. *Let S be a Γ-semiring and P be any semi-reduced prime ideal of S. Then $O^*(P)$ is a two sided ideal of S.*

Proof. The proof is similar to Lemma 4.4. □

Lemma 4.10. *Let S be a Γ-semiring and P be a semi-reduced prime ideal of S such that O^*_P is an ideal of S. Then:*

(i) If O^*_P has the IFP, then $\overline{O^*_P}$ is an ideal of S;

(ii) O^*_P is a completely semiprime ideal of S if and only if $O^*_P = \overline{O^*_P}$.

Proof. The proof is similar to Lemma 4.5 and is hence omited. □

Notation 4.11. We use $SSpec(S)$ and $mSSpec(S)$ to denote the set of all semi-reduced prime ideals and minimal semi-reduced prime ideals of S, respectively.

The following Theorem shows the existence of a minimal semi-reduced prime ideal of a Γ-semiring S.

Theorem 4.12. *For every semi-reduced prime ideal P of a Γ-semiring S, there exists a minimal semi-reduced prime ideal Q of S such that $Q \subseteq P$.*

Proof. Let S be a Γ-semiring. Let $\mathcal{P} = \{P_\alpha : P_\alpha$ is a semi-reduced prime ideal of $S\}$. Here $\mathcal{P} \neq \phi$. $\mathcal{P}$ is a poset with respect to set inclusion. Let $\mathcal{C}$ be a chain and $Q = \bigcap_{P_\alpha \in \mathcal{C}} P_\alpha$. Then Q is an ideal of S. Let A and B are two ideals of S such that $A\Gamma B \subseteq Q$. Suppose $A \nsubseteq Q$ and $B \nsubseteq Q$. Then $A \nsubseteq Q_i$

and $B \nsubseteq Q_j$ for some i and j. Since $\mathcal{C}$ is a chain, either $Q_i \subseteq Q_j$ or $Q_j \subseteq Q_i$. Hence $A, B \nsubseteq Q_i$ or $A, B \nsubseteq Q_j$, a contradiction as $Q \subseteq Q_i$ or $Q \subseteq Q_j$ and they are prime. So we must have either $A \subseteq Q$ or $B \subseteq Q$. Therefore, Q is a prime ideal of S. Since Q is a prime k-ideal of S, S/Q is a prime Γ-semiring. If possible, let S/Q has a nonzero nil ideal say I/Q. Then $Q \subset I \subseteq S$. Now we show that, $I \nsubseteq Q_i$, for some i. If not, $I \subseteq Q_i$, for all i. Then $I \subseteq \cap Q_i = Q$, a contradiction. Hence we can proved $(I + Q_i)/Q_i$ is a nonzero nil ideal of S/Q_i. Let $(x + q)/Q_i \in (I + Q_i)/Q_i$. Then $x \in I$ and $q \in Q_i$. Then for any α there exists a natural number n such that $((x/Q)\alpha)^{n-1}(x/Q) = 0/Q$ i.e. $(x\alpha)^{n-1}x \in Q$ i.e. $(x\alpha)^{n-1}x \in Q_i$. Since Q_i is an ideal and $q \in Q_i$, $((x + q)\alpha)^{n-1}(x + q) \in Q_i$. Hence $(((x + q)\alpha)^{n-1}(x + q))/Q_i = 0/Q_i$ i.e. $(((x + q)/Q_i)\alpha)^{n-1}(x + q)/Q_i = 0/Q_i$ Thus $(I + Q_i)/Q_i$ is a nonzero nil ideal of S/Q_i, hence a contradiction. Therefore, S/Q is a semi-reduced prime Γ-semiring. Hence $Q \in \mathcal{P}$. So the chain $\mathcal{C}$ has a lower bound in $\mathcal{P}$. By Zorn's Lemma, $\mathcal{P}$ has a minimal element. □

Lemma 4.13. *Let S be an SN Γ-semiring. Then* $\mathcal{N}(S) \subseteq \bigcap_{P \in SSpec(S)} \overline{O_P^*} \subseteq \bigcap_{Q \in mSSpec(S)} \overline{O_Q^*}$, *where $\mathcal{N}(S)$ denotes the set of all nilpotent elements of S.*

Proof. We first notice that if P_1 and P_2 are two semi-reduced prime ideals of S such that $P_1 \subseteq P_2$, then $\overline{O_{P_2}^*} \subseteq \overline{O_{P_1}^*}$.

