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ON LOCALLY ϕ-CONFORMALLY SYMMETRIC ALMOST KENMOTSU MANIFOLDS WITH NULLITY DISTRIBUTIONS

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Abstract. The aim of this paper is to investigate locally ϕ-conformally symmetric almost Kenmotsu manifolds with its characteristic vector field ξ belonging to some nullity distributions. Also, we give an example of a 5-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$.

1. Introduction

In the study of Riemannian manifolds (M, g), Gray [13] and Tanno [17] introduced the notion of k-nullity distribution, which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\} \tag{1.1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Moreover, if k is a smooth function then the distribution is called generalized k-nullity distribution.

Recently, Blair, Koufogiorgos and Papantoniou [3] introduced a generalized notion of the k-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, namely (k,μ)-nullity distribution which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM^{2n+1} : R(X,Y)Z = {}& k[g(Y,Z)X - g(X,Z)Y] \\ & + \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{1.2}$$

where $h = \frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation.

Next, Dileo and Pastore [11] introduced another generalized notion of the k-nullity distribution which is named the $(k,\mu)'$-nullity distribution on an almost

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Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$
\begin{aligned}
N_p(k,\mu)' = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\
+ \mu[g(Y,Z)h'X - g(X,Z)h'Y]\},
\end{aligned} \tag{1.3}
$$

where $h' = h \circ \phi$.

On the other hand, Kenmotsu [14] introduced a new class of almost contact metric manifolds named Kenmotsu manifolds nowadays. Several authors have studied Kenmotsu manifolds considering some curvature conditions. For example, De et al. [8] studied ϕ-recurrent Kenmotsu manifolds. In [7], De studied ϕ-symmetric Kenmotsu manifolds. However, not many investigations were performed on almost Kenmotsu manifolds with the characteristic vector field ξ belonging to some nullity distributions so far. Recently, some important results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions were obtained by Dileo et al. [10, 11, 12] and Wang et al. [18, 19, 20, 21, 22]. In [10], Dileo and Pastore obtained some important results on locally symmetric almost Kenmotsu manifolds.

The Weyl conformal curvature tensor C on a $(2n+1)$-dimensional manifold is defined by [23]

$$
\begin{aligned}
C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\
- g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\},
\end{aligned} \tag{1.4}
$$

where X, Y, Z are any vector fields, S is the Ricci tensor of type $(0,2)$, Q the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and r the scalar curvature.

The paper is organized as follows: In Section 2, we recall some basic formulas on almost Kenmotsu manifolds. In Section 3, we present some key lemmas on almost Kenmotsu manifolds with ξ belonging to some nullity distributions. Next, we study locally ϕ-conformally symmetric almost Kenmotsu manifolds with ξ belonging to some nullity distributions. Finally, we give an example of a 5-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$.

2. Almost Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold admitting a (ϕ, ξ, η)-structure or an almost contact structure, where ϕ is an $(1,1)$ tensor field, ξ a characteristic vector field and η an 1-form such that ([1, 2])

$$
\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}
$$

where I denote the identity endomorphism. It is customary to include also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1).

If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y of T_pM^{2n+1}, then M is said to have an almost contact metric structure (ϕ, ξ, η, g). The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y of T_pM^{2n+1}. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The normality of an almost contact metric manifold is given by vanishing the $(1, 2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y. It is well known [14] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and f is the warping function defined by $f = ce^t$ for some positive constant c. Let $\mathcal{D}$ be the distribution orthogonal to ξ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold $\mathcal{D}$ is an integrable distribution as η is closed. Let the two tensor fields $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on an almost Kenmotsu manifold M^{2n+1}. The tensor fields l and h are symmetric and satisfy the following relations [15]

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \tag{2.5}$$

for any vector fields X, Y. In the study of Sasakian manifolds, Takahashi [16] introduced the notion of locally ϕ-symmetric manifold as a weaker version of local symmetry. Later Boeckx et al. [5] studied ϕ-symmetric contact metric manifolds. In [4], Boeckx studied locally ϕ-symmetric contact metric manifolds. In [9], De et al. obtained some important theorems on 3-dimensional locally ϕ-symmetric normal almost contact metric manifolds. Also Boeckx and Cho [6] obtained some important classification theorems on locally symmetric contact metric manifolds. According to Takahashi [16] we have the following definition:

Definition 2.1. An almost Kenmotsu manifold is said to be ϕ-symmetric if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = 0$$

for any vector fields $W, X, Y, Z \in T_pM$. In addition, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-symmetric.

