

ON CONFORMALLY RECURRENT SPACES OF SECOND ORDER

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Introduction

In a recent paper [1] Adati and Miyazawa studied conformally recurrent spaces, that is, Riemannian spaces defined by $C_{ijk,l}^h = \lambda_l C_{ijk}^h$ where C_{ijk}^h is the conformal curvature tensor:

$$(1) \quad C_{ijk}^h = R_{ijk}^h - \frac{1}{n-2} (R_k^h g_{ij} - R_j^h g_{ik} + R_{ij} \delta_k^h - R_{ik} \delta_j^h) + \frac{R}{(n-1)(n-2)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}),$$

λ_l is a non-zero vector and comma denotes covariant differentiation with respect to the metric tensor g_{ij} . The present paper is concerned with non-flat Riemannian spaces $V_n (n > 3)$ defined by

$$(2) \quad C_{ijk,lm}^h = a_{lm} C_{ijk}^h$$

where a_{lm} is a tensor not identically zero. We shall call a Riemannian space defined by (2) a conformally recurrent space of second order and shall denote an n -space of this kind by $C(^2K_n)$. A Riemannian space whose curvature tensor satisfies $R_{ijk,lm}^h = \bar{a}_{lm} R_{ijk}^h$ was called a Recurrent space of second order by A. Lichnerowicz [2]. Such an n -space shall be denoted by 2K_n . Evidently every 2K_n is a $C(^2K_n)$ but the converse is not necessarily true. Sections 2 and 3 of this paper deal with Einstein and 2-Ricci-recurrent $C(^2K_n)$ respectively while section 4 deals with $C(^2K_n)$ admitting a parallel vector field. In the last section it will be shown that a Riemannian space satisfying $W_{ijk,lm}^h = a'_{lm} W_{ijk}^h$ where W_{ijk}^h is Weyl's projective curvature tensor is a $C(^2K_n)$.

1. Tensor of recurrence in a $C(^2K_n)$

We have

$$(C_{hijk} C^{hijk}),_{lm} = 2C_{hijk,lm} C^{hijk} + 2C^{hijk},_l C_{hijk,m}$$

Therefore, in a $C(^2K_n)$

$$(C_{hijk}C^{hijk})_{,lm} = 2a_{lm}C_{hijk}C^{hijk} + 2C^{hijk}_{,i}C_{hijk,m}$$

Hence

$$2(a_{lm} - a_{ml})C_{hijk}C^{hijk} = 0$$

So either

- (i) $C_{hijk}C^{hijk} = 0$ or
- (ii) $a_{lm} = a_{ml}$

If the space is of positive definite metric and not conformally flat, then (i) cannot hold and therefore a_{lm} is symmetric.

Again, if in a $C(^2K_n)$, $R_{ij} = 0$, then from (1) and (2) it follows that $R^h_{ijk,lm} = a_{lm}R^h_{ijk}$, that is, the space is a 2K_n . It is already known that for a 2K_n the tensor of recurrence is symmetric. Hence if for a $C(^2K_n)$, $R_{ij} = 0$, then its tensor of recurrence is symmetric. We can therefore state the following theorems:

THEOREM 1. *If a $C(^2K_n)$ with positive definite metric is not conformally flat, then its tensor of recurrence is symmetric.*

THEOREM 2. *If for a $C(^2K_n)$ the Ricci tensor is a zero tensor, then its tensor of recurrence is symmetric.*

2. Einstein $C(^2K_n)$

If a $C(^2K_n)$ is an Einstein space, defined by $R_{ij} = (R/n)g_{ij}$, then

$$(2.1) \quad R_{ij,lm} = 0$$

Let us suppose that an Einstein $C(^2K_n)$ is a 2K_n . Then $R^h_{ijk,lm} = d_{lm}R^h_{ijk}$ for a non-zero tensor d_{lm} . Consequently $R_{ij,lm} = d_{lm}R_{ij}$. Therefore in virtue of (2.1) $R_{ij} = 0$, because $d_{lm} \neq 0$. Hence $R = 0$.

