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On Conformally flat Almost Pseudo Ricci Symmetric Manifolds

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Dedicated to Professor Lajos Tamassy

Abstract. The object of the present paper is to study conformally flat almost pseudo Ricci symmetric manifolds. The existence of a conformally flat almost pseudo Ricci symmetric manifold with non-zero and non-constant scalar curvature is shown by a non-trivial example. We also show the existence of an n-dimensional non-conformally flat almost pseudo Ricci symmetric manifold with vanishing scalar curvature.

1. Introduction

The Einstein equations [12](p. 337), imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied [4] if the energy-momentum tensor is covariant-constant. In the paper [4] M. C. Chaki and Sarbari Ray had shown that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime. If however, $\nabla S \neq 0$, then such a spacetime may be called pseudo Ricci symmetric. We may say that the Ricci symmetric condition is only a special case of the pseudo Ricci symmetric condition. It is, therefore, meaningful to study the properties of pseudo Ricci symmetric spacetimes in general relativity.

In 1967, R. N. Sen and M. C. Chaki [15] studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expressions of the covariant derivative of Ricci tensor :

$$R_{ij,l} = 2\lambda_l R_{ij} + \lambda_i R_{lj} + \lambda_j R_{il} \ , \tag{1.1}$$

where λ_i is a non-zero covariant vector and ',' denotes covariant differentiation with respect to the metric tensor g_{ij} .

Later in 1988 M. C. Chaki [2] called a non-flat Riemannian manifold a pseudo

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Ricci symmetric manifold if its Ricci tensor satisfies (1.1). In index free notation this can be stated as follows:

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0,2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2G(X)S(Y,Z) + G(Y)S(X,Z) + G(Z)S(X,Y), \tag{1.2}$$

where ∇ denotes the Levi-Civita connection and G is a non-zero 1-form such that

$$g(X,\rho) = G(X), \tag{1.3}$$

for all vector fields X; ρ being the vector field corresponding to the associated 1-form G. If in (1.2) the 1-form $G = 0$, then the manifold reduces to Ricci symmetric manifold $(\nabla S = 0)$. This notion of pseudo Ricci symmetry is different from that of R. Deszcz [9].

Also in [13] S. Ray-Guha proved that a perfect fluid pseudo Ricci symmetric spacetime is a quasi Einstein manifold with each of its associated scalars equal to $r/3$ and a conformally flat perfect fluid pseudo Ricci symmetric spacetime obeying Einstein equation without cosmological constant and having the basic vector field of pseudo Ricci symmetric spacetime as the velocity vector field of the fluid is infinitesimally spatially isotropic relative to the velocity vector field.

In a recent paper [8] we have shown that a pseudo Ricci symmetric quasi Einstein perfect fluid spacetime represents the equation of state in the radiation era in the evolution of our universe.

So pseudo Ricci symmetric manifolds have some importance in the general theory of relativity. Considering this aspect M. C. Chaki and T. Kawaguchi [3] motivated to generalize pseudo Ricci symmetric manifold and introduced the notion of almost pseudo Ricci symmetric manifold.

A non-flat Riemannian manifold $(M^n, g), (n > 3)$, is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X) + B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \tag{1.4}$$

where A and B are two 1-forms and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In such a case A and B are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$.

If $B = A$, then the equation (1.4) reduces to (1.2), that is, $A(PRS)_n$ reduces to a pseudo Ricci symmetric manifold [2]. Thus pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$. In 1993 Tamassy and Binh [17] introduced the notion of weakly Ricci symmetric manifold which is the generalization of pseudo Ricci symmetric manifold in the sense of Chaki. It may be mentioned that an $A(PRS)_n$ is not a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [17]. In [3] Chaki and Kawaguchi proved that in a conformally flat $A(PRS)_n, (n > 3)$, the scalar curvature, r, can not be zero.

Let $g(X, P) = A(X)$ and $g(X, Q) = B(X)$, for all X. Then P, Q are called basic vector fields of the manifold corresponding to the associated 1-forms A and B, respectively.