Let P be any semi-reduced prime ideal of S, then by Theorem 4.12, there exists a minimal semi-reduced prime ideal Q of S such that $Q \subseteq P$. Therefore $\bigcap_{P \in SSpec(S)} \overline{O_P^*} \subseteq \bigcap_{Q \in mSSpec(S)} \overline{O_Q^*}$.

Let $a \in \mathcal{N}(S)$. Since S is an SN Γ-semiring, $a \in \mathcal{N}_\Gamma(S)$. So $(a\Gamma)^{n-1}a = 0$, for some positive integer n. Then $(a\Gamma)^{n-1}a\Gamma y = 0$ for each $y \in S - P$, where P is a semi-reduced prime ideal of S. So $(a\Gamma)^{n-1}a \subseteq O_P^*$, for each semi-reduced prime ideal P of S i.e. $a \in \overline{O_P^*}$ for each semi-reduced prime ideal P of S, which implies that $a \in \bigcap_{P \in SSpec(S)} \overline{O_P^*}$. Hence $\mathcal{N}(S) \subseteq \bigcap_{P \in SSpec(S)} \overline{O_P^*} \subseteq \bigcap_{Q \in mSSpec(S)} \overline{O_Q^*}$. □

For the subset $SSpec(S)$ of a Γ-semiring S, we have the following Theorem.

Theorem 4.14. *Let S be a Γ-semiring. Then* $\mathcal{N}^*(S) = \bigcap_{P \in SSpec(S)} N^*(P) =$

$\bigcap_{Q\in mSSpec(S)} N^*(Q)$.

Proof. Let $a \in \mathcal{N}^*(S)$. As $\mathcal{N}^*(S)$ is an ideal, $a\Gamma S \subseteq \mathcal{N}^*(S)$. Since P is proper, $S \setminus P$ is nonempty. Let $y \in S \setminus P$. Then $a\Gamma S\Gamma y \subseteq \mathcal{N}^*(S)$ for every semi-reduced prime ideal P of S. This implies $a \in N^*(P)$ for every semi-reduced prime ideal P of S, that is, $a \in \bigcap_{P\in SSpec(S)} N^*(P)$. Hence, $\mathcal{N}^*(S) \subseteq \bigcap_{P\in SSpec(S)} N^*(P)$. Again $N^*(P) \subseteq P$ for any semi-reduced prime ideal P of S. This proves $\bigcap_{P\in SSpec(S)} N^*(P) \subseteq \bigcap_{P\in SSpec(S)} P = \mathcal{N}^*(S)$. Hence, $\mathcal{N}^*(S) = \bigcap_{P\in SSpec(S)} N^*(P)$. Now for each $P \in SSpec(S)$ there exists $Q \in mSSpec(S)$ such that $Q \subseteq P$ and $Q \subseteq P \Rightarrow N^*(P) \subseteq N^*(Q)$, $\mathcal{N}^*(S) = \bigcap_{P\in SSpec(S)} N^*(P) \subseteq \bigcap_{Q\in mSSpec(S)} N^*(Q)$. Again $\mathcal{N}^*(S) = \bigcap_{Q\in mSSpec(S)} Q$ and $\bigcap_{Q\in mSSpec(S)} N^*(Q) \subseteq \bigcap_{Q\in mSSpec(S)} Q$. Therefore, we conclude that $\bigcap_{Q\in mSSpec(S)} N^*(Q) \subseteq \mathcal{N}^*(S)$. Hence $\mathcal{N}^*(S) = \bigcap_{P\in SSpec(S)} N^*(P) = \bigcap_{Q\in mSSpec(S)} N^*(Q)$. □

5. NI Γ-Semirings

Our aim inthis section is to characterize the NI Γ-semirings.