Again, if in an Einstein $C(^2K_n)$, $R = 0$ then $R_{ij} = 0$ and therefore the space is a 2K_n . In an Einstein $C(^2K_n)$ of zero scalar curvature

$$(2.2) \quad R^h_{ijk,lm} = a_{lm}R^h_{ijk}$$

Making use of (2.2) and the Bianchi identity we get

$$(2.3) \quad a_{lm}R^h_{ijk} + a_{jm}R^h_{ikl} + a_{km}R^h_{ilt} = 0$$

Multiplying (2.3) by a^l_i where $a^l_i = g^{lp}a_{pi}$ we have

$$(2.4) \quad a^l_i a_{lm}R^h_{ijk} + a^l_i a_{jm}R^h_{ikl} + a^l_i a_{km}R^h_{ilt} = 0$$

$R_{ij} = 0$ implies $a_{hm}R^h_{ijk} = 0$ by contracting h and k in (2.3). Hence $a^l_i R^p_{kit} = 0$. Using this (2.4) reduces to $a^l_i a_{lm}R^h_{ijk} = 0$. Since the space is not flat, $a^l_i a_{lm} = 0$. Thus we have the following theorems:

THEOREM 3. *A necessary and sufficient condition that an Einstein $C(2K_n)$ may be a $2K_n$ is that its scalar curvature is zero.*

THEOREM 4. *In an Einstein $C(2K_n)$ of zero scalar curvature $a^l_i a_{lm} = 0$. We now consider an Einstein $C(2K_n)$ of non-zero scalar curvature. From (2.1) as well as (1) and (2) we have*

$$(2.5) \quad R_{hijk,lm} = a_{lm} T_{hijk}$$

where

$$T_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{hk}g_{ij} - g_{hj}g_{ik})$$

Using (2.5) Walker's Lemma 1 [3], namely

$$(2.6) \quad R_{hijk,lm} - R_{hijk,mi} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0$$

reduces to

$$(2.7) \quad b_{lm} T_{hijk} + b_{hi} T_{jklm} + b_{jk} T_{lmhi} = 0$$

where

$$(2.8) \quad b_{lm} = a_{lm} - a_{mi}.$$

Since $T_{hijk} = T_{jklm}$, by Walker's Lemma 2 [3] we have from (2.7) either $b_{lm} = 0$ or $T_{hijk} = 0$. Hence we have the following theorem:

THEOREM 5. *If a $C(2K_n)$ is an Einstein space of non-zero scalar curvature, then either its tensor of recurrence is symmetric or it is a space of constant curvature.*

3. 2-Ricci-recurrent $C(2K_n)$

In a previous paper [4] we called a non-flat Riemannian space a Ricci-recurrent space of second order, or briefly a 2-Ricci-recurrent space if its Ricci tensor satisfies

$$(3.1) \quad R_{ij,kl} = a^*_{kl} R_{ij}$$

and $R_{ij} \neq 0$ for a non-zero tensor a^*_{kl} .

We put

$$(3.2) \quad \Pi_{ij} = \frac{1}{n-2} \left[R_{ij} - \frac{R}{2(n-1)} g_{ij} \right]$$

and

$$(3.3) \quad D^h_{ijk} = \Pi^h_k g_{ij} - \Pi^h_j g_{ik} + \Pi_{ij} \delta^h_k - \Pi_{ik} \delta^h_j$$

where $\Pi^h_k = g^{ht} \Pi_{tk}$. Then

$$(3.4) \quad C^h_{ijk} = R^h_{ijk} - D^h_{ijk}$$

Moreover,

$$(3.5) \quad \Pi = g^{ij} \Pi_{ij} = \frac{R}{2(n-1)}$$

and

$$D_{hijk} = -D_{ihjk} = -D_{hikj} = D_{jkh i}$$

where $D_{hijk} = g_{ht} D^t_{ijk}$. Let us suppose that Π_{ij} is a non-zero tensor satisfying

$$(3.6) \quad \Pi_{ij,kl} = a^*_{kl} \Pi_{ij}$$

where a^*_{kl} is a non-zero tensor. Then $\Pi_{,kl} = a^*_{kl} \Pi$ or,

$$(3.7) \quad R_{,kl} = a^*_{kl} R$$

From (3.2) we have

$$\Pi_{ij,kl} = \frac{1}{n-2} \left[R_{ij,kl} - \frac{R_{,kl}}{2(n-1)} g_{ij} \right].$$