In 1972 B. Y. Chen and K. Yano [5] introduced the notion of quasi-constant curvature as follows:

A non-flat Riemannian manifold $(M^n, g)(n > 3)$ is said to be quasi-constant curvature if its curvature tensor $\tilde{R}$ of type $(0, 4)$ satiesfies the condition

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + q[g(X,W)E(Y)E(Z) + g(Y,Z)E(X)E(W) \\ &\quad - g(X,Z)E(Y)E(W) - g(Y,W)E(X)E(Z)], \end{aligned} \tag{1.5}$$

where $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, R is the curvature tensor of type $(1, 3)$, p, q are scalar functions of which $q \neq 0$ and E is a non-zero 1-form defined by $g(X, \tilde{\xi}) = E(X)$ for all X, $\tilde{\xi}$ being a unit vector field. In such a case p and q were called associated scalars, E was called the associated 1–form and $\tilde{\xi}$ was called the generator of the manifold.

In 1956 S. S. Chern [7] studied a type of Riemannian manifold whose curvature tensor $\tilde{R}$ of type $(0, 4)$ satisfies the condition

$$\tilde{R}(X,Y,Z,W) = F(Y,Z)F(X,W) - F(X,Z)F(Y,W), \tag{1.6}$$

where F is a symmetric tensor of type $(0, 2)$. Such an n–dimensional manifold was called a special manifold with the associated symmetric tensor F and was denoted by $\psi(F)_n$.

Such a manifold is important for the following reasons:

Firstly, for possessing some remarkable properties relating to curvature and characterstic classes and secondly, for containing a manifold of quasi-constant curvature [5] as a subclass.

The paper is organised as follows:

After preliminaries in section 3 we first prove that a conformally flat almost pseudo Ricci symmetric manifold is a quasi Einstein manifold. Then we prove that in this manifold the vector field Q corresponding to the 1-form B is an eigen vector of the Ricci tensor S corresponding to the eigen value $t = \dfrac{S(Q,Q)}{\tilde{B}(Q)}$. We also show that such a manifold is a manifold of quasi-constant curvature and hence a subclass of $\psi(F)_n$. Next we prove that in a conformally flat almost pseudo Ricci symmetric manifold the vector field Q is a concircular vector field and a conformally flat almost pseudo Ricci symmetric manifold is a locally product manifold. We close this section by proving that a conformally flat almost pseudo Ricci symmetric manifold can be expressed as a warped product $IX_{e^q}M^*$ where M^* is an Einstein manifold. In section 4 we prove that every simply connected conformally flat $A(PRS)_n, (n > 3)$, satisfying $r > 2t$, can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface. In section 5 we prove the existence of a conformally flat almost pseudo

Ricci symmetric manifolds by constructing a non-trivial concrete example. Section 6 deals with an example of an n-dimensional non-conformally flat $A(PRS)_n$.

2. Preliminaries

Let (M^n, g) $(n > 3)$ be an almost pseudo Ricci symmetric manifold. Also let $g(LX, Y) = S(X, Y)$, for all X, Y. We take $A(LX) = \bar{A}(X)$ and $B(LX) = \bar{B}(X)$. Then $\bar{A}$ and $\bar{B}$ are called auxiliary 1-forms corresponding to the 1-forms A and B respectively. We get from (1.4) that

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y) = B(X)S(Y, Z) - B(Z)S(X, Y). \tag{2.1}$$

Now contracting (2.1) over Y, Z we get

$$dr(X) = 2rB(X) - 2\bar{B}(X), \tag{2.2}$$

where r is the scalar curvature. Next, contracting (1.4) over Y, Z we obtain

$$dr(X) = [A(X) + B(X)]r + 2\bar{A}(X). \tag{2.3}$$

3. Conformally flat $A(PRS)_n (n > 3)$

In this section we assume that the manifold $A(PRS)_n$ is conformally flat. Then $divC = 0$ where C denotes the Weyl's conformal cuvature tensor and $'div'$ denotes divergence. Hence we have [10]

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y) = \frac{1}{2(n-1)}[g(Y, Z)dr(X) - g(X, Y)dr(Z)]. \tag{3.1}$$

