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On characterizing radio k-coloring problem by path covering problem

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ABSTRACT

Let G be a finite simple graph. For an integer $k \geq 1$, a radio k-coloring of G is an assignment f of non-negative integers to the vertices of G satisfying the condition $|f(u) - f(v)| \geq k + 1 - d(u, v)$ for any two distinct vertices u, v of G. The span of f is the largest integer assigned to a vertex of G by f and radio k-chromatic number of G, denoted by $rc_k(G)$, is the minimum span over all radio k-colorings of G. For $k = 2$, the radio k-coloring becomes $L(2, 1)$ coloring problem. On the other hand, path covering problem deals with finding minimum number of vertex disjoint paths required to exhaust all the vertices of G. Georges et al. (1994) explored an elegant relation between $L(2, 1)$-coloring problem and path covering problem. As an extension of their work, we characterize the radio k-coloring problem for any $k \geq 2$ of a graph G by the path covering problem of G^c, where either G is triangle free or there is a Hamiltonian path in each component of G^c. As an application, for any such graph, if the exact value or an upper bound is known for any $rc_p(G)$, $p \geq 2$, we can get the exact value or an upper bound of $rc_k(G)$ for all $k \geq 2$. Determination of radio k-chromatic numbers of complete multi-partite graphs, a certain family of circulant graphs and join of circulant graphs of a certain family are among some other applications.

1. Introduction

Many graph coloring problems stem from a problem widely known as the frequency assignment problem (FAP) in communication network. In FAP, frequencies (non-negative integers) are assigned to the transmitters in a wireless network in an economic way. But as the proximity of transmitters increases, the mutual differences among the frequencies allotted to them should be greater to avoid interference. So the task is to minimize the span, i.e., the maximum frequency assigned, while satisfying the interference constraints. Hale [11] modelled this as a graph coloring problem. Roberts [24] proposed a variation of this problem taking a cue from which Griggs and Yeh [10] introduced $L(p_1, p_2, \ldots, p_m)$-coloring of a simple graph $G = (V, E)$ which is a function $f : V \to \mathbb{N}$ such that $|f(u) - f(v)| \geq p_i$ when $d(u, v) = i$, for $i = 1, 2, \ldots, m$, where $\mathbb{N}$ is the set of all non-negative integers. Interestingly, for a simple finite graph G and for any $k \geq 1$, if $p_i = k - i + 1,\ 1 \leq i \leq m$, the problem becomes the radio k-coloring problem introduced by Chartrand et al. [4,6] which finds motivation in FM channel assignments. In other words, if $\mathbb{N}$ is the set of all non-negative integers, then for any positive integer $k \geq 1$, a radio k-coloring f of a finite simple graph G is a mapping $f : V \to \mathbb{N}$ such that for any two vertices u, v in G,

$$|f(u) - f(v)| \geq k + 1 - d(u, v). \qquad (1)$$

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The span of a radio k-coloring f, denoted by *span(f)*, is $(\max_{v\in V} f(v) - \min_{v\in V} f(v))$ and the *radio k-chromatic number* of G, denoted by $rc_k(G)$ is defined as $\min_f\{span(f) : f \text{ is a radio } k\text{-coloring of } G\}$. Without loss of generality, we shall assume $\min_{v\in V} f(v) = 0$ for any radio k-coloring f on G. Any radio k-coloring f on G with span $rc_k(G)$ is referred as $rc_k(G)$-coloring or simply rc_k-coloring (when there is no confusion regarding the underlying graph).

So far radio k-coloring of graphs has been studied for $k \geq 2\cdot diam(G)-2$, $k = diam(G)$, $k = diam(G)-1$, $k = diam(G)-2$, $k = 3$, $k = 2$. For $k = diam(G)$, $k = diam(G) - 1$ and $k = diam(G) - 2$, the radio k-coloring is referred as *radio coloring*, *antipodal coloring* and *near-antipodal coloring* respectively while the corresponding radio k-chromatic numbers are known as *radio number*, *antipodal number* and *near-antipodal number* of G respectively. When $k = 2$, the problem reduces to the $L(2, 1)$-coloring problem introduced by Griggs and Yeh [10]. Note that $rc_2(G)$ is sometimes denoted as $\lambda_{2,1}(G)$ or $\lambda(G)$.