Therefore

$$\frac{1}{n-2} R_{ij,kl} = a^*_{kl} \left[\Pi_{ij} + \frac{R}{2(n-1)(n-2)} g_{ij} \right] = a^*_{kl} \frac{1}{n-2} R_{ij}$$

Hence

$$(3.8) \quad R_{ij,kl} = a^*_{kl} R_{ij}$$

Conversely, if (3.8) holds, then

$$\Pi_{ij,kl} = a^*_{kl} \Pi_{ij}$$

We can therefore state the following lemma:

LEMMA. *If in a Riemannian space, the tensor Π_{ij} , defined by (3.2), is a non-zero tensor, then the space is 2-Ricci-recurrent if and only if $\Pi_{ij,kl} = a^*_{kl} \Pi_{ij}$ for a non-zero tensor a^*_{kl} .*

We now suppose that in a $C(2K_n)$, (3.6) holds. Differentiating (3.3) covariantly we have

$$D^h_{ijk,lm} = \Pi^h_{k,lm} g_{ij} - \Pi^h_{j,lm} g_{ik} + \Pi_{ij,lm} \delta^h_k - \Pi_{ik,lm} \delta^h_j$$

Using (3.6) we get

$$(3.9) \quad D^h_{ijk,lm} = a^*_{lm} D^h_{ijk}.$$

From (3.4) we have

$$\begin{aligned} R^h_{ijk,lm} &= C^h_{ijk,lm} + D^h_{ijk,lm} \\ &= a_{lm} C^h_{ijk} + a^*_{lm} D^h_{ijk}. \end{aligned}$$

Therefore

$$(3.10) \quad R_{hijk,lm} - R_{hijk,ml} = b_{lm}C_{hijk} + c'_{lm}D_{hijk}$$

where b_{lm} is given by (2.8) and $c'_{lm} = a^*_{lm} - a^*_{ml}$. Now using (3.10) Walker's lemma (2.6) can be written as

$$(3.11) \quad \begin{aligned} b_{lm}C_{hijk} + b_{hi}C_{jklm} + b_{jk}C_{lmhi} + c'_{lm}D_{hijk} \\ + c'_{hi}D_{jklm} + c'_{jk}D_{lmhi} = 0. \end{aligned}$$

Let us suppose that a_{lm} is symmetric. Then $b_{lm} = 0$. Hence (3.11) reduces to

$$(3.12) \quad c'_{lm}D_{hijk} + c'_{hi}D_{jklm} + c'_{jk}D_{lmhi} = 0.$$

Since

$$C_{ij} = 0, \quad \Pi_{ij} = \frac{1}{n-2} \left[D_{ij} - \frac{D}{2(n-1)} g_{ij} \right].$$

Hence $\Pi_{ij} \neq 0$ implies $D_{ijkl} \neq 0$. Also $D_{hijk} = D_{jghi}$. Hence applying Walker's Lemma 2 to (3.12) we have $c'_{lm} = 0$. Hence a^*_{lm} is symmetric. Next, we suppose that a^*_{lm} is symmetric. Then $c'_{lm} = 0$ and it follows from (3.11) that

$$(3.13) \quad b_{lm}C_{hijk} + b_{hi}C_{jklm} + b_{jk}C_{lmhi} = 0.$$

Hence if $C_{hijk} \neq 0$, it follows from (3.13) that $b_{lm} = 0$ whence a_{lm} is symmetric.

