Research Article

Uday Chand De and Abdul Haseeb*

On generalized Sasakian-space-forms with *M*-projective curvature tensor

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Abstract: The object of the present paper is to study generalized Sasakian-space-forms satisfying the curvature condition $P(\xi, Y) \cdot W = 0$. Moreover, ϕ-M-projectively semisymmetric and ϕ-pseudo-projectively semisymmetric generalized Sasakian-space-forms are also studied.

Keywords: Generalized Sasakian-space-form, M-projective curvature tensor, η-Einstein manifold, ϕ-M-projectively semisymmetric manifold, ϕ-pseudo-projectively semisymmetric manifold

MSC 2010: 53D10, 53D15, 53C25

1 Introduction

The notion of generalized Sasakian-space-forms was introduced and studied by P. Alegre, D. E. Blair and A. Carriazo [1] as the almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{1.1}$$

where f_1, f_2, f_3 are some differential functions on M^{2n+1}. The above equation can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3,$$

where

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \end{aligned}$$

for any vector fields X, Y, Z on M. In such a case, we denote the manifold as $M(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well-known Sasakian-space-forms which can be obtained as a particular case of generalized Sasakian-space-forms by taking $f_1 = \frac{c+3}{2}$, $f_2 = f_3 = \frac{c-1}{4}$.

The Ricci tensor S, the Ricci operator Q and the scalar curvature r of the manifold of dimension $(2n + 1)$ are given respectively by

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y), \tag{1.2}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi, \tag{1.3}$$

$$r = 2n(2n + 1)f_1 + 6f_2 - 4nf_3$$

for all vector fields X, Y, Z on M^{2n+1}, where f_1, f_2, f_3 are smooth functions on the manifold [1].

Uday Chand De: Department of Pure Mathematics, University of Calcutta, 35 B.C. Road, Kolkata 700019, West Bengal, India, e-mail: uc_de@yahoo.com
***Corresponding author: Abdul Haseeb:** Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia, e-mail: malikhaseeb80@gmail.com

In 2008, F. Gherib, M. Gorine and M. Belkhelfa [9] have studied parallel and semisymmetry of projective, Weyl-conformal and concircular curvature tensors in generalized Sasakian-space-forms, while U. C. De and P. Majhi [7] studied ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian-space-forms. Recently, J. P. Singh [15] studied M-projectively locally symmetric and locally ϕ-symmetric generalized Sasakian-space-forms, while Venkatesha and B. Sumangala [16] have studied M-projectively flat generalized Sasakian-space-forms and they proved that the generalized Sasakian-space-form is irrotational if and only if it is M-projectively flat. It is known that any three-dimensional (α, β)-trans-Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-form [2]. Generalized Sasakian-space-forms have also been studied by several authors in the papers [3, 6, 8, 10, 14] and many others.

In 1971, G. P. Pokhariyal and R. S. Mishra [12] introduced a new curvature tensor and called it M-projective curvature tensor. In 1975, R. H. Ojha [11] studied the properties of the m-projective curvature tensor in a Kaehler manifold. In 2010, S. K. Chaubey and R. H. Ojha [5] studied the properties of the M-projective curvature tensor of a Kenmotsu manifold.

Motivated by the above studies, in this paper we study generalized Sasakian-space-forms with M-projective curvature tensor. The paper is organized as follows: In Section 2, we review some preliminary results. In Section 3, we study generalized Sasakian-space-forms satisfying the curvature condition $P(\xi, Y) \cdot W = 0$. Sections 4 and 5 deal with ϕ-M-projectively semisymmetric and ϕ-pseudo-projectively semisymmetric generalized Sasakian-space-forms, respectively.

2 Preliminaries

If, on an odd-dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exist a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi(\xi) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifold [4] and the structure (ϕ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of (2.1)–(2.3), we have

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0,$$
$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y).$$

By virtue of equations (1.1)–(1.3), we have

$$\begin{aligned} R(X, Y)\xi &= (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \\ R(\xi, X)Y &= -R(X, \xi)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \\ \eta(R(X, Y)Z) &= (f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \\ S(X, \xi) &= 2n(f_1 - f_3)\eta(X), \quad S(\xi, \xi) = 2n(f_1 - f_3), \\ Q\xi &= 2n(f_1 - f_3)\xi, \end{aligned} \tag{2.4}$$

where $g(QX, Y) = S(X, Y)$.