On the other hand, a *path covering* of a graph G is a set of vertex disjoint paths through all the vertices of G. A path covering of minimum cardinality of G is called a *minimum path covering* of G and its size is called the *path covering number* of G, denoted by $c(G)$. Finding a minimum path covering has applications in establishing ring protocols, codes optimization and mapping parallel programs to parallel architectures [1,2,22,23].

2. Previous works

So far, the radio k-chromatic numbers are known for very few families of graphs for specified values of k. Chartrand et al. [4,5] studied the radio numbers of paths and cycles while Liu and Zhu [21] obtained their exact values. The radio k-chromatic number of path P_n has been obtained in [21,13,15] for $k = n-1$, $n-2$, $n-3$ respectively. Kola and Panigrahi [16] have determined $rc_{n-4}(P_n)$ for an odd integer n. Liu generalized the results for paths to spider, i.e., trees with at most one vertex of degree greater than two, and obtained exact radio numbers in some specific cases [18]. Li et al. determined the radio number of a complete m-ary tree in [17]. Khennoufa et al. in [14] determined the radio number and the antipodal number of any hypercube by using generalized binary Gray codes. Moreover for hypercubes, upper bounds and lower bounds for radio k-chromatic numbers when $k \geq 2$ and their exact values when $k \geq 2\cdot$diameter-2 were obtained in [12]. Liu et al. in [19,20] studied radio numbers of squares of cycles and paths respectively. For powers of cycles i.e. C_n^r, Saha et al. obtained antipodal numbers for some values of n and r and bounds for the remaining cases [25].

It may be perceived that finding radio k-chromatic numbers, for any $k \geq 2$, even for paths, cycles and their powers is a challenging task. Since almost all graphs are asymmetric (i.e. its automorphism group is the identity group) [9], finding radio k-chromatic number for general graphs for any $k \geq 2$ is arguably a much more difficult job. Even in the literature, up to the best of our knowledge, there is no theoretical upper bound of $rc_k(G)$ for any graph G and for any $k \geq 3$. For any graph G, it was proved in [3] that $rc_2(G) \leq \Delta^2 + \Delta$ and conjectured in [10] that $rc_2(G) \leq \Delta^2$, Δ being the maximum degree in G.

3. Our contributions

In [8], Georges et al. investigated the relationship between $rc_2(G)$ and $c(G^c)$, where G^c is the complement of the graph G, and established the following beautiful results. In that paper, $rc_2(G)$ was denoted by $\lambda(G)$.

Theorem 3.1 (*[8]*)**.** *Let G be a graph with n vertices.*
(i) $\lambda(G) \leq (n-1)$ *if and only if* $c(G^c) = 1$.
(ii) *Let r be an integer,* $r \geq 2$*. Then* $\lambda(G) = n + r - 2$ *if and only if* $c(G^c) = r$.

Let $\mathcal{G}^*$ be the collection of all finite simple graphs. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two families of graphs defined by $\mathcal{G}_1 = \{G \in \mathcal{G}^* \mid G \text{ is triangle free}\}$ and $\mathcal{G}_2 = \{G \in \mathcal{G}^* \mid \text{each component in } G^c \text{ has a Hamiltonian path}\}$. In this paper we extend the above result for any $k \geq 2$ and for any graph which is either in $\mathcal{G}_1$ or in $\mathcal{G}_2$. In fact we have obtained an upper bound for $rc_k(G)$ in terms of $c(G^c)$ for any graph G, where G^c is the complement of the graph G. For a graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ and for $k \geq 2$, we show this upper bound to be a characterization for the existence of a Hamiltonian path in G^c and otherwise, i.e., if there is no Hamiltonian path in G^c, we then obtain a closed formula for $rc_k(G)$. Consequently for any graph G in $\mathcal{G}_1 \cup \mathcal{G}_2$, if the exact value or an upper bound is known for any $rc_p(G)$, $p \geq 2$, we can get the exact value or an upper bound of $rc_k(G)$ for all $k \geq 2$. Other applications include determining radio k-chromatic numbers of complete multi-partite graphs, certain family of circulant graphs and join of circulant graphs of a certain family.

