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ON MAXIMAL k-IDEALS OF SEMIRINGS

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ABSTRACT. For a semiring S with commutative addition, conditions are considered such that S has nontrivial k-ideals or maximal k-ideals, among others, by the help of the congruence class semiring S/A defined by an ideal A of S. Moreover, all maximal k-ideals of the semiring of nonnegative integers are described.

1. PRELIMINARIES

A *semiring* S is defined as an algebra $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in S$. A semiring S may have an *identity* 1 [a zero o], defined by $1a = a1 = a[o + a = a + o = a]$ for all $a \in S$. If there is an element $O \in S$ satisfying $Oa = aO = O$ for all $a \in S$, it is called *multiplicatively absorbing* or simply *absorbing*. Such an element satisfies $O + O = O$, but it need not be a zero of S, whereas a zero o of S need not even satisfy $oo = o$. Clearly, a semiring has an *absorbing zero* iff it has elements O and o which coincide.

A subset $A \neq \varnothing$ of a semiring S is called an *ideal* of S iff $a + b \in A$, $sa \in A$, and $as \in A$ hold for all $a, b \in A$ and all $s \in S$. An ideal A of S is called *proper* iff $A \subset S$ holds, where $\subset$ denotes proper inclusion, and a proper ideal A is called *maximal* iff there is no ideal B of S satisfying $A \subset B \subset S$. Obviously, a semiring S contains an ideal A consisting of one element iff S has an absorbing element O, and then $A = \{O\}$ is the only ideal of this kind. Finally, an ideal A of S is called *trivial*, iff $A = S$ holds or $A = \{O\}$, the latter clearly if S has an absorbing element. To deal with both cases simultaneously, we introduce the notion S' by $S' = S \backslash \{O\}$ if S has an absorbing element, and $S' = S$ otherwise.

In this paper we only consider semirings S for which $(S, +)$ is commutative. If also $(S, \cdot)$ is commutative, S is called a *commutative semiring*. Moreover, to avoid trivial exceptions, *each semiring S is assumed to have at least two elements.*

Using only commutativity of addition, the following concepts and statements, essentially due to [1, 2, 4], are well known. For each ideal A of a semiring S

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the *k-closure* $\overline{A}$ of A defined by

$$\overline{A} = \{\overline{a} \in S | \overline{a} + a_1 = a_2 \text{ for some } a_i \in A\}$$

is an ideal of S satisfying $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$. An ideal A of S is called a *k-ideal* of S iff $A = \overline{A}$ holds. Clearly, S is a k-ideal for each semiring S; however, if S has an absorbing element O, the ideal $\{O\}$ need not be a k-ideal of S. There are examples for $\{O\} \subset \overline{\{O\}} \subset S$ and $\{O\} \subset \overline{\{O\}} = S$, whereas $\{O\} = \overline{\{O\}}$ holds if O is an absorbing zero of S. A k-ideal $A \subset S$ is called a *maximal k-ideal* of S if there is no k-ideal B of S satisfying $A \subset B \subset S$. Note that a maximal k-ideal of S need not be a maximal ideal of S (cf. Remark 4.2).

Moreover, each ideal A of S defines a congruence ρ_A on $(S, +, \cdot)$ by

$$\rho_A = \{(x, y) \in S \times S | x + a_1 = y + a_2 \text{ for some } a_i \in A\}.$$

The corresponding congruence class semiring S/ρ_A, consisting of the classes $x\rho_A$, is also denoted by S/A. The k-closure $\overline{A}$ of A is such a congruence class, and $\overline{A}$ is the absorbing zero of S/A, regardless of whether S has a zero o or an absorbing element O (which implies $o\rho_A = \overline{A}$ or $O\rho_A = \overline{A}$, respectively). Moreover, ρ_A and $\rho_{\overline{A}}$, and hence S/A and $S/\overline{A}$ coincide.

2. Maximal k-ideals

Theorem 2.1. *Let S be a semiring such that $S = (a_1, \dots, a_n)$ is a finitely generated ideal of S. Then each proper k-ideal A of S is contained in a maximal k-ideal of S.*

Proof. Let $\mathfrak{B}$ be the set of all k-ideals B of S satisfying $A \subseteq B \subset S$, partially ordered by inclusion. Consider a chain $\{B_i | i \in I\}$ in $\mathfrak{B}$. One easily checks that $B = \bigcup_{i \in I} B_i$ is a k-ideal of S, and $S = (a_1, \dots, a_n)$ implies $B \neq S$, and hence $B \in \mathfrak{B}$. So by Zorn's lemma, $\mathfrak{B}$ has a maximal element as we were to show.