Using (2.1) and (2.2) in (3.1) we get

$$\begin{aligned} B(X)S(Y, Z) - B(Z)S(X, Y) &= \frac{r}{(n-1)}[B(X)g(Y, Z) - B(Z)g(X, Y)] \\ &\quad -\frac{1}{(n-1)}[\bar{B}(X)g(Y, Z) - \bar{B}(Z)g(X, Y)]. \end{aligned} \tag{3.2}$$

Now putting $Y = Q$ in (3.2) we get

$$B(X)\bar{B}(Z) - B(Z)\bar{B}(X) = 0. \tag{3.3}$$

Again putting $X = Q$ in (3.3) we have

$$\bar{B}(Z) = tB(Z), \tag{3.4}$$

where $t = \dfrac{\bar{B}(Q)}{B(Q)}$ is a scalar. So, from (3.4) and (2.2) we obtain

$$dr(X) = 2(r - t)B(X). \tag{3.5}$$

Since $B \neq 0$, putting $X = Q$ in (3.2) and using (3.4) we get

$$S(Y,Z) = ag(Y,Z) + bT(Y)T(Z), \tag{3.6}$$

where $a = \frac{r-t}{(n-1)}$, $b = \frac{nt-r}{(n-1)}$ are scalars and $T(X) = \frac{B(X)}{\sqrt{B(Q)}}$.

A Riemannian manifold is said to be a quasi-Einstein manifold if its Ricci tensor is of the form (3.6).

Hence we have the following theorem:

Theorem 3.1. *A conformally flat $A(PRS)_n$ is a quasi-Einstein manifold.*

Now from (3.6) we have

$$S(Y,Z) = \frac{r-t}{(n-1)}g(Y,Z) + \frac{nt-r}{(n-1)B(Q)}B(Y)B(Z). \tag{3.7}$$

Putting $Z = Q$ in (3.7) we get

$$S(Y,Q) = tB(Y) = tg(Y,Q). \tag{3.8}$$

Thus we can state the following:

Corollary 3.1. *The vector field Q corresponding to the 1-form B is an eigen vector of the Ricci tensor corresponding to the eigen value t.*

In a conformally flat Riemannian manifold the curvature tensor $\tilde{R}$ of type $(0,4)$ satiesfies the condition [10]

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \frac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\quad + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\ &\quad - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned} \tag{3.9}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the Riemannian curvature tensor of type $(1,3)$, and r is the scalar curvature. Now using (3.6) in (3.9) we get

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + q[g(X,W)T(Y)T(Z) + g(Y,Z)T(X)T(W) \\ &\quad - g(X,Z)T(Y)T(W) - g(Y,W)T(X)T(Z)],\end{aligned} \tag{3.10}$$

where $p = \frac{r-2t}{(n-1)(n-2)}$ and $q = \frac{nt-r}{(n-1)(n-2)}$. This implies that the manifold is of quasi-constant curvature. Thus we can state the following theorem:

Theorem 3.2. *A conformally flat $A(PRS)_n$ is a manifold of quasi-constant curvature.*

Now we shall prove that a $\psi(F)_n$ contains a manifold of quasi-constant curvature as a subclass.

For this let us choose

$$F(X,Y) = \sqrt{p}g(X,Y) + \frac{q}{\sqrt{p}}E(X)E(Y).$$

Then from (1.5) it follows that

$$\tilde{R}(X,Y,Z,W) = F(Y,Z)F(X,W) - F(X,Z)F(Y,W).$$

Hence a manifold of quasi-constant curvature is a $\psi(F)_n$. So we have the following:

Proposition 1. *A manifold of quasi-constant curvature is a* $\psi(F)_n$.

From this Proposition 1 and Theorem 3.2 we can conclude that

Corollary 3.2. *A conformally flat* $A(PRS)_n$ *is a* $\psi(F)_n$.