4. Preliminaries

Throughout this paper, unless otherwise stated, graphs are taken as finite and simple with at least two vertices. Let L be a rc_k-coloring on a graph $G = (V, E)$. An integer $i \in \{0, 1, \ldots, rc_k(G)\}$ is a hole in L if i is not assigned to any vertex of G by L. Let $L_i^k(G) = \{v \in V \mid L(v) = i\}$ and $l_i^k(G) = |L_i^k(G)|$. We replace $L_i^k(G)$ by L_i and $l_i^k(G)$ by l_i if there is no confusion regarding G and k. For a fixed k, the vertices of L_i^k are represented by $v_j^{i(k)}$ (or v_j^i when there is no confusion regarding k), $1 \leq j \leq l_i$, and if $l_i = 1$, we replace $v_j^{i(k)}$ by $v^{i(k)}$ (or simply v_j^i by v^i if there is no confusion regarding k). In a rc_k-coloring L on G, a color i is referred as a multiple color if $l_i \geq 2$.

For definitions of Hamiltonian path, connectivity and independence number of a graph and disjoint union and join of two graphs, the reader is referred to [26]. Note that connectivity and independence number of a graph G and disjoint union and join of two graphs G, H are denoted by $\kappa(G)$, $\alpha(G)$, $G + H$ and $G \vee H$ respectively. The reader is further referred to [9] for definition of circulant graph of order n and connection set S which has been denoted here by $Cay(\mathbb{Z}_n, S)$. The complete graph, cycle and path with n vertices are denoted as K_n, C_n and P_n respectively.

We state the following two results without proof which will play important role in proving our main results.

Lemma 4.1 ([26]). *A graph G has a Hamiltonian path if and only if the graph $G \vee K_1$ has a Hamiltonian cycle.*

Theorem 4.1 ([7]). *If $\kappa(G) \geq \alpha(G)$, then G has a Hamiltonian cycle (unless $G = K_2$).*

5. Preparatory results

We develop some lemmas and theorems which are important to establish our main results. The following lemma gives the maximum number of consecutive holes in a rc_k-coloring on G.

Lemma 5.1. *Let L be a rc_k-coloring of a graph G. Then L cannot have k consecutive holes.*

Proof. If possible, let $i, i+1, i+2, \ldots, i+k-1$ be k consecutive holes. Define a new radio k-coloring $\hat{L}$ given by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } L(u) \leq i-1 \\ L(u)-1, & \text{if } L(u) \geq i+k. \end{cases}$$

Note that for any vertex $x \in A = \{u \in V(G) : \hat{L}(u) \leq i-1\}$ and any vertex $y \in B = \{u \in V(G) : \hat{L}(u) \geq i+k-1\}$, $\hat{L}(y) - \hat{L}(x) \geq k$. Also as L is a proper radio k-coloring, $\hat{L}$ is also a proper radio k-coloring with span $rc_k - 1$, a contradiction. □

Lemma 5.2. *Let L be a rc_k-coloring of a graph $G = (V, E)$, for $k \geq 2$. Let u, v be two distinct vertices of G such that $u \in L_i$ and $v \in L_j$. If $0 \leq |i-j| \leq k-1$, then u and v are adjacent in G^c and if $0 \leq |i-j| \leq k-2$, then for any vertex $w \in V \setminus \{u, v\}$, w is adjacent to either of u, v in G^c.*

Proof. Let $|L(u) - L(v)| = |i-j| \leq k-1$. Then $d(u, v) \geq 2$ in G. So u and v are adjacent in G^c.

Let $|L(u) - L(v)| = |i-j| \leq k-2$. Then $d(u, v) \geq 3$ in G. If possible let $\exists\, w \in V(G) \setminus \{u, v\}$ which is adjacent to neither of u and v in G^c. Then $d(u, v) = 2$ in G, a contradiction. □

Theorem 5.1. *Let G be a graph of order n and L be a rc_k-coloring of G ($k \geq 2$) with $rc_k(G) \leq (n-1)(k-1)$. Then G^c is connected.*

Proof. We consider two cases.

Case I: Let L have a multiple color, say, i. Then there are two vertices u, v ($u \neq v$) in L_i. Therefore, by Lemma 5.2, G^c is connected.

Case II: Let L have no multiple color. Let ρ_L be the number of holes in L. Then $rc_k = n + \rho_L - 1$. Therefore $n + \rho_L - 1 \leq (n-1)(k-1)$, i.e. $\rho_L \leq (n-1)(k-2)$.

If any two successive colors i, j have exactly $(k-2)$ successive holes between them then $|i-j| = k-1$, implying v^i and v^j are adjacent in G^c. Thus we get a path P in G^c given by $P : v^0, v^{k-1}, v^{2k-2}, \ldots, v^{(n-1)(k-1)}$. Hence G^c is connected.

