ON CONFORMALLY-FLAT RIEMANNIAN SPACE OF CLASS ONE

R. N. SEN

1. The purpose of this paper is two-fold; first, to obtain necessary and sufficient conditions that a conformally-flat orientable Riemannian space C_n^1 with $n \geqq 3$ be of class one; second, to obtain a normal form for the metric of such a space. A Riemannian space V_n is a conformally-flat space C_n if there exists a scalar function σ such that the product σg_{ij} of σ and the fundamental tensor g_{ij} has zero curvature; it is of class one if it is isometrically embeddable as a hypersurface in a Euclidean space. The conformal flatness property can be expressed by the condition that $s_i = \frac{1}{2}\partial_i \log \sigma$ is related to the curvature tensor by

$$R_{hijk} + g_{hk}s_{ij} + g_{ij}s_{hk} - g_{hj}s_{ik} - g_{ik}s_{hj} = 0, \tag{1}$$

where

$$s_{ij} = \nabla_i s_j - s_i s_j + \tfrac{1}{2} g_{ij} s_k s^k. \tag{2}$$

The condition of class one, for an orientable space, implies the existence of a (second fundamental) symmetric tensor b_{ij} such that

$$R_{hijk} = b_{hj}b_{ik} - b_{hk}b_{ij}; \qquad \nabla_i b_{jk} - \nabla_j b_{ik}. \tag{3}$$

The converse is true in the local sense.

The algebraic relations (1), (3) lead to a result of J. A. Schouten [1] which states that $n-1$ of the eigenvalues of b_{ij} at each point of a C_n^1 are equal. Denote this value by ρ, the remaining eigenvalue by $\tilde{\rho}$ and denote by e_i the eigenvector of b_{ij} belonging to $\tilde{\rho}$. The quantities ρ, $\tilde{\rho}$ are also known as the principal normal curvatures and e_i the unit vector tangential to the line of curvature corresponding to $\tilde{\rho}$. Assume that $\tilde{\rho} \neq \rho \neq 0$. Then

$$b_{ij} = \rho g_{ij} + (\tilde{\rho} - \rho) e_i e_j \tag{4}$$

and by contraction of (3) we express the Ricci tensor in terms of g_{ij} and $e_i e_j$; or in g_{ij} and b_{ij}. We thus find (a), (b) below; by the second identity in (3) together with the property of conformal flatness we find (c) below.

$$b_{ij} = -\frac{1}{n-2}\left(\frac{1}{\rho} R_{ij} + \tilde{\rho} g_{ij}\right), \tag{a}$$

Received by the editors May 27, 1965.

$$\text{(b)} \qquad R_{hijk} = \rho^2(g_{hj}g_{ik} - g_{hk}g_{ij}) + \rho(\tilde{\rho} - \rho)(g_{hj}e_ie_k + g_{ik}e_he_j - g_{hk}e_ie_j - g_{ij}e_he_k),$$

$$\text{(c)} \qquad \partial_i\rho \text{ is proportional to } e_i.$$

These formulas are due to Verbickii [2]; he also showed that the existence of scalar functions ρ, $\tilde{\rho}$ and a unit vector field e_i such that (b), (c) hold is sufficient that V_n be locally a C_n^1.

2. The above results are most easily verified by choosing an orthonormal basis for the tangent space consisting of eigenvectors of b_{ij}; we choose e_i to be the first of these. Then g_{ij} and b_{ij} take diagonal forms with respect to this basis,

$$[g_{ij}] = \operatorname{diag}(1, 1, \cdots), \qquad [b_{ij}] = \operatorname{diag}(\tilde{\rho}, \rho, \rho, \cdots).$$

For brevity we only give the first two diagonal elements:

$$[g_{ij}] = \operatorname{diag}(1, 1); \quad [b_{ij}] = \operatorname{diag}(\tilde{\rho}, \rho); \quad [e_ie_j] = \operatorname{diag}(1, 0).$$

Then

$$[R_{ij}] = \operatorname{diag}(-(n-1)\rho\tilde{\rho}, \quad -\{(n-2)\rho^2 + \rho\tilde{\rho}\});$$

and among g_{ij}, b_{ij}, R_{ij}, e_ie_j any one can be written as a linear combination of any two. This is how (5) below is proved.

THEOREM 1. *If a V_n is a C_n^1, then there are scalars $E \neq 0$ and F such that*

$$\text{(5)} \qquad R_{hijk} = E(R_{hj}R_{ik} - R_{hk}R_{ij}) + F(g_{hj}g_{ik} - g_{hk}g_{ij}).$$

Conversely, if in a C_n scalars $E \neq 0$, F exist such that (5) *holds, where*

$$\text{(6)} \qquad R = -\frac{n-1}{(n-2)E} + (n-1)(n-2)F,$$

then C_n is a C_n^1.