Now putting $Z = Q$ in (1.4) and using (3.6) and (3.8) we get

$$\begin{aligned}(\nabla_X S)(Y,Q) &= aA(Q)g(X,Y) + t[A(X)B(Y) + A(Y)B(X)] \\ &\quad +[t + b\frac{A(Q)}{B(Q)}]B(X)B(Y). \end{aligned} \tag{3.11}$$

Again,

$$(\nabla_X S)(Y,Q) = \nabla_X S(Y,Q) - S(\nabla_X Y,Q) - S(Y,\nabla_X Q). \tag{3.12}$$

Using (3.6), (3.8), $(\nabla_X B)(Y) = g(Y,\nabla_X Q)$ and $B(\nabla_X Q) = g(Q,\nabla_X Q) = \frac{1}{2}(XB(Q))$ we get from (3.12) that

$$(\nabla_X S)(Y,Q) = (Xt)B(Y) + (t-a)g(Y,\nabla_X Q) + \frac{b}{2B(Q)}(XB(Q))B(Y). \tag{3.13}$$

Now from (3.6) we get

$$LX = aX + \frac{b}{B(Q)}B(X)Q. \tag{3.14}$$

So by (3.14)

$$\bar{A}(X) = A(LX) = aA(X) + \frac{bA(Q)}{B(Q)}B(X). \tag{3.15}$$

With the help of (3.5) and (3.15) we get from (2.3) that

$$A(X) = \lambda B(X), \tag{3.16}$$

where

$$\lambda = \frac{r - 2t + 2b\frac{A(Q)}{B(Q)}}{r(1+2a)} \tag{3.17}$$

is a scalar. From (3.11), (3.13) and (3.16) we get

$$g(Y, \nabla_X Q) = \frac{aA(Q)}{(t-a)} g(X,Y)$$
$$+ \frac{tA(X) + [t(1+\lambda) + b\frac{A(Q)}{B(Q)}]B(X) - \frac{b}{2B(Q)}(XB(Q)) - (Xt)}{(t-a)} B(Y),$$

which implies that

$$\nabla_X Q = -fX + \omega(X)Q, \tag{3.18}$$

where

$$f = \frac{aA(Q)}{(a-t)} \tag{3.19}$$

and

$$\omega(X) = \frac{tA(X) + [t(1+\lambda) + b\frac{A(Q)}{B(Q)}]B(X) - \frac{b}{2B(Q)}(XB(Q)) - (Xt)}{(t-a)} \tag{3.20}$$

are a scalar function and a 1-form respectively. Hence Q is a concircular vector field [16], [18]. Thus we have the following theorem:

Theorem 3.3. *In a conformally flat $A(PRS)_n (n > 3)$, the vector field Q is a concircular vector field.*

Let $Q^\perp$ denote the $(n-1)$-dimensional distribution in a conformally flat $A(PRS)_n$ orthogonal to Q. If X and Y belong to $Q^\perp$, then

$$g(X, Q) = g(Y, Q) = 0. \tag{3.21}$$

Since $(\nabla_X g)(Y, Q) = 0$, it follows from (3.18) and (3.21) that

$$-g(\nabla_X Y, Q) = g(Y, \nabla_X Q) = -fg(X,Y).$$

Similarly, we get

$$-g(\nabla_Y X, Q) = g(X, \nabla_Y Q) = -fg(X,Y).$$

Hence

$$g(\nabla_X Y, Q) = g(\nabla_Y X, Q). \tag{3.22}$$

Now $[X,Y] = \nabla_X Y - \nabla_Y X$ and therefore, by (3.22) we obtain

$$g([X,Y],Q) = g(\nabla_X Y - \nabla_Y X, Q) = 0.$$

Hence $[X,Y]$ is orthogonal to Q. That is $[X,Y]$ belongs to $Q^{\perp}$. Thus the distribution $Q^{\perp}$ is involutive [1]. Hence from Frobenius' theorem [1] it follows that $Q^{\perp}$ is integrable. This implies that a conformally flat $A(PRS)_n, n > 3$, is a product manifold. We can therefore, state the following theorem:

Theorem 3.4. *A conformally flat $A(PRS)_n(n > 3)$, is a locally product manifold.*

K. Yano [19] proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta,$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are the functions of x^γ only $(\alpha, \beta, \gamma, \delta = 2, 3, \cdots, n)$ and $q = q(x^1) \neq \mathit{constant}$ is a function of x^1 only. Thus if an $A(PRS)_n, n > 3$, is conformally flat, that is, if it satiesfies (3.1), it is a warped product $IX_{e^q}M^*$, where (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold. A. Gebarowski [11] proved that warped product $IX_{e^q}M^*$ satiesfies (3.1) if and only if M^* is an Einstein manifold. Thus if $A(PRS)_n$ satiesfies (3.1), it must be a warped product $IX_{e^q}M^*$ where M^* is an Einstein manifold. Thus we can state the following result:

Theorem 3.5. *A conformally flat $A(PRS)_n(n > 3)$, can be expressed as a warped product $IX_{e^q}M^*$ where M^* is an Einstein manifold.*

4. Special conformally flat $A(PRS)_n(n > 3)$

The notion of a special conformally flat manifold which generalizes the notion of a subprojective manifold was introduced by Chen and Yano [6]. According to them a conformally flat manifold is said to be a special conformally flat manifold if the tensor H of type $(0,2)$ defined by

$$H(X,Y) = -\frac{1}{(n-2)}S(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y), \tag{4.1}$$

is expressible in the form

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X.\alpha)(Y.\alpha), \tag{4.2}$$

where α and β are two scalars such that α is positive. In virtue of (3.6) we can express (4.1) as

$$H(X,Y) = -\frac{r-2t}{2(n-1)(n-2)}g(X,Y) + \frac{r-nt}{(n-1)(n-2)B(Q)}B(X)B(Y). \tag{4.3}$$

We now put

$$\alpha^2 = \frac{r - 2t}{(n-1)(n-2)}. \tag{4.4}$$

We may assume that t is constant and then taking covariant differentiation to the both sides of (4.4) with respect to X and using (3.5) we get

$$\alpha(X.\alpha) = \frac{r - t}{(n-1)(n-2)} B(X) \tag{4.5}$$

Then the equation (4.3) can be expressed as

$$H(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta(X.\alpha)(Y.\alpha), \tag{4.6}$$

where $\beta = \dfrac{(r - nt)(r - 2t)}{(r - t)^2 B(Q)}$.

Since $r \neq 0$, α is not zero. Suppose $r > 2t$ then from (4.4) it follows that α may be taken as positive. From (4.6) we conclude that the $A(PRS)_n$ under consideration is a special conformally flat manifold.

It is known from a theorem of Chen's and Yano's paper [6] that every simply connected special conformally flat manifold can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface. we can therefore state the following result:

Theorem 4.1. *Every simply connected conformally flat* $A(PRS)_n (n > 3)$*, satisfying* $r > 2t$*, can be isometrically immersed in a Euclidean space* E^{n+1} *as a hypersurface.*

5. Existence of conformally flat almost pseudo Ricci symmetric manifolds

Let us consider a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij} dx^i dx^j = (x^4)^{\frac{4}{3}} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \tag{5.1}$$

$(i, j = 1, 2, 3, 4)$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{1/3},$$
$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor and their covariant derivatives are:

$$\begin{aligned} R_{11} &= -\frac{2}{9(x^4)^{2/3}}, & R_{22} &= -\frac{2}{9(x^4)^{2/3}}, \\ R_{33} &= -\frac{2}{9(x^4)^{2/3}}, & R_{44} &= -\frac{2}{3(x^4)^2}, \\ R_{11,4} &= \frac{4}{9(x^4)^{5/3}}, & R_{22,4} &= \frac{4}{9(x^4)^{5/3}}, \\ R_{33,4} &= \frac{4}{9(x^4)^{5/3}}, & R_{44,4} &= \frac{4}{3(x^4)^3}. \end{aligned}$$

It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is $R = -\frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant. We shall now show that $\mathbb{R}^4$ is conformally flat. For this we shall prove that

$$C_{1441} = C_{2442} = C_{3443} = 0,$$

as all other components of the conformal curvature tensor are zero automatically.

$$\begin{aligned} C_{1441} &= R_{1441} - \frac{1}{2}[g_{11}R_{44} + g_{44}R_{11} - 2g_{14}R_{14}] + \frac{R}{3 \times 2}[g_{11}g_{44} - (g_{14})^2] \\ &= -\frac{2}{9(x^4)^{2/3}} - \frac{1}{2}[-(x^4)^{4/3} \times \frac{2}{3(x^4)^2} - \frac{2}{9(x^4)^{2/3}}] - \frac{2}{9(x^4)^2} \times (x^4)^{4/3} \\ &= -\frac{2}{9(x^4)^{2/3}} + \frac{1}{2}(\frac{2}{3} + \frac{2}{9})\frac{1}{(x^4)^{2/3}} - \frac{2}{9(x^4)^{2/3}} \\ &= 0. \end{aligned}$$

