

## On a type of almost Kenmotsu manifolds with nullity distributions

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**Abstract.** The object of the present paper is to characterize Weyl semisymmetric almost Kenmotsu manifolds with its characteristic vector field  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution and  $(k, \mu)$ -nullity distribution respectively. Also we characterize almost Kenmotsu manifolds satisfying the curvature condition  $C \cdot S = 0$ , where  $C$  and  $S$  are the Weyl conformal curvature tensor and Ricci tensor respectively with its characteristic vector field  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution. As a consequence of the main results we obtain several corollaries. Finally, we present an example to verify our results.

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### 1. INTRODUCTION

In the present time the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of  $k$ -nullity distribution ( $k \in \mathbb{R}$ ) was introduced by Gray [7] and Tanno [11] in the study of Riemannian manifolds  $(M, g)$ , which is defined for any  $p \in M$  and  $k \in \mathbb{R}$  as follows:

$$N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \quad (1.1)$$

for any  $X, Y \in T_pM$ , where  $T_pM$  denotes the tangent vector space of  $M$  at any point  $p \in M$  and  $R$  denotes the Riemannian curvature tensor of type  $(1, 3)$ .

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Recently Blair, Koufogiorgos and Papantoniou [3] introduced a generalized notion of the  $k$ -nullity distribution named the  $(k, \mu)$ -nullity distribution on a contact metric manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$ , which is defined for any  $p \in M^{2n+1}$  and  $k, \mu \in \mathbb{R}$  as follows:

$$N_p(k, \mu) = \{Z \in T_p M^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \quad (1.2)$$

where  $h = \frac{1}{2}\mathcal{L}_\xi \phi$  and  $\mathcal{L}$  denotes the Lie differentiation.

In [4], Dileo and Pastore introduced the notion of  $(k, \mu)'$ -nullity distribution, another generalized notion of the  $k$ -nullity distribution, on an almost Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$ , which is defined for any  $p \in M^{2n+1}$  and  $k, \mu \in \mathbb{R}$  as follows:

$$N_p(k, \mu)' = \{Z \in T_p M^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)h'X - g(X, Z)h'Y]\}, \quad (1.3)$$

where  $h' = h \circ \phi$ .

On the other hand, Kenmotsu [9] introduced a new type of contact metric manifolds named Kenmotsu manifolds nowadays. Let us consider  $M^{2n+1}$  be an almost contact metric manifold with almost contact structure  $(\phi, \xi, \eta, g)$  given by a  $(1, 1)$  tensor field  $\phi$ , a characteristic vector field  $\xi$ , a 1-form  $\eta$  and a compatible metric  $g$  satisfying the conditions [1,2]

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, & \phi(\xi) &= 0, & \eta(\xi) &= 1, & \eta \circ \phi &= 0, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for any vector fields  $X$  and  $Y$  of  $T_p M^{2n+1}$ . The fundamental 2-form  $\Phi$  is defined by  $\Phi(X, Y) = g(X, \phi Y)$  for any vector fields  $X$  and  $Y$  of  $T_p M^{2n+1}$ . The condition for an almost contact metric manifold being normal is equivalent to vanishing of the  $(1, 2)$ -type torsion tensor  $N_\phi$ , defined by  $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$ , where  $[\phi, \phi]$  is the Nijenhuis torsion of  $\phi$  [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold such that  $d\eta = 0$  and  $d\Phi = 2\eta \wedge \Phi$ . Also Kenmotsu manifolds can be characterized by  $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ , for any vector fields  $X, Y$ . It is well known [9] that a Kenmotsu manifold  $M^{2n+1}$  is locally a warped product  $I \times_f N^{2n}$  where  $N^{2n}$  is a Kähler manifold,  $I$  is an open interval with coordinate  $t$  and the warping function  $f$ , defined by  $f = ce^t$  for some positive constant  $c$ . Let us denote the distribution orthogonal to  $\xi$  by  $\mathcal{D}$  and defined by  $\mathcal{D} = Ker(\eta) = Im(\phi)$ . In an almost Kenmotsu manifold, since  $\eta$  is closed,  $\mathcal{D}$  is an integrable distribution.