Definition 2.2. An almost Kenmotsu manifold is said to be ϕ-conformally symmetric if it satisfies

$$\phi^2((\nabla_W C)(X,Y)Z) = 0$$

for any vector fields $W, X, Y, Z \in T_pM$. In addition, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-conformally symmetric.

3. Properties of the nullity conditions

In this section we present some key results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0. \tag{3.1}$$

We have the following results.

Lemma 3.1 (Prop. 3.1 and Prop. 5.1 of [15]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold satisfying either the generalized (k,μ)-nullity condition or the generalized $(k,\mu)'$-nullity condition (the term generalized means k, μ both are smooth functions), with $h \neq 0$. Then, one has*

$$h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2), \tag{3.2}$$

$$S(X,\xi) = 2nk\eta(X) \tag{3.3}$$

for any vector fields X on M^{2n+1}. Furthermore, in the case of generalized (k,μ)-nullity condition, one has

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX] \tag{3.4}$$

and in the case of generalized $(k,\mu)'$-nullity condition, one has

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X] \tag{3.5}$$

for any $X, Y \in T_pM$. In addition if $n > 1$, then one has

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X) - (\mu+2)\eta(X)h'Y \tag{3.6}$$

for any $X, Y \in T_pM$.

Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue λ. It follows from (3.2) that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces associated with h' corresponding to the non-zero eigenvalue λ and $-\lambda$ respectively.

Lemma 3.2 (Prop. 4.1 and Prop. 4.3 of [11]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are* *integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as following*:

(a) $K(X,\xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,

(b) $K(X,Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k-2n)$.

Lemma 3.3 (Lemma 3 of [20])**.** *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold with* ξ *belonging to the generalized* $(k,\mu)'$*-nullity distribution and* $h' \neq 0$*. If* $n > 1$*, then the Ricci operator* Q *of* M^{2n+1} *is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi + [\mu - 2(n-1)]h'. \tag{3.7}$$

Moreover, if both k *and* μ *are constant, then we have*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.8}$$

In both cases, the scalar curvature of M^{2n+1} *is* $2n(k-2n)$.

Lemma 3.4 (Theorem 5.1 and Proposition 5.2 of [15])**.** *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$, $(n > 1)$ *be an almost Kenmotsu manifold such that* $h' \neq 0$ *and* ξ *belongs to the generalized* $(k,\mu)'$*-nullity distribution. Then for any* $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ *and* $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$*, the Riemannian curvature tensor satisfies*:

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0,\\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0,\\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},\\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,\\
R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

Further, for the sectional curvature we have:

(a) $K(X,\xi) = k + \lambda\mu$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k - \lambda\mu$ *if* $X \in [-\lambda]'$,

(b) $K(X,Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k-2n)$.

Lemma 3.5 (Proposition 4.2 of [11])**.** *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold such that* $h' \neq 0$ *and* ξ *belongs to the* $(k,-2)'$*-nullity distribution. Then for any* $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ *and* $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$*, the Riemannian curvature tensor satisfies*:

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0,\\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0,
\end{aligned}$$

$$\begin{aligned}R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},\\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,\\ R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].\end{aligned}$$

Lemma 3.6 (Lemma 4.1 of [11]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to the $(k, -2)'$-nullity distribution. Then for any $X, Y \in T_pM$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{3.9}$$

Lemma 3.7 (Theorem 4.1 of [11]). *Let M be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characteristic vector field ξ belonging to the (k, μ)-nullity distribution. Then $k = -1$, $h = 0$ and M is locally a warped product of an open interval and an almost Kähler manifold.*