Otherwise, there are successive colors i, j having at most $(k-3)$ holes between them i.e., $0 \leq |i-j| \leq k-2$. Then by Lemma 5.2, we have G^c is connected. □

Now we investigate the connectedness of G^c if $rc_k(G) > (n-1)(k-1)$ for $k \geq 2$.

Theorem 5.2. *Let G be a simple graph of order n and L be a rc_k-coloring of G ($k \geq 2$) with $rc_k(G) = (n-1)(k-1) + t,\ t \geq 1$. Then G^c has at most $t+1$ components.*

Proof. If possible, let G^c have m components with $m \geq t+2$ and let the components of G^c be $H_1, H_2, \ldots, H_m$. Clearly then $G = H_1^c \vee H_2^c \vee \cdots \vee H_m^c$.

Also, each vertex of H_i^c is adjacent to each vertex of H_j^c ($i \neq j$) in G. Note that any two vertices of G are at most two distance apart.

Therefore colors of any two vertices in H_i^c must differ by at least $(k-1)$, for every i. Also colors of any vertex of H_i^c and any vertex of H_j^c ($i \neq j$) must differ by at least k. Moreover all the vertices must receive distinct colors. Hence we should have $rc_k(G) \geq (n-1)(k-1) + t + 1$, a contradiction. □

6. General upper bound

The following theorem provides a general upper bound for $rc_k(G)$ in terms of the number of vertices, k and the path covering number of the complement graph, i.e., $c(G^c)$.

Theorem 6.1. *Let G be a graph with $c(G^c) = r$. Then for any $k \geq 2$, $rc_k(G) \leq n(k-1) + r - k$.*

Proof. Let $P_1, P_2, \ldots, P_r$ be the disjoint paths covering the vertices of G^c. Let p_i be the number of vertices of P_i. Also we denote the j-th vertex $(1 \le j \le p_i)$ of the i-th path $(1 \le i \le r)$ as x^i_j.

Note that any two consecutive vertices in a path P_i are not adjacent in G. Again, any end vertex of P_i is not adjacent to any end vertex of P_j in G^c, for $i \ne j$, and hence they are adjacent in G, because otherwise this would reduce the size of minimum path covering of G^c, a contradiction. Therefore colors of any two consecutive vertices in a path P_i should differ by at least $(k-1)$. Also, colors of end vertices of any two different covering paths should differ by at least k. Regarding these constraints, we define a radio k-coloring L of G as $L(x^i_j) = (\Sigma_{t=1}^{i-1} p_t - i + j)(k-1) + (i-1)k$.

Thus we obtain $span(L) = (n-r)(k-1) + (r-1)k = n(k-1) + r - k$. Hence for $k \ge 2,\ rc_k(G) \le n(k-1) + r - k$. □

Remark 6.1. We will see later (Corollaries 8.4 and 8.5) that for any $k \ge 2$, the upper bound is attained when G is a complete multi-partite graph.

7. Characterizing radio *k*-chromatic number by path covering number

7.1. For triangle free graph G

The following theorem establishes an upper bound for $rc_k(G)$ obtained in Theorem 6.1 as a necessary and sufficient condition for having a Hamiltonian path in G^c, if $G \in \mathcal{G}_1$.

Theorem 7.1. *Let G be a triangle-free graph with n vertices. Then for any $k \ge 2,\ rc_k(G) \le (n-1)(k-1)$ if and only if G^c has a Hamiltonian path.*

Proof. Let G^c have a Hamiltonian path. So by Theorem 6.1, $rc_k(G) \le (n-1)(k-1)$.

Conversely let $rc_k(G) \le (n-1)(k-1)$. Then by Theorem 5.1, G^c is connected. Therefore $\kappa(G^c \vee K_1) \ge 2$. Since G is triangle-free, therefore $\alpha(G^c) \le 2$ and so $\alpha(G^c \vee K_1) \le 2$. Therefore $\kappa(G^c \vee K_1) \ge \alpha(G^c \vee K_1)$. So $G^c \vee K_1$ has a Hamiltonian cycle, by Theorem 4.1 and hence by Lemma 4.1, G^c has a Hamiltonian path. □

The next theorem provides a closed formula for $rc_k(G)$ when G^c has no Hamiltonian path and $G \in \mathcal{G}_1$.