A Riemannian manifold  $(M^{2n+1}, g)$  is called locally symmetric if its curvature tensor  $R$  is parallel, that is,  $\nabla R = 0$ , where  $\nabla$  is the Levi-Civita connection. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by  $R(X, Y) \cdot R = 0$ , where  $R(X, Y)$  acts on  $R$  as a derivation. A complete intrinsic classification of these manifolds was given by Szabó in [10]. A Riemannian manifold is said to be Weyl semisymmetric if the Weyl conformal curvature tensor  $C$  satisfies  $R \cdot C = 0$ . In a recent paper [8] Jun, De and Pathak studied Weyl semisymmetric Kenmotsu manifolds.

In [5], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. Moreover almost Kenmotsu manifolds satisfying some nullity conditions were also investigated by Dileo and Pastore [4]. We refer the reader to [5,4,6] for more related results on  $(k, \mu)'$ -nullity distribution and  $(k, \mu)$ -nullity distribution on almost Kenmotsu manifolds. In recent papers [12–15] Wang and Liu study almost Kenmotsu manifolds with nullity distributions.

In [13], Wang and Liu study  $\xi$ -Riemannian semisymmetric almost Kenmotsu manifolds with  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution and  $(k, \mu)$ -nullity distribution.

The paper is organized as follows:

In Section 2, we give a brief account on almost Kenmotsu manifolds with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution and  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution. Section 3 deals with Weyl semisymmetric almost Kenmotsu manifolds and almost Kenmotsu manifolds satisfying the curvature condition  $C \cdot S = 0$  with characteristic vector field  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution. Section 4 is devoted to study conformally flat almost Kenmotsu manifolds and Weyl semisymmetric almost Kenmotsu manifolds with characteristic vector field  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution. As a consequence of the main results we obtain several corollaries. In the final section, we present an example to verify our results.

### 2. ALMOST KENMOTSU MANIFOLDS

Let  $M^{2n+1}$  be an almost Kenmotsu manifold. We denote  $h = \frac{1}{2}\mathcal{L}_\xi \phi$  and  $l = R(\cdot, \xi)\xi$  on  $M^{2n+1}$ . The two  $(1, 1)$ -type tensors  $l$  and  $h$  are symmetric and satisfy [4]

$$h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0. \tag{2.1}$$

Besides the above we have the following results [4]

$$\nabla_X \xi = X - \eta(X)\xi - \phi hX (\Rightarrow \nabla_\xi \xi = 0), \tag{2.2}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.3}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.4}$$

for any vector fields  $X, Y$ . The  $(1, 1)$ -type symmetric tensor field  $h' = h \circ \phi$  is anticommuting with  $\phi$  and  $h'\xi = 0$ . Also it is clear that

$$h = 0 \iff h' = 0, \quad h'^2 = (k + 1)\phi^2 (\iff h^2 = (k + 1)\phi^2). \tag{2.5}$$

### 3. $\xi$ BELONGS TO THE $(k, \mu)'$ -NULLITY DISTRIBUTION

In this section we consider an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution. Let  $X \in \mathcal{D}$  be the eigenvector of  $h'$  corresponding to the eigenvalue  $\lambda$ . Then from (2.5) it is clear that  $\lambda^2 = -(k + 1)$ , a constant. Hence  $k \leq -1$  and  $\lambda = \pm\sqrt{-k - 1}$ . We denote the eigenspaces associated with  $h'$  by  $[\lambda]'$  and  $[-\lambda]'$  corresponding to the non-zero eigenvalues  $\lambda$  and  $-\lambda$  of  $h'$  respectively. To prove our main theorem in this section we recall some results:

**Lemma 3.1** (Prop. 4.1 and Prop. 4.3 of [4]). *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost Kenmotsu manifold such that  $\xi$  belongs to the  $(k, \mu)'$ -nullity distribution and  $h' \neq 0$ . Then  $k < -1$ ,  $\mu = -2$  and  $Spec(h') = \{0, \lambda, -\lambda\}$ , with 0 as simple eigenvalue and  $\lambda = \sqrt{-k - 1}$ . The distributions  $[\xi] \oplus [\lambda]'$  and  $[\xi] \oplus [-\lambda]'$  are integrable with totally geodesic leaves. The distributions  $[\lambda]'$  and  $[-\lambda]'$  are integrable with totally umbilical leaves. Furthermore, the sectional curvature is given as follows:*

- (a)  $K(X, \xi) = k - 2\lambda$  if  $X \in [\lambda]'$  and  
 $K(X, \xi) = k + 2\lambda$  if  $X \in [-\lambda]'$ ,
- (b)  $K(X, Y) = k - 2\lambda$  if  $X, Y \in [\lambda]'$ ;  
 $K(X, Y) = k + 2\lambda$  if  $X, Y \in [-\lambda]'$  and  
 $K(X, Y) = -(k + 2)$  if  $X \in [\lambda]'$ ,  $Y \in [-\lambda]'$ ,
- (c)  $M^{2n+1}$  has constant negative scalar curvature  $r = 2n(k - 2n)$ .