A generalized Sasakian-space-form is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = lg(X, Y) + m\eta(X)\eta(Y)$$

for arbitrary vector fields X and Y, where l and m are smooth functions on M^{2n+1}.

Definition 2.1 ([7]). For $n > 1$, an almost contact metric manifold (M^{2n+1}, g) is said to be ϕ-M-projectively semisymmetric if $W(X, Y) \cdot \phi = 0$ on M for all $X, Y \in \chi(M)$.

Definition 2.2 ([7]). For $n > 1$, an almost contact metric manifold (M^{2n+1}, g) is said to be ϕ-pseudo-projectively semisymmetric if $P(X, Y) \cdot \phi = 0$ on M for all $X, Y \in \chi(M)$.

For $n > 1$ and a $(2n + 1)$-dimensional almost contact metric manifold, the M-projective curvature tensor W (see [11, 15]) and the pseudo-projective curvature tensor P (see [13]) are given respectively by

$$W(X, Y)Z = R(X, Y)Z - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \tag{2.5}$$

and

$$P(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)[g(Y, Z)X - g(X, Z)Y], \tag{2.6}$$

where a and b are constants such that $a, b \neq 0$ and R, S and r are the Riemannian curvature tensor of type $(0, 2)$, the Ricci tensor and the scalar curvature of M, respectively. If $a = 1$, $b = -\frac{1}{2n}$, then (2.6) takes the form

$$P(X, Y)Z = \bar{P}(X, Y)Z,$$

where $\bar{P}$ is the projective curvature tensor.

For further use, we state the following lemma without proof.

Lemma 2.3. *Let M be a $(2n + 1)$-dimensional generalized Sasakian-space-form with M-projective curvature tensor. Then*

$$\begin{aligned}
W(\xi, Y)Z &= (f_1 - f_3)[g(Y, Z)\xi - \eta(Z)Y] \\
&\quad - \frac{1}{4n}[S(Y, Z)\xi - 2n(f_1 - f_3)\eta(Z)Y + 2n(f_1 - f_3)g(Y, Z)\xi - \eta(Z)QY], && (2.7)\\
\eta(W(\xi, Y)Z) &= \frac{(f_1 - f_3)}{2}g(Y, Z) - \frac{1}{4n}S(Y, Z), && (2.8)\\
\eta(W(\xi, Y)\xi) &= 0, \quad \eta(W(Y, Z)\xi) = 0, \quad \eta(W(\xi, \xi)\xi) = 0, && (2.9)\\
S(Y, W(\xi, Z)\xi) &= \frac{3f_2 + (2n-1)f_3}{4n}S(Y, Z) - \frac{(3f_2 + (2n-1)f_3)(f_1 - f_3)}{2}\eta(Y)\eta(Z),\\
S(\xi, W(\xi, Y)\xi) &= 0, && (2.10)\\
g(Y, W(\xi, Z)\xi) &= -\frac{(f_1 - f_3)}{2}g(Y, Z) + \frac{1}{4n}S(Y, Z),\\
\eta(W(QY, Z)\xi) &= 0, \quad \eta(W(\xi, QY)\xi) = 0,\\
\eta(W(\xi, Z)QY) &= \frac{1}{2}(f_1 - f_3)S(Y, Z) - \frac{1}{4n}S(U, QY), && (2.11)\\
P(\xi, Y)Z &= \Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, Z)\xi - \eta(Z)Y] + b[S(Y, Z)\xi - S(\xi, Z)Y]
\end{aligned}$$

for all $X, Y, Z \in \chi(M)$.