Theorem 7.2. *Let G be a triangle-free graph with n vertices. Then for any $k \ge 2,\ rc_k(G) = n(k-1) + r - k$ if and only if $c(G^c) = r$, when $r \ge 2$.*

Proof. We shall prove the result by induction on r. We first prove this for $r = 2$.

Let $c(G^c) = 2$. Since $G \in \mathcal{G}_1$ and $c(G^c) > 1$, by Theorem 7.1, $rc_k(G) \ge (n-1)(k-1) + 1$, i.e., $rc_k(G) \ge n(k-1) + 2 - k$. By Theorem 6.1, $rc_k(G) \le n(k-1) + 2 - k$. Hence $rc_k(G) = n(k-1) + 2 - k$.

Conversely let $rc_k(G) = n(k-1) + 2 - k$. Then $rc_k(G) > (n-1)(k-1)$. So, $c(G^c) \ge 2$, by Theorem 7.1. We construct a new graph H from G by introducing a new vertex x in such a way that $V(H) = V(G) \cup \{x\}$ and $E(H) = E(G)$.

Note that $rc_k(H) = rc_k(G) = n(k-1) + 2 - k \le n(k-1)$, since $k \ge 2$.

Since G is triangle-free, so is H. Also H is a graph with $n+1$ vertices. Therefore by Theorem 7.1, H^c has a Hamiltonian path and hence $c(G^c) \le 2$. Thus $c(G^c) = 2$.

Hence for any $k \ge 2,\ rc_k(G) = n(k-1) + 2 - k$ if and only if $c(G^c) = 2$.

We assume that for any triangle-free graph G and $2 \le t \le m-1$, $rc_k(G) = n(k-1) + t - k$ if and only if $c(G^c) = t$, when $k \ge 2$. We shall show that the hypothesis holds for $t = m$.

Let $c(G^c) = m$. Then $rc_k(G) \ge n(k-1) + m - k$, because otherwise we would have $rc_k(G) = n(k-1) + t - k$, for some t satisfying $2 \le t \le m-1$ which would imply $2 \le c(G^c) \le m-1$ by the induction hypothesis, a contradiction. By Theorem 6.1, $rc_k(G) \le n(k-1) + m - k$. Hence $rc_k(G) = n(k-1) + m - k$.

Conversely, let $rc_k(G) = n(k-1) + m - k$, for some triangle-free graph G, with $k \ge 2$. By the induction hypothesis, $c(G^c) \ge m$.

A new graph H is constructed from G by introducing $(m-1)$ new vertices $x_1, x_2, \ldots, x_{m-1}$ such that $V(H) = V(G) \cup \{x_1, x_2, \ldots, x_{m-1}\}$ and $E(H) = E(G)$. So $|V(H)| = n + m - 1$.

Now for $k \ge 2$ and $m \ge 3$,

$$\begin{aligned} rc_k(H) &= rc_k(G) = n(k-1) + m - k \\ &= (n+m-2)(k-1) + (m-1)(2-k) \\ &\le \{(n+m-1) - 1\}(k-1). \end{aligned}$$

So by Theorem 7.1, H^c has a Hamiltonian path. As H^c has $(m-1)$ more vertices than G which is an induced subgraph of H, $c(G^c) \le m$. Hence $c(G^c) = m$. □

7.2. For graph G with a Hamiltonian path in every component of G^c

We shall prove the results similar to the above subsection for graphs belonging to $\mathcal{G}_2$.

Theorem 7.3. *Let $G \in \mathcal{G}_2$ be a graph with n vertices. Then for any $k \geq 2$, $rc_k(G) \leq (n-1)(k-1)$ if and only if G^c has a Hamiltonian path.*

Proof. If G^c has a Hamiltonian path, by Theorem 6.1, we have $rc_k(G) \leq (n-1)(k-1)$.

Conversely, let $rc_k(G) \leq (n-1)(k-1)$. Then by Theorem 5.1, G^c is connected and hence has a Hamiltonian path as $G \in \mathcal{G}_2$. □

Remark 7.1. In Theorems 7.1 and 7.3, an upper bound for $rc_k(G)$ is proved to be a necessary and sufficient condition for the existence of a Hamiltonian path in G^c when $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. Now for any $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ with diameter 2 and a Hamiltonian path in G^c, this upper bound is attained for $k = 2$ (see Corollary 4.2 of [8]).