**Lemma 3.2** (Lemma 3 of [14]). Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution. If  $h' \neq 0$ , then the Ricci operator  $Q$  of  $M^{2n+1}$  is given by

$$Q = -2nid + 2n(k + 1)\eta \otimes \xi - 2nh'. \quad (3.1)$$

Moreover, the scalar curvature of  $M^{2n+1}$  is  $2n(k - 2n)$ .

**Lemma 3.3** (Proposition 4.2 of [4]). Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost Kenmotsu manifold such that  $h' \neq 0$  and  $\xi$  belongs to the  $(k, -2)$ -nullity distribution. Then for any  $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$  and  $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$ , the Riemannian curvature tensor satisfies:

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\ R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\ R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\ R(X_\lambda, Y_\lambda)Z_\lambda &= (k - 2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k + 2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]. \end{aligned}$$

From (1.3) we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \quad (3.2)$$

where  $k, \mu \in \mathbb{R}$ . Also we get from (3.2)

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \quad (3.3)$$

Contracting  $Y$  in (3.2) we have

$$S(X, \xi) = 2nk\eta(X). \quad (3.4)$$

The Weyl conformal curvature tensor  $C$  on a  $(2n + 1)$ -dimensional manifold is defined by [16]

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (3.5)$$

where  $X, Y, Z$  are any vector fields,  $S$  is the Ricci tensor of type  $(0, 2)$  and  $Q$  is the Ricci operator defined by  $S(X, Y) = g(QX, Y)$ . Using the results (3.1)–(3.4) one can easily obtain

the following:

$$C(\xi, Y)Z = \left( \mu + \frac{2n}{2n-1} \right) \{g(h'Y, Z)\xi - \eta(Z)h'Y\}, \tag{3.6}$$

$$C(X, Y)\xi = \left( \mu + \frac{2n}{2n-1} \right) \{\eta(Y)h'X - \eta(X)h'Y\}. \tag{3.7}$$

Now we are in a position to prove our main theorem.

**Theorem 3.1.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$  be an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution and  $h' \neq 0$ . If the manifold  $M^{2n+1}$  is Weyl semisymmetric then  $M^{2n+1}$  is locally isometric to the Riemannian product of an  $(n + 1)$ -dimensional manifold of constant sectional curvature  $-4$  and a flat  $n$ -dimensional manifold.*

**Proof.** We suppose that the manifold is conformally semisymmetric, that is,  $R \cdot C = 0$ . Then  $(R(X, Y) \cdot C)(U, V)W = 0$  for all vector fields  $X, Y, U, V, W$ , which implies

$$R(X, Y)C(U, V)W - C(R(X, Y)U, V)W - C(U, R(X, Y)V)W - C(U, V)R(X, Y)W = 0. \tag{3.8}$$

Substituting  $X = U = \xi$  in (3.8) we have,

$$R(\xi, Y)C(\xi, V)W - C(R(\xi, Y)\xi, V)W - C(\xi, R(\xi, Y)V)W - C(\xi, V)R(\xi, Y)W = 0. \tag{3.9}$$

Making use of (3.3) and (3.6) we get

$$\begin{aligned} R(\xi, Y)C(\xi, V)W &= k[g(Y, C(\xi, V)W)\xi - \eta(C(\xi, V)W)Y] \\ &\quad + \mu[g(h'Y, C(\xi, V)W)\xi - \eta(C(\xi, V)W)h'Y] \\ &= k \left( \mu + \frac{2n}{2n-1} \right) \{g(h'V, W)\eta(Y)\xi - g(h'V, W)Y - \eta(W)g(Y, h'V)\xi\} \\ &\quad - \mu \left( \mu + \frac{2n}{2n-1} \right) \{g(h'Y, h'V)\eta(W)\xi + g(h'V, W)h'Y\}, \end{aligned} \tag{3.10}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