4. Locally ϕ-conformally symmetric almost Kenmotsu manifolds

This section deals with locally ϕ-conformally symmetric almost Kenmotsu manifolds with some nullity distributions. At first we prove the following:

Theorem 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then the manifold M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. We suppose that the manifold M^{2n+1} is a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distribution. Then we have

$$\phi^2((\nabla_Z C)(X, Y)W) = 0 \tag{4.1}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Substituting $W = \xi$ in the above equation we get

$$\phi^2((\nabla_Z C)(X, Y)\xi) = 0. \tag{4.2}$$

Making use of (3.8) and (1.3) in (1.4) we obtain

$$C(X, Y)\xi = \left(\mu + \frac{2n}{2n-1}\right)\{\eta(Y)h'X - \eta(X)h'Y\} \tag{4.3}$$

for any vector fields $X, Y \in T_pM$. Taking the covariant differentiation along any arbitrary vector field $Z \in T_pM$ of (4.3) we have

$$\begin{aligned}(\nabla_Z C)(X, Y)\xi = &\left(\mu + \frac{2n}{2n-1}\right)\{(\nabla_Z\eta)Y(h'X) + \eta(Y)(\nabla_Z h')X\\ &- (\nabla_Z\eta)X(h'Y) - \eta(X)(\nabla_Z h')Y\}\end{aligned} \tag{4.4}$$

for any vector fields $X, Y \in T_pM$. Applying ϕ^2 on both sides of (4.4) gives

$$\begin{aligned}\phi^2((\nabla_Z C)(X,Y)\xi) = \left(\mu + \frac{2n}{2n-1}\right)\{&(\nabla_Z\eta)Y(-h'X) + \eta(Y)\phi^2((\nabla_Z h')X) \\ &- (\nabla_Z\eta)X(-h'Y) - \eta(X)\phi^2((\nabla_Z h')Y)\}\end{aligned} \tag{4.5}$$

for any vector fields $X, Y \in T_pM$. With the help of (2.3) we have from the above equation

$$\begin{aligned}\phi^2((\nabla_Z C)(X,Y)\xi) = \left(\mu + \frac{2n}{2n-1}\right)\{&-h'X[g(Y,Z) - \eta(Y)\eta(Z) + g(h'Z,Y)] \\ &+ h'Y[g(X,Z) - \eta(X)\eta(Z) + g(h'Z,X)] \\ &+ \eta(Y)\phi^2((\nabla_Z h')X) - \eta(X)\phi^2((\nabla_Z h')Y)\}\end{aligned} \tag{4.6}$$

for any vector fields $X, Y \in T_pM$. In view of (4.1), (4.6) and noticing that X, Y, Z are orthogonal to ξ implies

$$\left(\mu + \frac{2n}{2n-1}\right)\{-h'X[g(Y,Z) + g(h'Z,Y)] + h'Y[g(X,Z) + g(h'Z,X)]\} = 0. \tag{4.7}$$

In [11], Dileo and Pastore proved that if ξ belongs to the $(k,\mu)'$-nullity distribution then $\mu = -2$. Using this result and by the assumption $n > 1$, it follows from (4.7) that

$$-h'X[g(Y,Z) + g(h'Z,Y)] + h'Y[g(X,Z) + g(h'Z,X)] = 0 \tag{4.8}$$

for any vector fields X, Y, Z orthogonal to ξ. Letting $X, Y, Z \in [-\lambda]'$ in (4.8) implies that

$$\lambda(1-\lambda)[g(Y,Z)X - g(X,Z)Y] = 0. \tag{4.9}$$

Suppose $\lambda = 0$, then from the fact $\lambda^2 = -(k+1)$ we have $k = -1$ and hence $h' = 0$ from (3.2), which contradicts our assumption $h' \neq 0$. Therefore $\lambda \neq 0$, then it follows from (4.9) that $\lambda = 1$ and hence $k = -2$. Then we can write from Lemma 3.5 that

$$\begin{aligned}R(X_\lambda, Y_\lambda)Z_\lambda &= -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= 0\end{aligned}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also it follows from Lemma 3.2 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.2 we see that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [11] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [1]'$ and $[-1]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof. □