Definition 5.1. A Γ-semiring S is said to be a NI Γ-semiring if and only if $\mathcal{N}^*(S) = \mathcal{N}(S)$, where $\mathcal{N}^*(S)$ denotes the unique maximal nil ideal of the Γ-semiring S i.e. the nil radical of S and $\mathcal{N}(S)$ is used to denote the set of all nilpotent elements of S.

Definition 5.2. (see [9]) A Γ-semiring S is said to be *reduced* if it has no nonzero nilpotent elements.

The following Lemma is a crucial Lemma.

Lemma 5.3. *Every reduced* Γ*-semiring* S *is an NI* Γ*-semiring.*

In related to the strongly nilpotent elements of a Γ-semiring, we consider the SN Γ-semirings.

Recall that a Γ-semiring S is a SN Γ-semiring if $\mathcal{N}(S) = \mathcal{N}_\Gamma(S)$, where $\mathcal{N}_\Gamma(S)$ is the set of all strongly nilpotent elements of S.

Now we give some examples of SN Γ-semiring.

Example 5.4. Let D be a division semiring. Consider two sets as:

$$S = \left\{ \begin{pmatrix} d_1 & d_2 \\ 0 & d_3 \\ 0 & 0 \end{pmatrix} : d_1, d_2, d_3 \in D \right\}$$

and

$$\Gamma = \left\{ \begin{pmatrix} d_4 & d_5 & d_6 \\ 0 & d_7 & d_8 \end{pmatrix} : d_4, d_5, d_6, d_7, d_8 \in D \right\}.$$

Then S and Γ be two additive commutative semigroup under usual matrix addition. Now we define the mapping: $S \times \Gamma \times S \longrightarrow S$ by usual matrix multiplication i.e.

$$\begin{pmatrix} d_1 & d_2 \\ 0 & d_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_4 & d_5 & d_6 \\ 0 & d_7 & d_8 \end{pmatrix} \begin{pmatrix} d_9 & d_{10} \\ 0 & d_{11} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} d_1d_4d_9 & d_1d_4d_{10} + d_{11}(d_1d_5 + d_2d_7) \\ 0 & d_3d_7d_{11} \\ 0 & 0 \end{pmatrix}.$$

From the above product it follows that

$$\mathcal{N}(S) = \mathcal{N}_\Gamma(S) = \left\{ \begin{pmatrix} 0 & d_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : d_2 \in D \right\}.$$

Then S is an SN Γ-semiring.

Example 5.5. Let D be a division semiring. Consider two sets: $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \\ 0 & 0 \end{pmatrix} : d \in D \right\}$ and $\Gamma = \left\{ \begin{pmatrix} d_4 & d_5 & d_6 \\ 0 & d_7 & d_8 \end{pmatrix} : d_4, d_5, d_6, d_7, d_8 \in D \right\}$. Then S is a Γ-semiring with respect to usual matrix addition and matrix multiplication.

Here, $\mathcal{N}(S) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Hence, S is an SN Γ-semiring.

Example 5.6. Let $S = \{r\omega : r \in \mathbb{Z}\}$ and $\Gamma = \{r\omega^2 : r \in \mathbb{Z}\}$, where ω is a cube root of unity and $\mathbb{Z}$ is the set of all integers. Then S is a Γ-semiring with usual addition and multiplication. Here S has no nonzero nilpotent element. Therefore, S is an SN Γ-semiring.