Similarly, it follows from (3.3) and (3.6) that

$$\begin{aligned} C(R(\xi, Y)\xi, V)W &= k\eta(Y)C(\xi, V)W - kC(Y, V)W - \mu C(h'Y, V)W \\ &= k \left( \mu + \frac{2n}{2n-1} \right) \{g(h'V, W)\eta(Y)\xi - \eta(W)\eta(Y)h'V\} \\ &\quad - kC(Y, V)W - \mu C(h'Y, V)W, \end{aligned} \tag{3.11}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

With the help of (3.3) and (3.6) we obtain

$$\begin{aligned}
& C(\xi, R(\xi, Y)V)W \\
&= kg(Y, V)C(\xi, \xi)W - k\eta(V)C(\xi, Y)W \\
&\quad + \mu g(h'Y, V)C(\xi, \xi)W - \mu\eta(V)C(\xi, h'Y)W \\
&= -k \left( \mu + \frac{2n}{2n-1} \right) \{g(h'Y, W)\eta(V)\xi - \eta(W)\eta(V)h'Y\} \\
&\quad + \mu(k+1) \left( \mu + \frac{2n}{2n-1} \right) \{g(Y, W)\eta(V)\xi - \eta(W)\eta(V)Y\}, \tag{3.12}
\end{aligned}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

Again using (3.3), (3.6) and (3.7) we have

$$\begin{aligned}
& C(\xi, V)R(\xi, Y)W \\
&= kg(Y, W)C(\xi, V)\xi - k\eta(W)C(\xi, V)Y \\
&\quad + \mu g(h'Y, W)C(\xi, V)\xi - \mu\eta(W)C(\xi, V)h'Y \\
&= - \left( \mu + \frac{2n}{2n-1} \right) \{kg(Y, W)h'V + \mu g(h'Y, W)h'V + \mu g(h'V, h'Y)\eta(W)\xi\} \\
&\quad - k \left( \mu + \frac{2n}{2n-1} \right) \{g(h'V, Y)\eta(W)\xi - \eta(Y)\eta(W)h'V\}, \tag{3.13}
\end{aligned}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

Finally, substituting (3.10)–(3.13) in (3.9) yields

$$\begin{aligned}
& kC(Y, V)W + \mu C(h'Y, V)W + \left( \mu + \frac{2n}{2n-1} \right) \{-kg(h'V, W)Y \\
&\quad - \mu g(h'V, W)h'Y + kg(h'Y, W)\eta(V)\xi - k\eta(V)\eta(W)h'Y \\
&\quad - \mu(k+1)g(Y, W)\eta(V)\xi + \mu(k+1)\eta(V)\eta(W)Y \\
&\quad + kg(Y, W)h'V + \mu g(h'Y, W)h'V\} = 0, \tag{3.14}
\end{aligned}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

Substituting  $Y = h'Y$  in (3.14) and using the fact  $h'^2 = (k+1)\phi^2$  of (2.5) we get

$$\begin{aligned}
& kC(h'Y, V)W - \mu(k+1)C(Y, V)W + \left( \mu + \frac{2n}{2n-1} \right) \{-kg(h'V, W)h'Y \\
&\quad + \mu(k+1)g(h'V, W)Y - k(k+1)g(Y, W)\eta(V)\xi + k(k+1)\eta(V)\eta(W)Y \\
&\quad - \mu(k+1)g(h'Y, W)\eta(V)\xi + \mu(k+1)\eta(V)\eta(W)h'Y \\
&\quad + kg(h'Y, W)h'V - \mu(k+1)g(Y, W)h'V\} = 0, \tag{3.15}
\end{aligned}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

Subtracting  $\mu$  multiple of (3.15) from  $k$  multiple of (3.14) implies

$$\begin{aligned}
& (k+2)^2 C(Y, V)W + (k+2)^2 \left( \mu + \frac{2n}{2n-1} \right) \{g(h'Y, W)\eta(V)\xi \\
&\quad - \eta(V)\eta(W)h'Y - g(h'V, W)Y + g(Y, W)h'V\} = 0, \tag{3.16}
\end{aligned}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ . In [4], Dileo and Pastore proved that if  $\xi$  belongs to the  $(k, \mu)$ '-nullity distribution then  $\mu = -2$ . Using this result, Lemmas 3.2 and 3.3 and letting  $Y, V, W \in [-\lambda]'$  we have

$$C(Y, V)W = \frac{2nk - 2\lambda + 2n}{2n - 1} \{g(V, W)Y - g(Y, W)V\}, \tag{3.17}$$

for any vector fields  $Y, V, W$  on  $M^{2n+1}$ .