Theorem 7.4. *Let $G \in \mathcal{G}_2$ be a graph with n vertices. Then for any $k \geq 2$, $rc_k(G) = n(k-1) + r - k$ if and only if $c(G^c) = r$, when $r \geq 2$.*

Proof. We prove this theorem by induction on r.

Let $r = 2$. Then $rc_k(G) \leq (n-1)(k-1) + 1$ by using Theorem 6.1.

Note that if $rc_k(G) \leq (n-1)(k-1)$, then we would have $r = 1$, by Theorem 7.3, a contradiction. Therefore $rc_k(G) \geq (n-1)(k-1) + 1$. Hence $rc_k(G) = (n-1)(k-1) + 1$, i.e., $rc_k(G) = n(k-1) + 2 - k$.

Conversely, let $rc_k(G) = n(k-1) + 2 - k$. Then by Theorem 7.3, $c(G^c) \geq 2$. But by Theorem 5.2 and noting that $G \in \mathcal{G}_2$, $c(G^c) \leq 2$. Hence $c(G^c) = 2$.

Now assume that for $2 \leq r \leq m$, $rc_k(G) = n(k-1) + r - k$ if and only if $c(G^c) = r$.

Let $H \in \mathcal{G}_2$ be a graph with n vertices such that $c(H^c) = m + 1$. Then $rc_k(H) \leq n(k-1) + m + 1 - k$ by Theorem 6.1. Also $rc_k(H) \geq n(k-1) + m + 1 - k$, because otherwise by induction hypothesis, $c(H^c) \leq m$ as $H \in \mathcal{G}_2$, a contradiction. Hence $rc_k(H) = n(k-1) + m + 1 - k$.

Conversely, let $rc_k(H) = n(k-1)+m+1-k$. Then by Theorem 5.2 and noting the fact that $H \in \mathcal{G}_2$, we have $c(H^c) \leq m+1$. But by induction hypothesis, $c(H^c) < m+1$ would imply $rc_k(H) \leq n(k-1)+m-k$, a contradiction. Hence $c(H^c) = m+1$. □

8. Applications

Corollary 8.1. *Let $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ be a graph with n vertices. Then for any $p \geq q \geq 2$, $rc_p(G) \leq (n-1)(p-1)$ if and only if $rc_q(G) \leq (n-1)(q-1)$.*

Proof. By Theorems 7.1 and 7.3, $rc_p(G) \leq (n-1)(p-1)$ implies G^c has a Hamiltonian path and again by Theorems 7.1 and 7.3, $rc_q(G) \leq (n-1)(q-1)$. □

Corollary 8.2. *Let $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ be a graph with n vertices. Then for any $p \geq q \geq 2$, $rc_p(G) = n(p-1) + r - p$ if and only if $rc_q(G) = n(q-1) + r - q$, for $r \geq 2$.*

Proof. By Theorems 7.2 and 7.4, $rc_p(G) = n(p-1) + r - p$ implies $c(G^c) = r$ and again by Theorems 7.2 and 7.4, $rc_q(G) = n(q-1) + r - q$, for $r \geq 2$. □

Corollary 8.3. *If for any triangle free graph $G = (V, E)$ of order n, $rc_k(G) = (n-1)(k-1) - t$, for $k \geq 2$, and $n \geq \lfloor \frac{t}{k-1} \rfloor + 1$, then for every $\lfloor \frac{t}{k-1} \rfloor$-subset S of V, the subgraph of G^c induced by $V \setminus S$ has a Hamiltonian path.*

Proof. Let $t = q(k-1) + s$, where $0 \leq s \leq k-1$, i.e., $q = \lfloor \frac{t}{k-1} \rfloor$. Let S be any subset of V of size q.

Now $rc_k(G) = (n-1)(k-1) - t = (n-1)(k-1) - q(k-1) - s = (n-q-1)(k-1) - s$.