Theorem 5.7. *Let S be an SN Γ-semiring. The following statements are equivalent:*

(1) *S is a NI Γ-semiring.*

(2) *$\mathcal{N}^*(S)$ is a completely semiprime ideal of S.*

(3) *$\mathcal{N}^*(S)$ is a symmetric ideal of S.*

(4) *$\mathcal{N}^*(S)$ has IFP.*

(5) *$N^*(P)$ has IFP for each semi-reduced prime ideal P of S.*

(6) *$N^*(P) = N^*_P = \overline{N^*_P}$ for each semi-reduced prime ideal P of S.*

Proof. (1) ⇒ (2) Let $a\Gamma a \subseteq \mathcal{N}^*(S)$, where $a \in S$. Since S is an SN and a NI Γ-semiring, $a\Gamma a \subseteq \mathcal{N}_\Gamma(S)$. Then there exista a positive integer n such that $((a\Gamma a)\Gamma)^{n-1}(a\Gamma a) = 0$. This implies that $(a\Gamma)^{2n-1}a = 0$. So $a \in \mathcal{N}_\Gamma(S) = \mathcal{N}(S) = \mathcal{N}^*(S)$. Therefore $\mathcal{N}^*(S)$ is a completely semiprime ideal of S.

(2) ⇒ (3) Let $a\Gamma b\Gamma c \subseteq \mathcal{N}^*(S)$, where $a, b, c \in S$. Then $(c\Gamma a\Gamma b)\Gamma(c\Gamma a\Gamma b) = c\Gamma(a\Gamma b\Gamma c)\Gamma a\Gamma b \subseteq \mathcal{N}^*(S)$. Since $\mathcal{N}^*(S)$ is completely semiprime, $c\Gamma a\Gamma b \subseteq \mathcal{N}^*(S)$. Now $(a\Gamma b\Gamma a\Gamma c)\Gamma(a\Gamma b\Gamma a\Gamma c) = a\Gamma b\Gamma a\Gamma(c\Gamma a\Gamma b)\Gamma a\Gamma c \subseteq \mathcal{N}^*(S)$ as $\mathcal{N}^*(S)$ is an ideal of S. This implies $a\Gamma b\Gamma a\Gamma c \subseteq \mathcal{N}^*(S)$. Again by using similar argument, we have

$$\begin{aligned}(b\Gamma a\Gamma c\Gamma b\Gamma a)\Gamma(b\Gamma a\Gamma c\Gamma b\Gamma a) &= b\Gamma a\Gamma c\Gamma b\Gamma(a\Gamma b\Gamma a\Gamma c)\Gamma b\Gamma a \subseteq \mathcal{N}^*(S)\\ &\Rightarrow b\Gamma a\Gamma c\Gamma b\Gamma a \subseteq \mathcal{N}^*(S)\\ \Rightarrow (a\Gamma c\Gamma b\Gamma a\Gamma c\Gamma b)\Gamma(a\Gamma c\Gamma b\Gamma a\Gamma c\Gamma b) &= a\Gamma c\Gamma(b\Gamma a\Gamma c\Gamma b\Gamma a)\Gamma c\Gamma b\Gamma a\Gamma c\Gamma b \subseteq \mathcal{N}^*(S)\\ &\Rightarrow a\Gamma c\Gamma b \subseteq \mathcal{N}^*(S)\end{aligned}$$

as $\mathcal{N}^*(S)$ is completely semiprime. Hence, $\mathcal{N}^*(S)$ is a right symmetric ideal of S. Also $(b\Gamma a\Gamma c)\Gamma(b\Gamma a\Gamma c) = b\Gamma(a\Gamma c\Gamma b)\Gamma a\Gamma c \subseteq \mathcal{N}^*(S) \Rightarrow b\Gamma a\Gamma c \subseteq \mathcal{N}^*(S)$. This shows that $\mathcal{N}^*(S)$ is a left symmetric ideal of S. Similarly, $\mathcal{N}^*(S)$ is a right symmetric ideal of S.