With the help of (3.17) and noticing  $Y, V, W \in [-\lambda]'$  we obtain from (3.16)

$$(k + 2)^2(k + 1 - \lambda)\{g(V, W)Y - g(Y, W)V\} = 0. \tag{3.18}$$

Making use of (3.18) and the fact  $\lambda = \pm\sqrt{-k-1}$  yields

$$\lambda(\lambda - 1)^2(\lambda + 1)^3 = 0. \tag{3.19}$$

Since  $h' \neq 0$ , (2.5) implies  $k \neq -1$  and hence  $\lambda \neq 0$ . Then it follows from (3.19) that  $\lambda^2 = 1$  and consequently  $k = -2$ . Without loss of any generality we may choose  $\lambda = 1$ . Then we have from Lemma 3.3

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_\lambda &= -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= 0, \end{aligned}$$

for any  $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$  and  $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$ . Also noticing  $\mu = -2$  it follows from Lemma 3.1 that  $K(X, \xi) = -4$  for any  $X \in [\lambda]'$  and  $K(X, \xi) = 0$  for any  $X \in [-\lambda]'$ . Again from Lemma 3.1 we see that  $K(X, Y) = -4$  for any  $X, Y \in [\lambda]'$ ;  $K(X, Y) = 0$  for any  $X, Y \in [-\lambda]'$  and  $K(X, Y) = 0$  for any  $X \in [\lambda]'$ ,  $Y \in [-\lambda]'$ . As is shown in [4] that the distribution  $[\xi] \oplus [\lambda]'$  is integrable with totally geodesic leaves and the distribution  $[-\lambda]'$  is integrable with totally umbilical leaves by  $H = -(1 - \lambda)\xi$ , where  $H$  is the mean curvature vector field for the leaves of  $[-\lambda]'$  immersed in  $M^{2n+1}$ . Here  $\lambda = 1$ , then two orthogonal distributions  $[\xi] \oplus [\lambda]'$  and  $[-\lambda]'$  are both integrable with totally geodesic leaves immersed in  $M^{2n+1}$ . Then we can say that  $M^{2n+1}$  is locally isometric to  $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ . This completes the proof of our theorem.  $\square$

Since conformally symmetric manifold ( $\nabla C = 0$ ) implies  $R \cdot C = 0$ , therefore from Theorem 3.1 we can state the following:

**Corollary 3.1.** *A conformally symmetric almost Kenmotsu manifold  $M^{2n+1}$  ( $n > 1$ ) with  $\xi$  belonging to the  $(k, \mu)$ '-nullity distribution and  $h' \neq 0$  is locally isometric to the Riemannian product of an  $(n + 1)$ -dimensional manifold of constant sectional curvature  $-4$  and a flat  $n$ -dimensional manifold.*

Since  $R \cdot R = 0$  implies  $R \cdot C = 0$ , we obtain the following:

**Corollary 3.2.** *A semisymmetric almost Kenmotsu manifold  $M^{2n+1}$  ( $n > 1$ ) with  $\xi$  belonging to the  $(k, \mu)$ '-nullity distribution and  $h' \neq 0$  is locally isometric to the Riemannian product of an  $(n + 1)$ -dimensional manifold of constant sectional curvature  $-4$  and a flat  $n$ -dimensional manifold.*

The above corollary has been proved by Wang and Liu [13]. Obviously, [Theorem 3.1](#) generalizes the theorem of Wang and Liu [13].

Next we consider an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution and  $h' \neq 0$  satisfying the curvature condition  $C \cdot S = 0$ . Then  $(C(X, Y) \cdot S)(U, V) = 0$  for all vector fields  $X, Y, U, V$ , which implies

$$S(C(X, Y)U, V) + S(U, C(X, Y)V) = 0, \quad (3.20)$$

for any vector fields  $X, Y, U, V$  on  $M^{2n+1}$ .