3 Generalized Sasakian-space-forms satisfying $P(\xi, Y) \cdot W = 0$

Theorem 3.1. *For $n > 1$, a $(2n + 1)$-dimensional generalized Sasakian-space-form satisfying $P(\xi, Y) \cdot W = 0$ is an η-Einstein manifold.*

Proof. Let us consider a generalized Sasakian-space-form satisfying

$$P(X, Y) \cdot W(Z, U)V = 0.$$

This implies that

$$P(X, Y)W(Z, U)V - W(P(X, Y)Z, U)V - W(Z, P(X, Y)U)V - W(Z, U)P(X, Y)V = 0. \tag{3.1}$$

Replacing $X = \xi$ in (3.1), we have

$$P(\xi, Y)W(Z, U)V - W(P(\xi, Y)Z, U)V - W(Z, P(\xi, Y)U)V - W(Z, U)P(\xi, Y)V = 0. \tag{3.2}$$

In view of (2.11), we have

$$\begin{aligned} P(\xi, Y)W(Z, U)V &= \Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, M(Z, U)V)\xi - \eta(M(Z, U)V)Y] \\ &\quad + b[S(Y, W(Z, U)V)\xi - S(\xi, W(Z, U)V)Y], \qquad (3.3) \\ W(P(\xi, Y)Z, U)V &= \Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, Z)W(\xi, U)V - \eta(Z)W(Y, U)V] \\ &\quad + b[S(Y, Z)W(\xi, U)V - S(\xi, Z)W(Y, U)V], \qquad (3.4) \\ W(Z, P(\xi, Y)U)V &= \Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, U)W(Z, \xi)V - \eta(U)W(Z, Y)V] \\ &\quad + b[S(Y, U)W(Z, \xi)V - S(\xi, U)W(Z, Y)V] \qquad (3.5) \\ W(Z, U)P(\xi, Y)V &= \Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, V)W(Z, U)\xi - \eta(V)W(Z, U)Y] \\ &\quad + b[S(Y, V)W(Z, U)\xi - S(\xi, V)W(Z, U)Y]. \qquad (3.6) \end{aligned}$$

Now, putting (3.3)–(3.6) in (3.2) and taking the inner product with ξ, we have

$$\begin{aligned} &\Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, W(Z, U)V) - \eta(W(Z, U)V)\eta(Y) \\ &\quad - g(Y, Z)\eta(W(\xi, U)V) + \eta(Z)\eta(W(Y, U)V) - g(Y, U)\eta(W(Z, \xi)V) \\ &\quad + \eta(U)\eta(W(Z, Y)V) - g(Y, V)\eta(W(Z, U)\xi) + \eta(V)\eta(W(Z, U)Y)] \\ &\quad + b\big[S(Y, W(Z, U)V) - S(\xi, W(Z, U)V)\eta(Y) - S(Y, Z)\eta(W(\xi, U)V) \\ &\quad - S(Y, U)\eta(W(Z, \xi)V) - S(Y, V)\eta(W(Z, U)\xi) + S(\xi, Z)\eta(W(Y, U)V) \\ &\quad + S(\xi, U)\eta(W(Z, Y)V) + S(\xi, V)\eta(W(Z, U)Y)\big] = 0. \end{aligned} \tag{3.7}$$

Replacing again Z and V by ξ and using (2.2), equation (3.7) takes the form

$$\begin{aligned} &\Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, W(\xi, U)\xi) - \eta(W(\xi, U)\xi)\eta(Y) \\ &\quad - \eta(Y)\eta(W(\xi, U)\xi) + \eta(W(Y, U)\xi) - g(Y, U)\eta(W(\xi, \xi)\xi) + \eta(U)\eta(W(\xi, Y)\xi) \\ &\quad - \eta(Y)\eta(W(\xi, U)\xi) + \eta(W(\xi, U)Y)] + b\big[S(Y, W(\xi, U)\xi) - S(\xi, W(\xi, U)\xi)\eta(Y) \\ &\quad - S(Y, \xi)\eta(W(\xi, U)\xi) - S(Y, U)\eta(W(\xi, \xi)\xi) - S(Y, \xi)\eta(W(\xi, U)\xi) \\ &\quad + S(\xi, \xi)\eta(W(Y, U)\xi) + S(\xi, U)\eta(W(\xi, Y)\xi) + S(\xi, \xi)\eta(W(\xi, U)Y)\big] = 0. \end{aligned} \tag{3.8}$$