We can therefore state the following theorems:

THEOREM 6. *If a $C(2K_n)$ which is not conformally flat is a 2-Ricci-recurrent space, then its tensor of recurrence is symmetric if and only if the tensor of 2-Ricci-recurrence is symmetric.*

THEOREM 7. *If a $C(2K_n)$ is a 2-Ricci-recurrent space, then the tensor of 2-Ricci-recurrence is symmetric if the tensor of recurrence of $C(2K_n)$ is so.*

4. $C(2K_n)$ admitting a parallel vector field

Let us assume that there exists a parallel vector field v^t in a $C(2K_n)$. Then $v^t_{,i} = 0$. Therefore $v^i_{,lm} - v^i_{,ml} = 0$. Hence using the Ricci identity and the Bianchi identity we have

$$\begin{aligned} v^t R^h_{ilm} &= 0, & v^t R_{tl} &= 0 \\ v^t R^h_{ilm,n} &= 0, & v^t R_{tl,n} &= 0. \end{aligned}$$

Therefore,

$$(4.1) \quad v^t R^h_{ijk,t} = 0, \quad v^t R_{ij,t} = 0, \quad v^t R_{,t} = 0.$$

From (2) we have $v^l C^h_{ijk,lm} = v^l a_{lm} C^h_{ijk}$ or,

$$(4.2) \quad v^l \left[R^h_{ijk,lm} - \frac{1}{n-2} (R^h_{k,lm} g_{ij} - R^h_{j,lm} g_{ik} + R_{ij,lm} \delta^h_k - R_{ik,lm} \delta^h_j) + \frac{1}{(n-1)(n-2)} R_{,lm} (\delta^h_k g_{ij} - \delta^h_j g_{ik}) \right] = v^l a_{lm} C^h_{ijk}.$$

Using (4.1) the left hand side of (4.2) reduces to zero. Hence $v^l a_{lm} C^h_{ijk} = 0$.

Thus we have the following theorem:

THEOREM 8. *If a $C(2K_n)$ admits a parallel vector field v^i , then either the space is conformally flat or $v^l a_{lm} = 0$.*

5. Projective recurrent spaces of second order

A. Riemannian space V_n ($n \geq 3$) satisfying

$$(5.1) \quad W^h_{ijk,lm} = a'_{lm} W^h_{ijk}$$

for a non-zero tensor a'_{lm} where W^h_{ijk} is Weyl's projective curvature tensor

$$(5.2) \quad W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1} (\delta^h_k R_{ij} - \delta^h_j R_{ik})$$

shall be called a projective recurrent space of second order.

From (5.2) we have

$$(5.3) \quad W^h_{ijk,lm} = R^h_{ijk,lm} - \frac{1}{n-1} (\delta^h_k R_{ij,lm} - \delta^h_j R_{ik,lm}).$$

Substituting (5.2) and (5.3) in (5.1) we get

$$(5.4) \quad R^h_{ijk,lm} - \frac{1}{n-1} (\delta^h_k R_{ij,lm} - \delta^h_j R_{ik,lm}) = a'_{lm} \left[R^h_{ijk} - \frac{1}{n-1} (\delta^h_k R_{ij} - \delta^h_j R_{ik}) \right].$$

Therefore,

$$(5.5) \quad R^h_{k,lm} = a'_{lm} R^h_k + \frac{1}{n} (R_{,lm} - a'_{lm} R) \delta^h_k$$

and

$$(5.6) \quad R_{ij,lm} = a'_{lm} R_{ij} + \frac{1}{n} (R_{,lm} - a'_{lm} R) g_{ij}.$$

Again from (1) we have

$$(5.7) \quad C_{ijk,lm}^h = R_{ijk,lm}^h - \frac{1}{n-2} (R_{k,lm}^h g_{ij} - R_{j,lm}^h g_{ik} + R_{ij,lm} \delta_k^h - R_{ik,lm} \delta_j^h) + \frac{1}{(n-1)(n-2)} R_{,lm} (\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

Making use of (5.4), (5.5) and (5.6) we have from (5.7)

$$C_{ijk,lm}^h = a'_{lm} C_{ijk}^h.$$

Hence we have the following theorem:

THEOREM 9. *Every n -dimensional ($n > 3$) projective recurrent space of second order is a $C(2K_n)$.*

References

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