Substituting  $X = U = \xi$  in [\(3.20\)](#) we have,

$$S(C(\xi, Y)\xi, V) + S(\xi, C(\xi, Y)V) = 0. \quad (3.21)$$

Making use of [\(3.4\)](#) and [\(3.6\)](#) we get from [\(3.21\)](#)

$$\left(\mu + \frac{2n}{2n+1}\right) \{S(h'Y, V) - 2nkg(h'Y, V)\} = 0, \quad (3.22)$$

for any vector fields  $Y, V$  on  $M^{2n+1}$ . Since  $\xi$  belongs to the  $(k, \mu)'$ -nullity distribution, therefore  $\mu = -2$  [4].

Hence for  $2n + 1 \geq 5$ ,

$$S(h'Y, V) = 2nkg(h'Y, V), \quad (3.23)$$

for any vector fields  $Y, V$  on  $M^{2n+1}$ .

Replacing  $Y$  by  $h'Y$  in [\(3.23\)](#) and using [\(2.5\)](#) yields

$$(k + 1)\{S(Y, V) - 2nkg(Y, V)\} = 0, \quad (3.24)$$

for any vector fields  $Y, V$  on  $M^{2n+1}$ .

Suppose  $k + 1 = 0$ , that is,  $k = -1$ . Dileo and Pastore [4] prove that in an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution if  $k = -1$ , then  $h' = 0$  and the manifold  $M^{2n+1}$  is locally a warped product of an almost Kähler manifold and an open interval. Thus  $k + 1 = 0$  contradicts our hypothesis  $h' \neq 0$ . Therefore  $S(V, Y) = 2nkg(V, Y)$ , for any vector fields  $V, Y$  on  $M^{2n+1}$ . Thus the manifold is an Einstein manifold.

Conversely, if the manifold under consideration is an Einstein manifold, then from [\(3.20\)](#) it follows that  $C \cdot S = 0$  holds identically.

By the above discussions we can state the following:

**Theorem 3.2.** *An almost Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$ ,  $n > 1$ , with  $\xi$  belonging to the  $(k, \mu)'$ -nullity distribution and  $h' \neq 0$  satisfies the curvature condition  $C \cdot S = 0$  if and only if the manifold is an Einstein manifold.*

#### 4. $\xi$ BELONGS TO THE $(k, \mu)$ -NULLITY DISTRIBUTION

In this section we study the curvature properties  $C = 0$  and  $R \cdot C = 0$  on an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution, where  $C$  and  $R$  are the conformal curvature tensor and Riemannian curvature tensor respectively.



From (1.2) we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad (4.1)$$

where  $k, \mu \in \mathbb{R}$ .

Now we state the following:

**Lemma 4.1** (Theorem 4.1 of [4]). *Let  $M^{2n+1}$  be an almost Kenmotsu manifold of dimension  $2n + 1$ . Suppose the characteristic vector field  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution. Then  $k = -1$ ,  $h = 0$  and  $M^{2n+1}$  is locally a warped product of an open interval and an almost Kähler manifold.*

From (4.1) and Lemma 4.1 we have the following:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (4.2)$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \quad (4.3)$$

$$S(X, \xi) = -2n\eta(X), \quad (4.4)$$

for any vector fields  $X, Y$  on  $M^{2n+1}$ . Moreover, we have the following:

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (4.5)$$

for any vector fields  $X, Y$  on  $M^{2n+1}$ .

Let us consider the manifold  $M^{2n+1}$  be conformally flat, that is,

$$C(X, Y)Z = 0, \quad (4.6)$$

for any vector fields  $X, Y, Z$  on  $M^{2n+1}$ .

From (3.5) and (4.6) we have

$$\begin{aligned} R(X, Y)Z &= \frac{1}{2n-1} \{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY\} - \frac{r}{2n(2n-1)} \{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \quad (4.7)$$

Substituting  $Z = \xi$  in (4.7) and using (4.2), (4.4) yields

$$\eta(Y)QX - \eta(X)QY = \left(1 + \frac{r}{2n}\right) \{\eta(Y)X - \eta(X)Y\}. \quad (4.8)$$

Putting  $Y = \xi$  in (4.8) and using (4.4) we obtain

$$QX = \left(1 + \frac{r}{2n}\right) X - \left(1 + 2n + \frac{r}{2n}\right) \eta(X)\xi. \quad (4.9)$$

Taking inner product of (4.9) with  $Y$  we have

$$S(X, Y) = \left(1 + \frac{r}{2n}\right) g(X, Y) - \left(1 + 2n + \frac{r}{2n}\right) \eta(X)\eta(Y). \quad (4.10)$$