Next we prove the following:

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then the manifold M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. We suppose that the manifold M^{2n+1} is a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized $(k, \mu)'$-nullity distribution. This implies

$$\phi^2((\nabla_Z C)(X, Y)W) = 0 \tag{4.10}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Setting $W = \xi$ in the above equation gives

$$\phi^2((\nabla_Z C)(X, Y)\xi) = 0. \tag{4.11}$$

From (3.5) we have

$$R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\} \tag{4.12}$$

for any $X, Y \in T_pM$. Making use of (4.12) and (3.7) in (1.4) we get

$$C(X, Y)\xi = \left(\frac{2(n-1)(\mu+1)}{2n-1}\right)\{\eta(Y)h'X - \eta(X)h'Y\} \tag{4.13}$$

for any vector fields $X, Y \in T_pM$. Taking the covariant differentiation along any arbitrary vector field $Z \in T_pM$ of (4.13) we have

$$\begin{aligned}(\nabla_Z C)(X, Y)\xi = {}& \frac{2(n-1)}{2n-1}\{\eta(Y)h'X - \eta(X)h'Y\}Z(\mu) \\ & + \left(\frac{2(n-1)(\mu+1)}{2n-1}\right)\{(\nabla_Z\eta)Y(h'X) + \eta(Y)(\nabla_Z h')X \\ & - (\nabla_Z\eta)X(h'Y) - \eta(X)(\nabla_Z h')Y\}\end{aligned} \tag{4.14}$$

for any vector fields $X, Y \in T_pM$. Applying ϕ^2 on both sides of (4.14) gives

$$\begin{aligned}& \phi^2((\nabla_Z C)(X, Y)\xi) \\ = {}& \frac{2(n-1)}{2n-1}\{\eta(Y)(-h'X) - \eta(X)(-h'Y)\}Z(\mu) \\ & + \left(\frac{2(n-1)(\mu+1)}{2n-1}\right)\{\eta(Y)\phi^2((\nabla_Z h')X) - (\nabla_Z\eta)Y(h'X) \\ & - (\nabla_Z\eta)X(-h'Y) - \eta(X)\phi^2((\nabla_Z h')Y)\}\end{aligned} \tag{4.15}$$

for any vector fields $X, Y \in T_pM$. Using (2.3) the above equation implies

$$\begin{aligned}\phi^2((\nabla_Z C)(X,Y)\xi) = &\frac{2(n-1)}{2n-1}\{\eta(Y)(-h'X) - \eta(X)(-h'Y)\}Z(\mu)\\ &+ \left(\frac{2(n-1)(\mu+1)}{2n-1}\right)\{h'Y[g(X,Z) - \eta(X)\eta(Z)]\\ &+ h'Yg(h'Z,X) - h'X[g(Y,Z) - \eta(Y)\eta(Z) + g(h'Z,Y)]\\ &+ \eta(Y)\phi^2((\nabla_Z h')X) - \eta(X)\phi^2((\nabla_Z h')Y)\}\end{aligned} \tag{4.16}$$

for any vector fields $X, Y \in T_pM$. In view of (4.11), (4.16) and noticing that X, Y, Z are orthogonal to ξ implies

$$\begin{aligned}&\left(\frac{2(n-1)(\mu+1)}{2n-1}\right)\{h'Y[g(X,Z) + g(h'Z,X)]\\ &-h'X[g(Y,Z) + g(h'Z,Y)]\} = 0.\end{aligned} \tag{4.17}$$

Again noticing the assumption $n > 1$ it follows from (4.17) that

$$(\mu+1)\{h'Y[g(X,Z) + g(h'Z,X)] - h'X[g(Y,Z) + g(h'Z,Y)]\} = 0 \tag{4.18}$$

for any vector fields X, Y, Z orthogonal to ξ. Letting $X, Y, Z \in [-\lambda]'$ in (4.18) implies that

$$(\mu+1)\lambda(1-\lambda)[g(Y,Z)X - g(X,Z)Y] = 0. \tag{4.19}$$

Here we see that three possible cases arise:

`Case 1.` $\mu = -1$. In [15], Pastore and Saltarelli proved that in an almost Kenmotsu manifold with ξ belonging to the generalized $(k,\mu)'$-nullity distribution and $h' \neq 0$ one has

$$\nabla_\xi h' = -(\mu+2)h', \tag{4.20}$$

and for any $X, Y, Z \in \mathcal{D}$ one has

$$g((\nabla_X h')Y, Z) = 0. \tag{4.21}$$

Noticing $\mu = -1$, we have from (4.20)

$$\nabla_\xi h' = -h'. \tag{4.22}$$

Making use of (4.21) and (4.22) we have

$$g(h'Y, Z) = 0, \tag{4.23}$$

which implies that $h' = 0$, a contradiction. Thus $\mu = -1$ is not possible.

`Case 2.` $\lambda = 0$. Then from the fact $\lambda^2 = -(k+1)$ we have $k = -1$ and hence $h' = 0$ from (3.2), which again contradicts our assumption $h' \neq 0$. Therefore $\lambda \neq 0$.

`Case 3.` $\lambda = 1$. Hence $k = -2$. Then we can write from Lemma 3.4 that

$$\begin{aligned}R(X_\lambda, Y_\lambda)Z_\lambda &= -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= 0\end{aligned}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also it follows from Lemma 3.4 that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [11] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [1]'$ and $[-1]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof. □

Now we study locally ϕ-conformally symmetric almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity and generalized (k,μ)-nullity distributions respectively. We prove the following:

Theorem 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then the manifold M^{2n+1} is an Einstein one.*

Proof. Let the manifold M^{2n+1} be a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. This implies

$$\phi^2((\nabla_Z C)(X,Y)W) = 0 \tag{4.24}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Putting $X = \xi$ in the above equation we have

$$\phi^2((\nabla_Z C)(\xi,Y)W) = 0. \tag{4.25}$$

Making use of equation (1.2) and Lemma 3.7 in (1.4) yields

$$\begin{aligned} C(\xi,Y)W = & -\frac{2n}{2n-1}\{g(Y,W)\xi - \eta(W)Y\} \\ & -\frac{1}{2n-1}\{S(Y,W)\xi - \eta(W)QY\} \end{aligned} \tag{4.26}$$

for any $Y, W \in T_pM$. Taking the covariant differentiation along any vector field $Z \in T_pM$ of (4.26) we get

$$\begin{aligned} (\nabla_Z C)(\xi,Y)W = & -\frac{2n}{2n-1}\{g(Y,W)(\nabla_Z\xi) - ((\nabla_Z\eta)W)Y\} \\ & -\frac{1}{2n-1}\{S(Y,W)(\nabla_Z\xi) - QY(\nabla_Z\eta)W \\ & + (\nabla_Z S)(Y,W)\xi - \eta(W)(\nabla_Z Q)Y\} \end{aligned} \tag{4.27}$$

for any $Z, Y, W \in T_pM$. Using (2.3) in the above equation implies

$$(\nabla_Z C)(\xi,Y)W = -\frac{2n}{2n-1}\{g(Y,W)[Z - \eta(Z)\xi] - [g(Z,W)$$

$$- \eta(Z)\eta(W)]Y\} - \frac{1}{2n-1}\{S(Y,W)[Z - \eta(Z)\xi] - [g(Z,W) - \eta(Z)\eta(W)]QY - \eta(W)(\nabla_Z Q)Y + g((\nabla_Z Q)Y, W)\xi\} \tag{4.28}$$

for any $Z, Y, W \in T_pM$. Applying ϕ^2 on both sides of the equation (4.28) gives