By similar calculations it can be shown that $C_{2442} = C_{3443} = 0$. We shall now show that $\mathbb{R}^4$ is an $A(PRS)_n$. Let us choose the associated 1-forms as

$$A_i(x) = \begin{cases} -\frac{3}{x^4} & \text{for i=4} \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

$$B_i(x) = \begin{cases} \frac{1}{x^4} & \text{for i=4} \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

at any point $x \in \mathbb{R}^4$. Now, (1.4) reduces to the equations

$$R_{11,4} = (A_4 + B_4)R_{11} + 2A_1R_{14}, \tag{5.4}$$

$$R_{22,4} = (A_4 + B_4)R_{22} + 2A_2R_{24}, \tag{5.5}$$

$$R_{33,4} = (A_4 + B_4)R_{33} + 2A_3R_{34}, \tag{5.6}$$

$$R_{44,4} = (A_4 + B_4)R_{44} + 2A_4R_{44}, \tag{5.7}$$

since for the other cases (1.4) holds trivially. By (5.2) and (5.3) we get

$$\begin{aligned}\text{R.H.S. of (5.4)} &= (A_4+B_4)R_{11}+2A_1R_{14} = (-\frac{3}{x^4}+\frac{1}{x^4})(-\frac{2}{9(x^4)^{2/3}})\\ &= \frac{4}{9(x^4)^{5/3}} = R_{11,4}\\ &= \text{L.H.S. of (5.4).}\end{aligned}$$

By similar argument it can be shown that (5.5), (5.6) and (5.7) are true. So, $(\mathbb{R}^4, g)$ is a conformally flat $A(PRS)_n$ whose scalar curvature is non-zero and non-constant. It is to be noted that (1.4) can be satisfied by a number of 1-forms A, B namely by those which fulfil (5.4), (5.5), (5.6) and (5.7). Thus we can state the following :

Theorem 5.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2+(dx^2)^2+(dx^3)^2]+(dx^4)^2$$

$(i, j = 1, 2, 3, 4)$. Then (M^4, g) is a conformally flat $A(PRS)_n$ with non-zero and non-constant scalar curvature.

6. Example of an n-dimensional non-conformally flat $A(PRS)_n$

In this section we want to construct an example of an n-dimensional non-conformally flat almost pseudo Ricci symmetric manifold.

On coordinate space $\mathbb{R}^n$ (with coordinates $x^1, x^2, \cdots, x^n$) we define a Riemannian space V_n. We calculate the components of the curvature tensor, the Ricci tensor and of its covariant derivatives , the conformal curvature tensor and then we verify the relation (1.4).

Let each Latin index runs over $1, 2, \cdots, n$ and each Greek index over $2, 3, \cdots, (n-1)$. We define a Riemannian metric on the $\mathbb{R}^n (n \geq 4)$ by the formula

$$ds^2 = \phi(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1dx^n \ , \tag{6.1}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant and ϕ is a function of $x^1, x^2, \cdots, x^{n-1}$ and independent of x^n. In the metric considered, the only non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are [14]

$$\begin{gathered}\Gamma^\beta_{11} = -\frac{1}{2}K^{\alpha\beta}\phi_{.\alpha} \ , \quad \Gamma^n_{11} = \frac{1}{2}\phi_{.1} \ , \quad \Gamma^n_{1\alpha} = \frac{1}{2}\phi_{.\alpha} \ , \\ R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta} \ , \quad R_{11} = \frac{1}{2}K^{\alpha\beta}\phi_{.\alpha\beta} \ ,\end{gathered} \tag{6.2}$$

where '.' denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$.