PROOF OF THE CONVERSE. Contraction of (5) with g^{hk} gives

$$R_{ij} = ER_j^kR_{ik} - ERR_{ij} - (n-1)Fg_{ij}.$$

Hence, every eigenvalue λ of R_{ij} satisfies

$$\lambda = E\lambda^2 - ER\lambda - (n-1)F;$$

which, by (6), has as its solutions

$$\lambda = \frac{-1}{(n-2)E}, \quad \tilde{\lambda} = (n-1)(n-2)F.$$

By (6), λ has multiplicity $n-1$; $\tilde{\lambda}$ has multiplicity 1. The situation is now easily reduced to that of a C_n involving a second fundamental tensor b_{ij} which is a linear combination of g_{ij} and e_ie_j, where e_i is a unit eigenvector of R_{ij} associated with $\tilde{\lambda}$. It is a simple exercise to relate the λ, $\tilde{\lambda}$ above with ρ, $\tilde{\rho}$ resulting in

$$\lambda = -\left\{(n-2)\rho^2 + \rho\tilde{\rho}\right\}, \quad \tilde{\lambda} = -(n-1)\rho\tilde{\rho}.$$

We thus obtain

THEOREM 2. *If a V_n is a C_n^1, then*

$$(7) \qquad R_{hijk} = \frac{R_{hj}R_{ik} - R_{hk}R_{ij}}{(n-2)\left\{(n-2)\rho^2 + \rho\tilde{\rho}\right\}} - \frac{\rho\tilde{\rho}}{n-2}(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

where $\tilde{\rho}\neq\rho\neq 0$ are scalars. Conversely if a C_n satisfies (7), *then C_n is a C_n^1 if $R = -(n-1)\left\{(n-2)^2\rho^2 + 2\rho\tilde{\rho}\right\}$.*

3. Theorem 1 of §2 can be applied to find the metric of a C_n^1. This can be done by taking the fundamental tensor of a C_n in the form $g_{ii} = 1/\phi^2$, $g_{ij} = 0$, $(i\neq j)$, and looking for the general form of ϕ for which the equations (5) and (6) are satisfied. The fundamental tensor is then obtained in a canonical form as

$$(8) \qquad \begin{aligned} &g_{ii} = 1/[f(U)]^2, \quad g_{ij} = 0, \quad (i \neq j), \qquad \text{where } U = \sum_i (X^i)^2 + c \\ &\text{and } X^i = ax^i + b^i \text{ with } a \neq 0,\ b,\ c \text{ constants,} \end{aligned}$$

where f is any real analytical function of U subject to a restriction stated below. The normal form of the metric is now obtained by taking $a=1$, $b^i = c = 0$ in (8).

This metric and some properties which have been obtained in previous papers [3], [4] are stated in the following theorem:

THEOREM 3. *The coordinates of any C_n^1 may be so chosen that its metric assumes the normal form*

$$(9) \qquad ds^2 = \sum_i (dx^i)^2/[f(\theta)]^2, \qquad \theta = \sum_i (x^i)^2,$$

where f is any real analytic function of θ subject to the restriction

$$(n-1)ff' + \theta ff'' - (n-1)\theta f'^2 \neq 0, \qquad (f' = df/d\theta, \text{ etc.}).$$

If $\rho\neq 0$ and $\tilde{\rho}$ are the eigenvalues of multiplicity $n-1$ and 1 respectively of the second fundamental tensor of the space (9), then

$$(10) \qquad \rho^2 = 4f'(f - \theta f'), \quad \rho\tilde{\rho} = 4(ff' + \theta ff'' - \theta f'^2).$$

The eigenvector $e_i = x^i/\theta^{1/2}f$ corresponding to $\tilde{\rho}$ is orthogonal to the hypersurface having constant curvature $\bar{k}^2 = f^2/\theta$. If the C_n^1 is symmetric in the sense of Cartan, then either $f = a\theta + b$ (a space of constant curvature) or $f = c\theta^{1/2}$, where a, b, c are nonzero constants. In the second case e_i is a parallel vector field and the C_n^1 is reducible.

I am thankful to the referee for his helpful suggestion in the matter of presentation of the paper.

Department of Pure Mathematics, Calcutta University

References

1. J. A. Schouten, *Über die konforme Abbildung* n-*dimensionaler Mannigfaltigkeiten mit quadratischer etc.*, Math. Z. **11** (1921), 58–88.

2. L. L. Verbickii, *Geometry of conformal Euclidean spaces of class one*, Vol. 9, Transactions of the seminar of vector and tensor analysis (translated from Russian), 1952, pp. 146–182.

3. R. N. Sen, *On a type of Riemannian space conformal to a flat space*, J. Indian Math. Soc. **21** (1957), 105–114.

4. ———, *Conformally Euclidean space of class one*, Indian J. Math. **6** (1964), 93–103.