(3) ⇒ (4) Let $a\Gamma b \subseteq \mathcal{N}^*(S)$, where $a, b \in S$. Suppose that $s \in S$. Then $s\Gamma a\Gamma b \subseteq \mathcal{N}^*(S)$. As $\mathcal{N}^*(S)$ is left symmetric, $a\Gamma s\Gamma b \subseteq \mathcal{N}^*(S)$. Therefore, $a\Gamma S\Gamma b \subseteq \mathcal{N}^*(S)$. Hence $\mathcal{N}^*(S)$ has IFP.

(4) ⇒ (5) Let $a\Gamma b \subseteq N^*(P)$, where $a, b \in S$ for every semi-reduced prime ideal P of S. Then $(a\Gamma b)S\Gamma Sy \subseteq \mathcal{N}^*(S)$, where $y \in S - P$. Since $\mathcal{N}^*(S)$

has IFP, $a\Gamma S\Gamma bS\Gamma Sy \subseteq \mathcal{N}^*(S)$. Since $y \in S - P$, by Definition, we have $a\Gamma S\Gamma b \subseteq N^*(P)$. Hence, $N^*(P)$ has IFP.

(5) $\Rightarrow$ (1). It is clear that $\mathcal{N}^*(S) \subseteq \mathcal{N}(S)$always. Let $x \in \mathcal{N}(S)$. Since S is an SN Γ-semiring, $x \in \mathcal{N}_\Gamma(S)$. Hence, exists a positive integer n such that $(x\Gamma)^{n-1}x = 0$. Suppose if possible, let $x \notin \mathcal{N}^*(S)$. Then $x \notin P$ for some semi-reduced prime ideal P of S. Hence $x \in S - P$. Since P is prime, $S \setminus P$ is an m-system. Clearly, there exists $s_1 \in S$ $\alpha_1, \beta_1 \in \Gamma$ such that $x\alpha_1 s_1\beta_1 x \in S \setminus P$. Again since $x\alpha_1 s_1\beta_1 x, x \in S \setminus P$, by the property of m-system, we have $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x \in S \setminus P$, for some $\alpha_2, \beta_2 \in \Gamma$ and $s_2 \in S$. Applying the property of m-system again after a number of finite steps, we have $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x.....\alpha_{n-1}s_{n-1}\beta_{n-1}x \in S \setminus P$ for some $s_i \in S, \alpha_i, \beta_i \in \Gamma$, where $i = 1, 2, ...(n-1)$. Also since $N^*(P)$ has the IFP, and $(x\Gamma)^{n-1}x = 0 \subseteq N^*(P)$ implies $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x...x\alpha_{n-1}s_{n-1}\beta_{n-1}x \in N^*(P)$ i.e. $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x...x\alpha_{n-1}s_{n-1}\beta_{n-1}x \in P[N^*(P) \subseteq P]$, a contradiction. Thus $x \in \mathcal{N}^*(S)$, and so $\mathcal{N}^*(S) = \mathcal{N}(S)$. Hence, we have proved that S is a NI Γ-semiring.

(1) $\Rightarrow$ (6) Let P be a semi-reduced prime ideal of S. Let $a \in \overline{N_P^*}$. Then $(a\Gamma)^{n-1}a \subseteq N_P^*$, for some positive integer n. Hence there exists $b \in S - P$ such that $(a\Gamma)^{n-1}a\Gamma b \subseteq \mathcal{N}^*(S)$. By (3), we see that $\mathcal{N}^*(S)$ is a left and right symmetric ideal of S. Thus $(a\Gamma b)\Gamma(a\Gamma b)\Gamma(a\Gamma b)\Gamma.....\Gamma(a\Gamma b)$(n-times)$\subseteq \mathcal{N}^*(S)$. Again by (2) $\mathcal{N}^*(S)$ is a completely prime ideal of S. Then we have $a\Gamma b \subseteq \mathcal{N}^*(S)$. Now by (4), $\mathcal{N}^*(S)$ has the IFP. Therefore $a\Gamma S\Gamma b \subseteq \mathcal{N}^*(S)$ which implies that $a \in N^*(P)$. Consequently, we deduce that $\overline{N_P^*} \subseteq N^*(P)$.