Corollary 2.2. *Let S be a semiring with identity* 1. *Then each proper k-ideal of S is contained in a maximal k-ideal of S.*

The proof is immediate by $S = (1)$.

Definition 2.3. A semiring S is said to satisfy condition (C) iff for all $a \in S'$ and all $s \in S$ there are $s_1, s_2 \in S$ such that

$$s + s_1 a = s_2 a$$

holds. Clearly, if S has an identity 1, then (C) is equivalent to the following condition (C′):

$$1 + s_1 a = s_2 a$$

holds for each $a \in S'$ and suitable $s_1, s_2 \in S$.

Example 2.4. Let P be the set of all nonnegative rational numbers. Then $(P, +, \cdot)$ with the usual operations, as well as $(P', +, \cdot)$, are semirings with 1 as identity satisfying condition (C′). The same is true, more generally, for each positive cone P of a totally ordered skew-field (cf. [3, Chapter VI]).

Example 2.5. Let $\mathbb{N}$ be the set of all nonnegative integers. Define $a + b = \max\{a, b\}$, and denote by $a \cdot b$ the usual multiplication. Then $(\mathbb{N}, +, \cdot)$ is a semiring with 1 as identity, which satisfies (C′) since $1 + a = a$ holds for all $a \in S'$.

Lemma 2.6. *If a semiring S with an absorbing zero O satisfies condition* (C), *then $ab = O$ for $a, b \in S$ implies $a = O$ or $b = O$.*

Proof. By way of contradiction, assume $ab = O$ and $a \neq O \neq b$. Then $s + s_1 a = s_2 a$, according to (C), yields $sb + s_1 ab = s_2 ab$, i.e., $sb = O$ for all $s \in S$. Consequently, $x + s_3 b = s_4 b$ implies $x = O$ for all $s_3, s_4 \in S$, which contradicts (C) applied to the element $b \in S'$.

Theorem 2.7. *Let S be a semiring. Then condition* (C) *implies that S contains only trivial k-ideals. The converse is true if $(S, \cdot)$ is commutative, and, provided that S has an absorbing element O, $Sa = \{sa | s \in S\} \neq \{O\}$ holds for all $a \in S'$.*

Proof. Assume that S satisfies (C). Let A be a k-ideal of S which contains at least one element $a \in S'$. Then $s + s_1 a = s_2 a$, according to (C), implies $s \in A$ for each $s \in S$, i.e., $A = S$. For the converse, our supplementary assumptions on S yield that Sa is an ideal of S and that $Sa \neq \{O\}$ holds for each $a \in S'$ if S has an absorbing element O. Now assume that S has only trivial k-ideals. Then the k-ideal $\overline{Sa}$ coincides with S for each $a \in S'$, regardless of whether S has an element O or not. Now,

$$\overline{Sa} = \{s \in S | s + s_1 a = s_2 a \text{ for some } s_i \in S\} = S$$

states that S satisfies condition (C).

Corollary 2.8. *Let S be a commutative semiring with identity. Then S has only trivial k-ideals iff it satisfies condition* (C′).

Proof. It was already stated that (C′) is equivalent to (C) if S has an identity 1, and $a = 1a \in Sa$ implies $Sa \neq \{O\}$ for all $a \in S'$ in the case that S has an absorbing element O. Hence the corollary follows from Theorem 2.7.

Theorem 2.9. *Let S be a commutative semiring with identity* 1 *and A a proper k-ideal of S. Then A is maximal iff the semiring $S/A = S/\rho_A$ satisfies condition* (C′).

Proof. Suppose A is a maximal k-ideal of S. Then A is the absorbing zero of S/A and $1\rho_A$ is its identity. Consider any $c\rho_A \in (S/A)'$. Then $c \notin A$ holds, and the smallest ideal B of S containing c and A consists of all elements sc, a, and $sc + a$ for $s \in S$ and $a \in A$. From $A \subset B$ it follows $\overline{B} = S$, and hence $1 + b_1 = b_2$ for suitable elements $b_1, b_2 \in B$. To avoid the discussion of different cases, we add $1c + a$ with an arbitrary element $a \in A$ to $1 + b_1 = b_2$ and obtain

$$1 + s_1 c + a_1 = s_2 c + a_2, \quad \text{i.e., } 1\rho_A + (s_1\rho_A)(c\rho_A) = (s_2\rho_A)(c\rho_A)$$

for suitable $s_i \in S$ and $a_i \in A$. This shows that S/A satisfies (C′).