Now substituting the values of  $QX$  and  $S(X, Y)$  in the expression of the conformal curvature tensor and considering the hypothesis  $C(X, Y)Z = 0$ , we get

$$\begin{aligned} R(X, Y)Z &= \left( \frac{r + 4n}{2n(2n - 1)} \right) \{g(Y, Z)X - g(X, Z)Y\} \\ &\quad - \frac{1}{2n - 1} \left( 1 + 2n + \frac{r}{2n} \right) \{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \quad (4.11)$$

In [4], Dileo and Pastore prove that in an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution the sectional curvature  $K(X, \xi) = -1$ . From this we get in an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution the scalar curvature  $r = -2n(2n + 1)$ .

Thus from (4.11) we obtain

$$R(X, Y)Z = -\{g(Y, Z)X - g(X, Z)Y\},$$

which implies that the manifold is of constant curvature  $-1$ .

Conversely, if the manifold  $M^{2n+1}$  is of constant curvature  $-1$ , then it can be easily shown that the manifold under consideration is conformally flat.

Hence we can state the following:

**Proposition 4.1.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution. Then  $M^{2n+1}$  is conformally flat if and only if the manifold is of constant curvature  $-1$ .*

Using (4.3), (4.4) and (3.5) one can easily verify the following:

$$\begin{aligned} C(\xi, Y)Z &= \left( \frac{r + 2n}{2n(2n - 1)} \right) \{g(Y, Z)\xi - \eta(Z)Y\} \\ &\quad - \frac{1}{2n - 1} \{S(Y, Z)\xi - \eta(Z)QY\}, \end{aligned} \quad (4.12)$$

for any vector field  $Y, Z$  on  $M^{2n+1}$ .

Now we prove the following:

**Proposition 4.2.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost Kenmotsu manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution. Then  $M^{2n+1}$  is Weyl semisymmetric if and only if the manifold is conformally flat.*

**Proof.** Let  $M^{2n+1}$  be a Weyl semisymmetric almost Kenmotsu manifold with  $\xi$  belongs to the  $(k, \mu)$ -nullity distribution. Therefore  $(R(X, Y) \cdot C)(U, V)W = 0$  for all vector fields  $X, Y, U, V, W$ , which implies

$$\begin{aligned} R(X, Y)C(U, V)W - C(R(X, Y)U, V)W \\ - C(U, R(X, Y)V)W - C(U, V)R(X, Y)W = 0. \end{aligned} \quad (4.13)$$

Substituting  $X = U = \xi$  in (4.13) we have,

$$\begin{aligned} R(\xi, Y)C(\xi, V)W - C(R(\xi, Y)\xi, V)W \\ - C(\xi, R(\xi, Y)V)W - C(\xi, V)R(\xi, Y)W = 0. \end{aligned} \quad (4.14)$$

Making use of (4.3) and (4.12) we get

$$\begin{aligned}
 R(\xi, Y)C(\xi, V)W &= \left( \frac{r + 2n}{2n(2n - 1)} \right) \{g(V, Y)\eta(W)\xi - g(V, W)\eta(Y)\xi \\
 &\quad + g(V, W)Y - \eta(W)\eta(V)Y\} \\
 &\quad + \frac{1}{2n - 1} \{S(V, W)\eta(Y)\xi - S(V, Y)\eta(W)\xi \\
 &\quad - S(V, W)Y - 2n\eta(W)\eta(V)Y\}, \tag{4.15}
 \end{aligned}$$

for any vector field  $Y, V, W$  on  $M^{2n+1}$ .

Again using (4.3) and (4.12) we obtain

$$\begin{aligned}
 C(R(\xi, Y)\xi, V)W &= C(Y, V)W - \left( \frac{r + 2n}{2n(2n - 1)} \right) \\
 &\quad \times \{g(V, W)\eta(Y)\xi - \eta(W)\eta(Y)V\} \\
 &\quad + \frac{1}{2n - 1} \{S(V, W)\eta(Y)\xi - \eta(Y)\eta(W)QV\}, \tag{4.16}
 \end{aligned}$$

for any vector field  $Y, V, W$  on  $M^{2n+1}$ .