By Proposition 3.19, $N^*(P) \subseteq N_P^* \subseteq \overline{N_P^*}$ and by above $\overline{N_P^*} \subseteq N^*(P)$ for each semi-reduced prime ideal P of S. Therefore, $N^*(P) = N_P^* = \overline{N_P^*}$.

(6) $\Rightarrow$ (1) Let $N^*(P) = N_P^* = \overline{N_P^*}$ for each semi-reduced prime ideal P of S. Then by Definition, $\mathcal{N}^*(S) \subseteq \mathcal{N}(S)$. Let $a \in \mathcal{N}(S)$. Since S is an SN Γ-semiring, $a \in \mathcal{N}_\Gamma(S)$. This implies that $(a\Gamma)^{n-1}a = 0$ for some positive integer n, and so $\Rightarrow (a\Gamma)^{n-1}a \subseteq N_P^* \Rightarrow a \in \overline{N_P^*} \Rightarrow a \in N^*(P) \Rightarrow a \in P(N^*(P) \subseteq P)$ for each prime ideal P. Hence, by Proposition 3.17, $a \in \mathcal{N}^*(S)$. Thus, $\mathcal{N}(S) \subseteq \mathcal{N}^*(S)$. This shows that S is an NI Γ-semiring. ☐

As an application of the above Theorem, we have the following Lemma.

Lemma 5.8. *Let S be a NI Γ-semiring and P a semi-reduced prime ideal of S such that $N^*(P) = P$. Then P is a completely prime ideal of S.*

Proof. Let $x\Gamma y \subseteq P$ and $y \notin P$. Since $N^*(P) = P$, $x\Gamma y \subseteq N^*(P)$, there exists $b \in S - P$ such that $x\Gamma y\Gamma S\Gamma b \subseteq \mathcal{N}^*(S)$. As S is NI Γ-semiring, $\mathcal{N}^*(S)$

has IFP. Thus $x\Gamma S\Gamma y\Gamma S\Gamma S\Gamma b \subseteq \mathcal{N}^*(S) \subseteq P$. Since P is prime and $y, b \notin P$, $x \in P$, the result follows. □

For SN Γ-semirings, we also have the following Theorem.

Theorem 5.9. *Let S be an SN Γ-semiring. Then the following statements hold.*

(1) If $O^*(P)$ has the IFP for each minimal semi-reduced prime ideal P of S, then S is a NI Γ-semiring.

(2) If S is a NI Γ-semiring and $O^*_P = P$ for a semi-reduced prime ideal P of S, then P is a completely prime ideal of S, in particular, $O^*(P)$ has IFP.

Proof. (1) Suppose $O^*(P)$ has IFP for each minimal semi-reduced prime ideal P of S. To prove that S is a NI Γ-semiring, it suffices to show that $\mathcal{N}(S) \subseteq \mathcal{N}^*(S)$. Let $a \in \mathcal{N}(S)$. Since S is an SN Γ-semiring, $(a\Gamma)^{n-1}a = 0$, for some positive integer n. Suppose if possible, let $a \notin \mathcal{N}^*(S)$, then there exists a semi-reduced prime ideal P of S such that $a \notin P$ i.e. $a \in S - P$. Since P is a semi-reduced prime ideal of S, P is a prime ideal of S. So $S - P$ is an m-system of S. Then there exist $s_1 \in S, \alpha_1, \beta_1 \in \Gamma$ such that $a\alpha_1 s_1\beta_1 a \in S\backslash P$. Again since $a\alpha_1 s_1\beta_1 a, a \in S\backslash P$, applying m-system property $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a \in S \setminus P$, for some $\alpha_2, \beta_2 \in \Gamma$ and $s_2 \in S$. Applying m-system property after finite step, we have $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a.....\alpha_{n-1}s_{n-1}\beta_{n-1}a \in S \setminus P$ for some $s_i \in S, \alpha_i, \beta_i \in \Gamma$, where $i = 1, 2, ...(n-1)$. Also since $O^*(P)$ has IFP, and $(a\Gamma)^{n-1}a(= 0) \subseteq O^*(P)$ implies that $a\Gamma S\Gamma a.....a\Gamma S\Gamma a$, that is,in particular $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a...a\alpha_{n-1}s_{n-1}\beta_{n-1}a \in O^*(P)$. As $O^*(P) \subseteq P$, $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a...a\alpha_{n-1}s_{n-1}\beta_{n-1}a \in P$, a contradiction. This proves that $a \in \mathcal{N}^*(S)$. Hence $\mathcal{N}(S) \subseteq \mathcal{N}^*(S)$, and so $\mathcal{N}^*(S) = \mathcal{N}(S)$, i.e. S is a NI Γ-semiring.