By making use of (2.8), equation (3.8) reduces to

$$\Big[a(f_1 - f_3) - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big][g(Y, W(\xi, U)\xi) + \eta(W(\xi, U)Y)] + b[S(Y, W(\xi, U)\xi) + S(\xi, \xi)\eta(W(\xi, U)Y)] = 0,$$

which in view of (2.4), (2.7), (2.9) and (2.10) gives

$$b\Big[\frac{3f_2 + (4n-1)f_3 - 2nf_1}{4n}S(Y, U) + n(f_1 - f_3)^2 g(Y, U) - \frac{(f_1 - f_3)(3f_2 + (2n-1)f_3)}{2}\eta(Y)\eta(U)\Big] = 0. \tag{3.9}$$

By simplifying, (3.9) takes the form

$$S(Y, U) = \frac{4n^2(f_1 - f_3)^2}{2nf_1 - (4n-1)f_3 - 3f_2}g(Y, U) - \frac{2n(f_1 - f_3)(3f_2 + (2n-1)f_3)}{2nf_1 - (4n-1)f_3 - 3f_2}\eta(Y)\eta(U), \quad \text{where } b \neq 0. \qquad \square$$

4 ϕ-M-projectively semisymmetric generalized Sasakian-space-forms

For $n > 1$, let M be a $(2n+1)$-dimensional ϕ-M-projectively semisymmetric generalized Sasakian-space-form. Therefore, $W(X, Y) \cdot \phi = 0$ turns into

$$(W(X, Y) \cdot \phi)Z = W(X, Y)\phi Z - \phi W(X, Y)Z = 0 \tag{4.1}$$

for any vector fields $X, Y, Z \in \chi(M)$. Now from (2.5) we have

$$W(X, Y)\phi Z = R(X, Y)\phi Z - \frac{1}{4n}[S(Y, \phi Z)X - S(X, \phi Z)Y + g(Y, \phi Z)QX - g(X, \phi Z)QY]. \tag{4.2}$$

In view of (1.1), (1.2), (1.3) and (2.2), equation (4.2) yields

$$\begin{aligned} W(X, Y)\phi Z = {} & [g(Y, \phi Z)X - g(X, \phi Z)Y]\Big(\frac{f_3 - 3f_2}{2n}\Big) + f_2[-g(X, Z)\phi Y + \eta(X)\eta(Z)\phi Y + g(Y, Z)\phi X \\ & - \eta(Y)\eta(Z)\phi X - 2g(X, \phi Y)Z + 2\eta(Z)g(X, \phi Y)\xi] \\ & + [g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi]\Big(\frac{2nf_3 - 3f_2 + f_3}{4n}\Big). \end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned} \phi W(X, Y)Z = {} & [g(Y, Z)\phi X - g(X, Z)\phi Y]\Big(\frac{f_3 - 3f_2}{2n}\Big) + f_2[g(Y, \phi Z)X - g(X, \phi Z)Y + g(X, \phi Z)\eta(Y)\xi \\ & - g(Y, \phi Z)\eta(X)\xi - 2g(X, \phi Y)Z + 2g(X, \phi Y)\eta(Z)\xi] \\ & + [\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z)\Big(\frac{2nf_3 - 3f_2 + f_3}{4n}\Big). \end{aligned} \tag{4.4}$$

From (4.1), (4.3) and (4.4) we have

$$\begin{aligned} & [g(Y, \phi Z)X - g(X, \phi Z)Y - g(Y, Z)\phi X + g(X, Z)\phi Y]\Big(\frac{f_3 - 3f_2 - 2nf_2}{2n}\Big) \\ & + [g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi]\Big(\frac{2nf_3 - 3f_2 + f_3 - 2nf_2}{2n}\Big) \\ & + [\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X]\Big(\frac{4nf_2 - 2nf_3 + 3f_2 - f_3}{4n}\Big) = 0. \end{aligned} \tag{4.5}$$

Putting $Y = \xi$ in (4.5), we obtain

$$f_3 g(X, \phi Z)\xi + \eta(Z)\phi X\Big(\frac{2nf_3 - f_3 + 3f_2}{4n}\Big) = 0.$$

Now, considering Z to be orthogonal to ξ, we have $\eta(Z) = 0$ and $g(X, \phi Z) \neq 0$, which implies that $f_3 = 0$ and therefore $f_2 = 0$. Thus we have the following proposition.

Proposition 4.1. *Let $n > 1$ and let M be a $(2n+1)$-dimensional ϕ-M-projectively semisymmetric generalized Sasakian-space-form. Then $f_2 = f_3 = 0$ holds.*

Suppose $f_3 = 0$. Therefore $W = 0$, and hence $W(X, Y) \cdot \phi = 0$. Thus in view of Proposition 4.1, we can state the following theorem.