$$\phi^2((\nabla_Z C)(\xi, Y)W) = -\frac{2n}{2n-1}\{g(Y,W)[-Z + \eta(Z)\xi] - [g(Z,W) - \eta(Z)\eta(W)][-Y + \eta(Y)\xi\} - \frac{1}{2n-1}\{S(Y,W)[-Z + \eta(Z)\xi] - [g(Z,W) - \eta(Z)\eta(W)][-QY + \eta(QY)\xi] - \eta(W)\phi^2((\nabla_Z Q)Y)\} \tag{4.29}$$

for any $Z, Y, W \in T_pM$. From (4.29) and (4.25) and using the fact Y, Z, W are orthogonal to ξ, we have

$$2n\{g(Y,W)Z - g(Z,W)Y\} + S(Y,W)Z - g(Z,W)QY = 0 \tag{4.30}$$

for any $Z, Y, W \in T_pM$. Taking inner product of (4.30) with arbitrary vector field U yields

$$2n\{g(Y,W)g(Z,U) - g(Z,W)g(Y,U)\} + S(Y,W)g(Z,U) - g(Z,W)S(Y,U) = 0 \tag{4.31}$$

for any $Z, Y, W, U \in T_pM$. Let $\{e_i : i = 1, 2, \ldots, 2n+1\}$ be a local orthonormal basis of tangent space at each point of the manifold M^{2n+1}. Setting $Y = U = e_i$ in (4.31) and taking summation over $i : 1 \leq i \leq 2n+1$, we get

$$S(Z,W) = (r + 4n^2)g(Z,W) \tag{4.32}$$

for any $Z, W \in T_pM$. In [11], Dileo and Pastore prove that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the sectional curvature $K(X,\xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$. Hence, from (4.32) we have $S(Z,W) = -2ng(Z,W)$, for any $Z, W \in T_pM$. Therefore the manifold M^{2n+1} is an Einstein one. This completes the proof. □

Theorem 4.4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (k,μ)-nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} reduces to an Einstein almost Kenmotsu manifold with ξ belonging to the generalized k-nullity distribution.*

Proof. Let us suppose that M^{2n+1} be a locally ϕ-conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (k,μ)-nullity distribution. This implies

$$\phi^2((\nabla_W C)(X,Y)Z) = 0 \tag{4.33}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Substituting $X = \xi$ in (4.33) we have

$$\phi^2((\nabla_W C)(\xi, Y)Z) = 0. \tag{4.34}$$

In [15], Pastore and Saltarelli prove that in an almost Kenmotsu manifold $M^{2n+1}, n > 1$ satisfying the generalized (k, μ)-nullity distribution with $h \neq 0$, the scalar curvature is given by $r = 2n(k - 2n)$. Using this result and (3.4) we have from (1.4)

$$\begin{aligned} C(\xi, Y)Z = & \frac{-2n}{2n-1}\{g(Y, Z)\xi - \eta(Z)Y\} + \mu\{g(hY, Z)\xi - \eta(Z)hY\} \\ & - \frac{1}{2n-1}\{S(Y, Z)\xi - \eta(Z)QY\} \end{aligned} \tag{4.35}$$

for any $Y, Z \in T_pM$. Taking the covariant differentiation along arbitrary vector field $W \in T_pM$ of (4.35) we get

$$\begin{aligned} (\nabla_W C)(\xi, Y)Z = & \frac{-2n}{2n-1}\{g(Y, Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)Y\} + W(\mu)\{g(hY, Z)\xi \\ & - \eta(Z)hY\} + \mu\{g(hY, Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)hY \\ & - \eta(Z)(\nabla_W h)Y\} - \frac{1}{2n-1}\{S(Y, Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)QY \\ & - \eta(Z)(\nabla_W Q)Y + (\nabla_W S)(Y, Z)\xi\}. \end{aligned} \tag{4.36}$$

After using (2.3) and applying ϕ^2 on both sides of (4.36) and noticing W, Y, Z are orthogonal to ξ implies

$$\begin{aligned} \phi^2((\nabla_W C)(\xi, Y)Z) = & \frac{-2n}{2n-1}\{g(Y, Z)[-W + \phi hW] + [g(Z, W) + g(h\phi W, Z)]Y\} \\ & + \mu\{g(hY, Z)[-W + \phi hW] + g(W, Z)hY + g(h\phi W, Z)hY\} \\ & - \frac{1}{2n-1}\{S(Y, Z)[-W + \phi hW] \\ & + [g(W, Z) + g(h\phi W, Z)]QY\}. \end{aligned} \tag{4.37}$$