(2) Let $x\Gamma y \subseteq P = O^*_P$. Then there exists $b \in S - P$ such that $(x\Gamma y)\Gamma b = 0$. If possible, let $x \notin P$. Since S is a NI Γ-semiring, by Theorem 5.7, $\mathcal{N}^*(S)$ has the IFP. Therefore $(x\Gamma y)\Gamma b(= 0) \subseteq \mathcal{N}^*(S)$ implies that $(x\Gamma S\Gamma y)\Gamma S\Gamma b \subseteq \mathcal{N}^*(S) \subseteq P$. Since P is prime and $x \notin P$, $y\Gamma S\Gamma b \subseteq P$. Again since $b \notin P$, $y \in P$ as P is a prime ideal of S. Therefore, either $x \in P$ or $y \in P$. Hence P is a completely prime ideal of S.

Let $x\Gamma y \subseteq O^*_P$. Since $O^*_P = P$ and P is a completely prime ideal of S, either $x \in P$ or $y \in P$. Again since P is an ideal of S, $x\Gamma S\Gamma y \subseteq P$ i.e. $x\Gamma S\Gamma y \subseteq O^*_P$(since $O^*_P = P$). Hence O^*_P has the IFP. □

We now give a characerization theorem for an SN Γ-semiring S.

Theorem 5.10. *Let S be an SN Γ-semiring S. Then the following statements are equivalent:*

(1) *S is NI Γ-semiring.*

(2) $\overline{O_P^*} \subseteq P$ *for each* $P \in SSpec(S)$.

(3) $\mathcal{N}(S) = \bigcap_{P \in SSSpec(S)} \overline{O_P^*} = \mathcal{N}^*(S)$.

Proof. (1) $\Rightarrow$ (2): Let $a \in \overline{O_P^*}$. Then there exists a positive integer n such that $(a\Gamma)^{n-1}a \subseteq O_P^*$. Hence $(a\Gamma)^{n-1}a\Gamma b = 0$ for some $b \in S \setminus P$ i.e. $(a\Gamma)^{n-1}a\Gamma b \subseteq \mathcal{N}^*(S)$, for some $b \in S \setminus P$, which implies $(a\Gamma)^{n-1}a \subseteq N_P^*$ i.e. $a \in \overline{N_P^*}$. Hence, $\overline{O_P^*} \subseteq \overline{N_P^*}$ for each semi-reduced prime ideal P of S. Also by Theorem 5.7, $\overline{N_P^*} = N^*(P)$ for each semi-reduced prime ideal P of S. Again since $N^*(P) \subseteq P$ for each semi-reduced prime ideal P of S. Thus $\overline{O_P^*} \subseteq \overline{N_P^*} = N^*(P) \subseteq P$ for each semi-reduced prime ideal P of S.