But $rc_k(G-S) \leq rc_k(G) = (n-q-1)(k-1) - s \leq (n-q-1)(k-1)$. Hence $(G-S)^c$ has a Hamiltonian path, by Theorem 7.1. This completes the proof. □

Corollary 8.4. *For any bipartite graph G of order n, $rc_k(G) \leq n(k-1) + 2 - k$ when $k \geq 2$. The upper bound is attained only when the graph is a complete bipartite.*

Proof. Every bipartite graph is triangle free. Note that if G be a non-complete bipartite graph with partite sets V and U, then G^c is $K_{|V|} + K_{|U|}$ with at least one additional edge which means G^c has a Hamiltonian path. Therefore by Theorem 7.1, $rc_k(G) \leq (n-1)(k-1)$. But if $G = V|U$ is a complete bipartite graph, then G^c is simply $K_{|V|} + K_{|U|}$ and hence $c(G^c) = 2$. Therefore by Theorem 7.2, $rc_k(G) = n(k-1) + 2 - k$. This completes the proof. □

Corollary 8.5. *Let $G = K_{m_1,m_2,\ldots,m_r}$ be a complete multi-partite graph of order n. Then $rc_k(G) = n(k-1) + r - k$, for $k \geq 2$.*

Proof. For any complete multi-partite graph $G = K_{m_1,m_2,\ldots,m_r}$, $G^c = K_{m_1} + K_{m_2} + \cdots + K_{m_r}$ and thus $G \in \mathcal{G}_2$. Hence the proof follows by Theorem 7.4. □

Next we determine the radio k-coloring number for a certain family of circulant graphs.

Corollary 8.6. *Let $G = Cay(\mathbb{Z}_n, S)$ where $S = \mathbb{Z}_n \setminus \{\overline{0}, \overline{m}, -\overline{m}\}$ such that $gcd(m, n) > 1$. Then for any $k \geq 2$, $rc_k(G) = n(k-1) + gcd(m, n) - k$, when $k \geq 2$.*

Proof. Clearly $G^c = Cay(\mathbb{Z}_n, \{\pm\overline{m}\})$. Since the order of $\overline{m}$ is $\frac{n}{gcd(m,n)}$ in the group $(\mathbb{Z}_n, +)$, therefore $\mathbb{Z}_n \neq \langle\overline{m}\rangle$. In fact the cyclic subgroup $\langle\overline{m}\rangle$ has $gcd(m, n)$ number of cosets in $\mathbb{Z}_n$. Note that the components of G^c are exactly the subgraphs induced by the cosets of $\langle\overline{m}\rangle$ in $\mathbb{Z}_n$. If $a + \langle\overline{m}\rangle$ is a coset of $\langle\overline{m}\rangle$ in $\mathbb{Z}_n$, then $P : a, a + \overline{m}, a + 2\overline{m}, \ldots, a + t\overline{m}$, where $t = \frac{n}{gcd(m,n)} - 1$, is a path exhausting all the vertices of the component of G^c corresponding to the coset. Hence $G \in \mathcal{G}_2$ and $c(G^c) = gcd(m, n)$. The proof follows from Theorem 7.4. □

We now construct a family of graphs. Let $G = Cay(\mathbb{Z}_m, S)$ such that $\exists\, \bar{i} \in \mathbb{Z}_m$ with $gcd(i, m) = 1$ and $\bar{i} \notin S$. Let $\mathcal{C}$ be the collection of all such circulant graphs for all $m \geq 5$. Let $\mathcal{P}_i = \{G_1, G_2, \ldots, G_i\} \subseteq \mathcal{C},\ i \geq 1$ and let $H(\mathcal{P}_i) = G_1 \vee G_2 \vee \cdots \vee G_i$ be a graph with n vertices.

Corollary 8.7. (i) $rc_k(H(\mathcal{P}_i)) \leq (n-1)(k-1)$, *if* $i = 1$, *for* $k \geq 2$.
(ii) $rc_k(H(\mathcal{P}_i)) = n(k-1) + i - k$, *if* $i \geq 2$, *for* $k \geq 2$.

Proof. For any graph $G \in \mathcal{C}$, $G^c = Cay(\mathbb{Z}_m, \overline{S})$, where $\exists\, \bar{i} \in \overline{S}$ such that $gcd(i, m) = 1$. Therefore $\mathbb{Z}_m = \langle\overline{S}\rangle$ and hence G^c has a Hamiltonian path. For $i = 1$, $H(\mathcal{P}_i) = G$ for some $G \in \mathcal{C}$. Again, for $i \geq 2$, $H(\mathcal{P}_i)^c = G_1^c + G_2^c + \cdots + G_i^c$, where $\{G_1, G_2, \ldots, G_i\} \subseteq \mathcal{C}$. Hence $H(\mathcal{P}_i) \in \mathcal{G}_2$ and the proof follows by Theorems 7.3 and 7.4. □

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