In view of (4.34) and (4.37) we have

$$\begin{aligned} & \frac{-2n}{2n-1}\{g(Y, Z)[-W + \phi hW] + [g(Z, W) + g(h\phi W, Z)]Y\} \\ & + \mu\{g(hY, Z)[-W + \phi hW] + g(W, Z)hY + g(h\phi W, Z)hY\} \\ & - \frac{1}{2n-1}\{S(Y, Z)[-W + \phi hW] + [g(W, Z) + g(h\phi W, Z)]QY\} = 0 \end{aligned} \tag{4.38}$$

for any $Y, Z, W \in T_pM$. Taking inner product of (4.38) with any vector field U we obtain

$$\begin{aligned} & \frac{-2n}{2n-1}\{g(Y, Z)[-g(W, U) + g(\phi hW, U)] + [g(Z, W) + g(h\phi W, Z)]g(Y, U)\} \\ & + \mu\{g(hY, Z)[-g(W, U) + g(\phi hW, U)] + [g(W, Z) + g(h\phi W, Z)]g(hY, U)\} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2n-1}\{S(Y,Z)[-g(W,U)+g(\phi hW,U)] \\
&+[g(W,Z)+g(h\phi W,Z)]S(Y,U)\}=0
\end{aligned} \tag{4.39}
$$

for any $Y,Z,W,U \in T_pM$. Let us consider $\{e_i : i = 1,2,\dots,2n+1\}$ be a local orthonormal basis of tangent space at each point of the manifold M^{2n+1}. Setting $Y = U = e_i$ in (4.39) and taking summation over $i : 1 \leq i \leq 2n+1$, implies

$$
\begin{aligned}
&\frac{-2n}{2n-1}\{2ng(W,Z)-2ng(\phi hW,Z)\}+\mu\{-g(hZ,W)+g(hZ,\phi hW)\} \\
&-\frac{1}{2n-1}\{S(Z,\phi hW)-S(Z,W)+2n(k-2n)[g(W,Z)+g(h\phi W,Z)]\} \\
&=0
\end{aligned} \tag{4.40}
$$

for any $Z,W \in T_pM$. Substituting $W = \phi hW$ in (4.40) and using (3.2) we get

$$
\begin{aligned}
&\frac{-2n}{2n-1}\{2ng(\phi hW,Z)+2n(k+1)g(W,Z)\} \\
&+\mu\{-g(hZ,\phi hW)-(k+1)g(hZ,W)\} \\
&-\frac{1}{2n-1}\{-S(Z,\phi hW)-(k+1)S(Z,W) \\
&+2n(k-2n)[g(\phi hW,Z)+(k+1)g(W,Z)]\}=0
\end{aligned} \tag{4.41}
$$

for any $Z,W \in T_pM$. Adding (4.40) and (4.41) one can easily get

$$(k+2)\{S(Z,W)-2nkg(Z,W)-(2n-1)\mu g(hZ,W)\}=0 \tag{4.42}$$

for any $Z,W \in T_pM$. From (4.42), we see that either $k=-2$ or,

$$S(Z,W)=2nkg(Z,W)+(2n-1)\mu g(hZ,W) \tag{4.43}$$

for any vector field $Z,W \in T_pM$. Suppose $k=-2$, a constant, then $\xi(k)=0$. Now we represent a result due to Pastore and Saltarelli [15]: In an almost Kenmotsu manifold with generalized (k,μ)-nullity distribution and $h \neq 0$, the relation $\xi(k)=-4(k+1)$ holds. Therefore substituting $k=-2$ in this result we get $\xi(k)=4$. Thus, we have $\xi(k)=0$ and $\xi(k)=4$, which is not possible. Hence, it follows from (4.42) that

