

ON A TYPE OF SASAKIAN SPACE

M. C. CHAKI and D. GHOSH

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Introduction

A Sasakian space [1] M^n ($n = 2m + 1$) is a Riemannian n -space with a positive definite metric tensor g_{ij} and a unit Killing vector field η which satisfies

$$(1) \quad \eta_{j,kl} = \eta_k g_{lj} - \eta_j g_{lk}$$

where the comma denotes covariant differentiation with respect to the metric tensor. In a recent paper [2] M. C. Chaki and A. N. Roy Chowdhury studied conformally recurrent spaces of second order, or briefly conformally 2-recurrent spaces, that is, non-flat Riemannian spaces V_n ($n > 3$) defined by

$$(2) \quad C_{kji}{}^h{}_{,lm} = a_{lm} C_{kji}{}^h$$

where $C_{kji}{}^h$ is the conformal curvature tensor:

$$(3) \quad C_{kji}{}^h = R_{kji}{}^h - \frac{1}{n-2} (g_{ji} R_k{}^h - g_{ki} R_j{}^h + R_{ji} \delta_k{}^h - R_{ki} \delta_j{}^h) \\ + \frac{R}{(n-1)(n-2)} (g_{ji} \delta_k{}^h - g_{ki} \delta_j{}^h)$$

and a_{lm} is a tensor not identically zero.

The present paper deals with conformally 2-recurrent Sasakian spaces, that is, Sasakian spaces in which (2) is satisfied. It is proved that such an n -space ($n > 3$) is of constant curvature.

1. Some formulas in a Sasakian Space

Since in a Sasakian space η is a unit vector field

$$(1.1) \quad \eta^r \eta_r = 1$$

Applying Ricci's identity to η_j we obtain

$$(1.2) \quad \eta_{j,kl} - \eta_{j,lk} = -\eta_r R_{lkj}{}^r$$

where R_{kji}^h is the Riemannian curvature tensor:

$$R_{kji}^h = \frac{\partial}{\partial x^k} \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - \frac{\partial}{\partial x^j} \left\{ \begin{matrix} h \\ ki \end{matrix} \right\} + \left\{ \begin{matrix} h \\ kr \end{matrix} \right\} \left\{ \begin{matrix} r \\ ji \end{matrix} \right\} - \left\{ \begin{matrix} h \\ jr \end{matrix} \right\} \left\{ \begin{matrix} r \\ ki \end{matrix} \right\}$$

Using (1) we can express (1.2) as

$$(1.3) \quad \eta_r R_{lkj}^r = \eta_l g_{jk} - \eta_k g_{jl}$$

Contracting (1.3) with g^{jl} we have

$$(1.4) \quad \eta_r R_k^r = (n-1)\eta_k$$

Thus in a Sasakian space the formulas (1.1), (1.3) and (1.4) hold.

2. Conformally 2-recurrent Sasakian space

Let us suppose that a Sasakian space is conformally 2-recurrent. If possible, let a conformally 2-recurrent Sasakian space be not conformally flat. It has been proved in theorem 1 of [2] that if a conformally 2-recurrent space with positive definite metric is not conformally flat, then its tensor of recurrence a_{lm} is symmetric. Hence if a conformally 2-recurrent Sasakian space be not conformally flat, then

$$(2.1) \quad C_{kji}^h{}_{,lm} - C_{kji}^h{}_{,ml} = 0$$

Applying Ricci's identity to the left-hand side of (2.1) we get

$$C_{kji}^r R_{mlr}^h - C_{kjr}^h R_{mli}^r - C_{kri}^h R_{mlj}^r - C_{rji}^h R_{mlk}^r = 0$$

Transvecting this with $\eta_h \eta^m$ and using (1.1) and (1.3) we have

$$(2.2) \quad C_{kji}^r - \eta_r \eta_l C_{kji}^r + \eta_r \eta_i C_{kjl}^r + \eta_j \eta_r C_{kli}^r + \eta_r \eta_k C_{lji}^r + g_{lj} \eta^r \eta_s C_{rki}^s - g_{lk} \eta^r \eta_s C_{rji}^s = 0$$

Contracting this with g^{lj} we get

$$(2.3) \quad \eta^r \eta_s C_{rki}^s = 0$$

Substituting this value in (2.2) we have

$$(2.4) \quad C_{kji}^r - \eta_r \eta_l C_{kji}^r + \eta_r \eta_i C_{kjl}^r + \eta_j \eta_r C_{kli}^r + \eta_k \eta_r C_{lji}^r = 0$$

Now,

$$(2.5) \quad C_{kji}^r \eta_r = \frac{1}{n-2} \left\{ \left(\frac{R}{n-1} - 1 \right) \eta_k g_{ji} - \eta_j g_{ki} \right\} - (\eta_k R_{ji} - \eta_j R_{ki})$$

Using (2.5) we can express (2.4) as

$$(2.6) \quad C_{kji} + \frac{1}{n-2} \left[\left(\frac{R}{n-1} - 1 \right) (\eta_k g_{ji} - \eta_j g_{ki}) - (\eta_k R_{ji} - \eta_j R_{ki}) \right] \eta_i = 0$$

Contracting this with g^{ik} we get

$$R_{ij} = \left(\frac{R}{n-1} - 1 \right) g_{ij} + \left(n - \frac{R}{n-1} \right) \eta_i \eta_j$$

Substituting this value in (2.6) we have $C_{kji} = 0$. But this is contrary to hypothesis.

Hence if a Sasakian space is conformally 2-recurrent, then it is conformally flat. It has been proved by Okumura [3] that a conformally flat Sasakian space is of constant curvature. We can therefore state the following theorem.

THEOREM. *A conformally 2-recurrent Sasakian space is of constant curvature.*

In theorem 9 of [2], it has been proved that every n -dimensional ($n > 3$) projective 2-recurrent space, that is a Riemannian space in which Weyl's projective curvature tensor:

$$W_{kji}{}^h = R_{kji}{}^h - \frac{1}{n-1} (\delta_k{}^h R_{ji} - \delta_j{}^h R_{ki})$$

satisfies the relation $W_{kji}{}^h,{}_{lm} = a'_{lm} W_{kji}{}^h$ for a non-zero tensor a'_{lm} , is a conformally 2-recurrent space. We have therefore the following corollary of the above theorem:

COROLLARY. *A projective 2-recurrent Sasakian space is of constant curvature.*

References

- [1] S. Sasaki, *Lecture note on almost contact manifolds* Tohoku University (1965).
- [2] M. C. Chaki and A. N. Roy Chowdhury, 'On conformally recurrent Spaces of Second order', *Journ. Australian Math. Soc.* 10 (1969), 155—161.
- [3] M. Okumura, 'Some remarks on space with a certain contact structure', *Tohoku Math. J.* 14 (1962), 135—145.

Department of Pure Mathematics
Calcutta University