Theorem 4.2. *For $n > 1$, a $(2n+1)$-dimensional generalized Sasakian-space-form M is ϕ-M projectively semisymmetric if and only if $f_2 = f_3 = 0$.*

5 ϕ-pseudo-projectively semisymmetric generalized Sasakian-space-forms

For $n > 1$, let M be a $(2n+1)$-dimensional ϕ-pseudo-projectively semisymmetric generalized Sasakian-space-form. Therefore, $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0 \tag{5.1}$$

for any vector fields $X, Y, Z \in \chi(M)$. Now from (2.6) we have

$$\begin{aligned}P(X,Y)\phi Z &= aR(X,Y)\phi Z + b[S(Y,\phi Z)X - S(X,\phi Z)Y]\\ &\quad - \frac{r}{(2n+1)}\Big(\frac{a}{2n+b}\Big)[g(Y,\phi Z)X - g(X,\phi Z)Y].\end{aligned} \tag{5.2}$$

In view of (1.1), (1.2) and (2.2), equation (5.2) yields

$$\begin{aligned}P(X,Y)\phi Z &= [g(Y,\phi Z)X - g(X,\phi Z)Y]\Big[af_1 + (2nf_1 + 3f_2 - f_3)b - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big]\\ &\quad + af_2[-g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y + g(Y,Z)\phi X\\ &\quad - \eta(Y)\eta(Z)\phi X - 2g(X,\phi Y)Z + 2\eta(Z)g(X,\phi Y)\xi]\\ &\quad + af_3[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi].\end{aligned} \tag{5.3}$$

Similarly,

$$\begin{aligned}\phi P(X,Y)Z &= [g(Y,Z)\phi X - g(X,Z)\phi Y]\Big[af_1 + (2nf_1 + 3f_2 - f_3)b - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big]\\ &\quad + af_2[g(Y,\phi Z)X - g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi\\ &\quad - g(Y,\phi Z)\eta(X)\xi - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi]\\ &\quad + [\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X][af_3 + (3f_2 + (2n-1)f_3)b].\end{aligned} \tag{5.4}$$

From (5.1), (5.3) and (5.4) we have

$$\begin{aligned}&[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y]\Big[af_1 + (2nf_1 + 3f_2 - f_3)b - \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big]\\ &\quad + af_2[g(Y,Z)\phi X - g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X] + a(f_3 - f_2)[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi]\\ &\quad - [a(f_3 - f_2) + b(3f_2 + (2n-1)f_3)][\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X] = 0.\end{aligned} \tag{5.5}$$

Putting $Y = \xi$ in (5.5), we obtain

$$\begin{aligned}&\Big[af_3 - af_1 - (2nf_1 + 3f_2 - f_3)b + \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big]g(X,\phi Z)\xi\\ &\quad + \Big[af_3 + (3f_2 + (2n-1)f_3)b - af_1 - (2nf_1 + 3f_2 - f_3)b + \frac{r}{(2n+1)}\Big(\frac{a}{2n} + b\Big)\Big]\eta(Z)\phi X = 0.\end{aligned}$$

Now, considering Z to be orthogonal to ξ, we obtain $\eta(Z) = 0$ and $g(X,\phi Z) \neq 0$, which implies that

$$f_3 = \frac{(a+2nb)}{(a+b)}f_1 + \frac{3b}{(a+b)}f_2 - \frac{r(a+2nb)}{2n(2n+1)(a+b)}. \tag{5.6}$$

Thus we have the following proposition.

Proposition 5.1. *For $n > 1$ and for a $(2n+1)$-dimensional ϕ-pseudo-projectively semisymmetric generalized Sasakian-space-form M, equation* (5.6) *holds.*

Suppose (5.6) holds. Therefore $P = 0$, and hence $P(X,Y)\cdot\phi = 0$. Thus in view of Proposition 5.1, we can state the following theorem.

Theorem 5.2. *For $n > 1$, a $(2n+1)$-dimensional generalized Sasakian-space-form is ϕ-pseudo-projectively semisymmetric if and only if* (5.6) *holds.*

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