$$S(Z,W)=2nkg(Z,W)+(2n-1)\mu g(hZ,W) \tag{4.44}$$

for any $Z,W \in T_pM$. Now, replacing Z by hZ in (4.44) and using (3.2) and also noticing Z,W are orthogonal to ξ yields

$$S(hZ,W)=2nkg(hZ,W)-(2n-1)\mu(k+1)g(Z,W) \tag{4.45}$$

for any $Z,W \in T_pM$. Setting $Z=W=\xi$ in (4.45) implies

$$\mu(k+1)=0. \tag{4.46}$$

Suppose $k+1=0$, that is, $k=-1$. Then from (3.2), we have $h=0$, which contradicts our assumption $h \neq 0$. Hence it follows from (4.46) that

$$\mu = 0.$$

Then ξ belongs to the generalized k-nullity distribution. Also substituting the above relation into (4.44) we get $S(Z,W) = 2nkg(Z,W)$ for any $Z, W \in T_pM$. Therefore the manifold M^{2n+1} is an Einstein one. Thus the manifold M^{2n+1} reduces to an Einstein almost Kenmotsu manifold with ξ belonging to the generalized k-nullity distribution. This completes the proof of our theorem. □

5. Example of a 5-dimensional almost Kenmotsu manifold

In [11], Dileo and Pastore construct an example of an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. With the help of that example here we give an example of an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. We consider a 5-dimensional manifold $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 are five vector fields in $\mathbb{R}^5$ which satisfies

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0,$$
$$[e_i, e_j] = 0,\ \text{where } i, j = 2, 3, 4, 5.$$

Let g be the Riemannian metric defined by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1$$
$$\text{and } g(\xi, e_i) = g(e_i, e_j) = 0 \text{ for } i \neq j;\ i, j = 2, 3, 4, 5.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \xi)$$

for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = e_4,\ \phi(e_3) = e_5,\ \phi(e_4) = -e_2,\ \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g we have

$$\eta(\xi) = 1,\ \phi^2 Z = -Z + \eta(Z)\xi$$

and

$$g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U)$$

for any $Z, U \in \chi(M)$. Moreover,

$$h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3,\ h'e_4 = -e_4,\ h'e_5 = -e_5.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

Using Koszul's formula we get the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\ \nabla_\xi e_4 = 0,\ \nabla_\xi e_5 = \xi,$$
$$\nabla_{e_2}\xi = 2e_2,\ \nabla_{e_2}e_2 = -2\xi,\ \nabla_{e_2}e_3 = 0,\ \nabla_{e_2}e_4 = 0,\ \nabla_{e_2}e_5 = 0,$$
$$\nabla_{e_3}\xi = 2e_3,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_3}e_3 = -2\xi,\ \nabla_{e_3}e_4 = 0,\ \nabla_{e_3}e_5 = 0,$$

$$\nabla_{e_4}\xi = 0,\ \nabla_{e_4}e_2 = 0,\ \nabla_{e_4}e_3 = 0,\ \nabla_{e_4}e_4 = 0,\ \nabla_{e_4}e_5 = 0,$$
$$\nabla_{e_5}\xi = 0,\ \nabla_{e_5}e_2 = 0,\ \nabla_{e_5}e_3 = 0,\ \nabla_{e_5}e_4 = 0,\ \nabla_{e_5}e_5 = 0.$$

In view of the above relations we have

$$\nabla_X\xi = -\phi^2 X + h'X$$

for any $X \in \chi(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$R(\xi, e_2)\xi = 4e_2,\ R(\xi, e_2)e_2 = -4\xi,\ R(\xi, e_3)\xi = 4e_3,\ R(\xi, e_3)e_3 = -4\xi,$$
$$R(\xi, e_4)\xi = R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0,$$
$$R(e_2, e_3)e_2 = 4e_3,\ R(e_2, e_3)e_3 = -4e_2,\ R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0,$$
$$R(e_2, e_5)e_2 = R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0,$$
$$R(e_3, e_5)e_3 = R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0.$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution, with $k = -2$ and $\mu = -2$.

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