Conversely, assume (C′) for S/A, and let B be a k-ideal of S satisfying $A \subset B$. Then there is an element $c \in B \setminus A$, and $c\rho_A \in (S/A)'$ yields $(1 + s_1 c)\rho_A = (s_2 c)\rho_A$ for suitable elements $s_i \in S$ by (C′). Hence $1 + s_1 c + a_1 = s_2 c + a_2$ holds for some $a_i \in A$, i.e., $1 + b_1 = b_2$ for $b_1, b_2 \in B$. This shows $\overline{B} = S$ and that A is a maximal k-ideal of S.

3. Completely prime k-ideals

Recall that an ideal A of a semiring S is called *completely prime* (cf., e.g., [5]) iff $ab \in A$ implies $a \in A$ or $b \in A$ for all $a, b \in S$.

Proposition 3.1. *Let S be a commutative semiring with identity. Then each maximal k-ideal A of S is completely prime.*

Proof. By Theorem 2.9, the semiring S/A satisfies the condition (C′) and hence (C). Since S/A has A as its absorbing zero, we can apply Lemma 2.6 and obtain that S/A has no zero-divisors. Hence $a\rho_A \neq A$ and $b\rho_A \neq A$ imply $(ab)\rho_A \neq A$, i.e., $a \notin A$ and $b \notin A$ imply $ab \notin A$ as we were to show.

Concerning the converse of Proposition 3.1, *we show that a completely prime ideal A of a commutative semiring S with identity need not be a k-ideal, and if it is one, A need not be a maximal k-ideal of S.*

Example 3.2. Let S be the set of all real numbers a satisfying $0 < a \leq 1$, and define $a + b = a \cdot b = \min\{a, b\}$ for all $a, b \in S$. Then $(S, +, \cdot)$ is easily checked to be a commutative semiring with 1 as identity. Each real number r such that $0 < r < 1$ defines an ideal $A = \{a \in S | a \leq r\}$ of S which is obviously completely prime. However, $r + 1 = r$ together with $r \in A$ and $1 \notin A$ show that A is not a k-ideal of S. The same is true if one includes 0 in these considerations (in this case 0 is an absorbing element but not a zero of $S \cup \{0\}$), but also if one adjoins 0 as an absorbing zero to S (cf., e.g., [7, Lemma 1.3]).

Example 3.3. The polynomial ring $\mathbb{Z}[x]$ over the ring $\mathbb{Z}$ of integers contains the subsemiring

$$S = \mathbb{N}[x] = \left\{ f(x) = \sum_{i=0}^{n} a_i x^i | a_i \in \mathbb{N} \right\},$$

which is clearly commutative and has $1 \in \mathbb{N}$ as its identity. The ideal $A = (x)$ of S consists of all $f(x) \in S$ such that $a_0 = 0$ holds. Obviously, A is completely prime and a k-ideal of S. Now consider the set B consisting of all $f(x) \in S$ for which a_0 is divisible by 2. Clearly, B is a k-ideal of S, and $A \subset B \subset S$ shows that A is not a maximal k-ideal.

4. Maximal k-ideals of $\mathbb{N}$

In this section we consider the semiring $(\mathbb{N}, +, \cdot)$ of nonnegative integers with respect to their usual operations.

Proposition 4.1. *The semiring $\mathbb{N}$ has exactly the k-ideals $(a) = \{na | n \in \mathbb{N}\}$ for each $a \in \mathbb{N}$. Consequently, the maximal k-ideals of $\mathbb{N}$ are given by (p) for each prime number p.*

Proof. Obviously, each ideal (a) of $\mathbb{N}$ is a k-ideal. Now assume that $A \neq (0)$ is a k-ideal of $\mathbb{N}$. Let a be the smallest positive integer contained in A, and b any element of A. Then $b = qa + r$ holds for some $q \in \mathbb{N}'$ and $r \in \mathbb{N}$ satisfying $0 \leq r < a$. Since r belongs to the k-ideal A, it follows that $r = 0$, and, hence, $A = (a)$. The last statement follows since $(a) \subseteq (b)$ holds iff b divides a.

Remark 4.2. None of the maximal k-ideals (p) of $\mathbb{N}$ is a maximal ideal of $\mathbb{N}$. This follows since each ideal $A = (p)$ is properly contained in the proper ideal $B = \{b \in \mathbb{N} | b \geq p\}$ of $\mathbb{N}$.

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