Here we consider $K_{\alpha\beta}$ as Kronecker symbol $\delta_{\alpha\beta}$ and

$$\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta (x^1)^{2/3} ,$$

where $M_{\alpha\beta}$ are constant and satisfy the relations

$$\begin{aligned} M_{\alpha\beta} &= 0\,, \text{ for } \alpha \neq \beta, \\ &\neq 0\,, \text{ for } \alpha = \beta, \\ \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} &= 0\,. \end{aligned} \tag{6.3}$$

Thus we have the following relations:

$$\phi_{.\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta})(x^1)^{2/3},$$

$$\delta_{\alpha\beta}\delta^{\alpha\beta} = n-2 \quad \text{and} \quad \delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0\,.$$

Therefore,

$$\begin{aligned} \delta^{\alpha\beta}\phi_{.\alpha\beta} &= 2(\delta^{\alpha\beta}M_{\alpha\beta} + \delta^{\alpha\beta}\delta_{\alpha\beta})(x^1)^{2/3} \\ &= 2(n-2)(x^1)^{2/3}\,. \end{aligned}$$

Since $\phi_{.\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components for R_{hijk} and R_{ij} in virtue of (6.2) are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\phi_{.\alpha\alpha} = (1 + M_{\alpha\alpha})(x^1)^{2/3}$$

and

$$R_{11} = \frac{1}{2}\phi_{.\alpha\beta}\delta^{\alpha\beta} = (n-2)(x^1)^{2/3}\,.$$

Also the only non-zero component of covariant derivative of the Ricci tensor is

$$R_{11,1} = \frac{2(n-2)}{3(x^1)^{1/3}}. \tag{6.4}$$

Again from (6.1) we obtain $g_{ni} = g_{in} = 0$ for $i \neq 1$ which implies $g^{11} = 0$. Hence $R = g^{ij}R_{ij} = g^{11}R_{11} = 0$. Therefore, V_n will be a space whose scalar curvature is zero. Hence the only non-zero components of the conformal curvature tensor C_{hijk} are

$$\begin{aligned} C_{1\alpha\alpha 1} &= R_{1\alpha\alpha 1} - \frac{1}{n-2}(g_{\alpha\alpha}R_{11}) \\ &= (1 + M_{\alpha\alpha})(x^1)^{2/3} - \frac{1}{n-2}(n-2)(x^1)^{2/3} \\ &= M_{\alpha\alpha}(x^1)^{2/3} \end{aligned} \tag{6.5}$$

which never vanish. Hence V_n is not conformally flat . We shall now show that V_n is an $A(PRS)_n$. Let us consider the associated 1-form as follows:

$$A_i(x) = \begin{cases} \dfrac{1}{9(x^1)^2} , & \text{for i=1} \\ 0 , & \text{otherwise,} \end{cases} \tag{6.6}$$

$$B_i(x) = \begin{cases} \dfrac{2x^1-1}{3(x^1)^2} , & \text{for i=1} \\ 0 , & \text{otherwise,} \end{cases} \tag{6.7}$$

at any point $x \in V_n$.

To verify the relation (1.4) it is sufficient to prove the following:

$$R_{11,1} = (3A_1 + B_1)R_{11} , \tag{6.8}$$

as for the case other than (6.8) the components of each term of (1.4) vanish identically and the relation (1.4) holds trivially. Now from (6.4), (6.6) and (6.7) we get the following relation for the right hand side (R.H.S.) and the left hand side (L.H.S.) of (6.8)

$$\begin{aligned} R.H.S.\ of\ (6.8) &= (3A_1+B_1)R_{11} = \left(3 \times \frac{1}{9(x^1)^2} + \frac{2x^1-1}{3(x^1)^2}\right) R_{11} \\ &= \frac{2}{3x^1} \times (n-2)(x^1)^{2/3} = \frac{2(n-2)}{3(x^1)^{1/3}} \\ &= R_{11,1} = L.H.S.\ of\ (6.8) . \end{aligned}$$

It is to be noted that (1.4) can be satisfied by a number of 1-forms A, B, namely, by those which fulfil (6.8). Thus we can state the following:

Theorem 6.1. *Let $V_n (n \geq 4)$ be a Riemannian space with the metric of the form*

$$\begin{aligned} ds^2 &= \phi(dx^1)^2 + \delta_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1dx^n , \\ \phi &= (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta (x^1)^{2/3} , \end{aligned}$$

where $M_{\alpha\beta}$ are constant defined by (6.3)*, then V_n is an almost pseudo Ricci symmetric space with zero scalar curvature which is not conformally flat.*

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