(2) $\Rightarrow$ (3) Since $\overline{O_P^*} \subseteq P$ for each semi-reduced prime ideal P of S,

$$\bigcap_{P \in SSpec(S)} \overline{O_P^*} \subseteq \bigcap_{P \in SSpec(S)} P = \mathcal{N}^*(S).$$

Now by Proposition 4.13, $\mathcal{N}(S) \subseteq \bigcap_{P \in SSpec(S)} \overline{O_P^*} \subseteq \mathcal{N}^*(S)$. Also $\mathcal{N}^*(S) \subseteq \mathcal{N}(S)$. Therefore $\mathcal{N}(S) = \bigcap_{P \in SSpec(S)} \overline{O_P^*} = \mathcal{N}(S)$.

(3) $\Rightarrow$ (1) This part is obvious. □

We now formulate another Theorem for the SN Γ-semirings.

Theorem 5.11. *If $\overline{O_P^*} = P$ for each $P \in SSpec(S)$ of an SN Γ-semiring S, then:*

(1) *S is NI Γ-semiring.*

(2) $\overline{O_P^*} = N^*(P)$ *for each* $P \in SSpec(S)$.

(3) *Every semi-reduced prime ideal of S is minimal and completely prime.*

Proof. (1) Since $\overline{O_P^*} = P$ and $\overline{O_P^*} \subseteq P$. Hence bythe above Theorem, $\mathcal{N}(S) = \mathcal{N}^*(S)$, that is, S is an NI Γ-semiring.

(2) Since $N^*(P) \subseteq P$ and $\overline{O_P^*} = P$ for each semi-reduced prime ideal P of S,we have $N^*(P) \subseteq \overline{O_P^*}$ for each semi-reduced prime ideal P of S. Since S is NI, by Theorem 4.4, $N^*(P) = \overline{N_P^*}$ for each semi-reduced prime ideal P of S.

Also $\overline{O_P^*} \subseteq \overline{N_P^*}$ [see he above theorem] for each semi-reduced prime ideal P of S. Thus, $\overline{O_P^*} \subseteq N^*(P)$ for each semi-reduced prime ideal P of S. Therefore $\overline{O_P^*} = N^*(P)$ for each semi-reduced prime ideal P of S.

(3) Let P be a semi-reduced prime ideal of S. From (ii) and the given condition $\overline{O_P^*} = P$, we get $N^*(P) = P$ for each semi-reduced prime ideal P of S. If Q is a minimal semi-reduced prime ideal of S contained in P, then $N^*(P) \subseteq N^*(Q) \subseteq Q \subseteq P = N^*(P)$. Thus $P = Q$ i.e. P is a minimal semi-reduced prime ideal of S.

Let $x\Gamma y \subseteq P = N^*(P)$ and $x \notin P$. Since $x\Gamma y \subseteq N^*(P)$, there exists $b \in S - P$ such that $(x\Gamma y)\Gamma S\Gamma b \subseteq \mathcal{N}^*(S)$ i.e. $x\Gamma(y\Gamma S\Gamma b) \subseteq \mathcal{N}^*(S)$. Since $\mathcal{N}^*(S)$ has the IFP (by Theorem 5.7), $x\Gamma S\Gamma(y\Gamma S\Gamma b) \subseteq \mathcal{N}^*(S) \subseteq P$. As $x \notin P$, $y\Gamma S\Gamma b \subseteq P$. Again since $b \notin P$, $y \in P$,either $x \in P$ or $y \in P$. This proves that P is a completely prime ideal of S. □

In closing this paper, we propose the following open problems.

Problems: (1) We know that in a ring or in a semiring with 1 there always exists a maximal ideal. We have proved this result in case of Γ-semiring with unity via operators. The readars are invited to provide a direct proof of this result without useing operators.

(2) In a ring or in a semiring with identity there always exists a nonzero element which is not nilpotent viz. the identity. Does the result hold in case of a Γ-semiring with unity?

If any one of (1) of (2) is true we can say every Γ-semiring with unity contains a semi-reduced prime ideal.

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