Similarly, it follows from (4.3) and (4.12) that

$$\begin{aligned}
 C(\xi, R(\xi, Y)V)W &= \left( \frac{r + 2n}{2n(2n - 1)} \right) \{g(Y, W)\eta(V)\xi - \eta(V)\eta(W)Y\} \\
 &\quad - \frac{1}{2n - 1} \{S(Y, W)\eta(V)\xi - \eta(W)\eta(V)QY\}, \tag{4.17}
 \end{aligned}$$

for any vector field  $Y, V, W$  on  $M^{2n+1}$ .

With the help of (4.3) and (4.12) we have

$$\begin{aligned}
 C(\xi, V)R(\xi, Y)W &= \left( \frac{r + 2n}{2n(2n - 1)} \right) \{g(Y, W)V - g(Y, W)\eta(V)\xi \\
 &\quad + g(Y, V)\eta(W)\xi - \eta(W)\eta(Y)V\} \\
 &\quad - \frac{1}{2n - 1} \{2ng(Y, W)\eta(V)\xi + g(Y, W)QV \\
 &\quad + S(V, Y)\eta(W)\xi - \eta(W)\eta(Y)QV\}, \tag{4.18}
 \end{aligned}$$

for any vector field  $Y, V, W$  on  $M^{2n+1}$ .

Finally, substituting (4.15)–(4.18) in (4.14) gives

$$\begin{aligned}
 &\left( \frac{r + 2n}{2n(2n - 1)} \right) \{g(V, W)Y - g(Y, W)V\} - C(Y, V)W \\
 &\quad + \frac{1}{2n - 1} \{S(Y, W)\eta(V)\xi - \eta(W)\eta(V)QY + 2ng(Y, W)\eta(V)\xi \\
 &\quad + g(Y, W)QV - S(V, W)Y - 2n\eta(V)\eta(W)Y\} = 0, \tag{4.19}
 \end{aligned}$$

for any vector field  $Y, V, W$  on  $M^{2n+1}$ .

Using (3.5) in (4.19) yields

$$\begin{aligned} R(Y, V)W &= \frac{1}{2n-1} \{g(V, W)Y - g(Y, W)V - 2n\eta(V)\eta(W)Y \\ &\quad + S(Y, W)\eta(V)\xi - \eta(V)\eta(W)QY + 2ng(Y, W)\eta(V)\xi \\ &\quad - S(Y, W)V + g(V, W)QY\}. \end{aligned} \quad (4.20)$$

Contracting  $Y$  in (4.20) it follows that

$$S(V, W) = \left(1 + \frac{r}{2n}\right)g(V, W) - \left(1 + 2n + \frac{r}{2n}\right)\eta(V)\eta(W), \quad (4.21)$$

for any vector field  $V, W$  on  $M^{2n+1}$ .

Taking inner product of (4.19) with respect to  $Z$  gives

$$\begin{aligned} &\left(\frac{r+2n}{2n(2n-1)}\right) \{g(V, W)g(Y, Z) - g(Y, W)g(V, Z)\} - g(C(Y, V)W, Z) \\ &\quad + \frac{1}{2n-1} \{S(Y, W)\eta(V)\eta(Z) - \eta(W)\eta(V)S(Y, Z) + 2ng(Y, W)\eta(V)\eta(Z) \\ &\quad + g(Y, W)S(V, Z) - S(V, W)g(Y, Z) - 2n\eta(V)\eta(W)g(Y, Z)\} = 0. \end{aligned} \quad (4.22)$$

Putting the value of  $S(V, W)$  in (4.22) one can easily obtain

$$g(C(Y, V)W, Z) = 0, \quad (4.23)$$

that is,  $C(Y, V)W = 0$ , for any vector field  $Y, V, W$  on  $M^{2n+1}$ . Hence the manifold is conformally flat.

Conversely, if the manifold is conformally flat then obviously it is Weyl semisymmetric. This completes the proof of the proposition.  $\square$

From Propositions 4.1 and 4.2 we obtain the following:

**Theorem 4.1.** *An almost Kenmotsu manifold  $M^{2n+1}$  ( $n > 1$ ) with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution is Weyl semisymmetric if and only if the manifold is of constant curvature  $-1$ .*

Since conformally symmetric manifold ( $\nabla C = 0$ ) implies  $R \cdot C = 0$ , therefore from Theorem 4.1 we can state the following:

**Corollary 4.1.** *An almost Kenmotsu manifold  $M^{2n+1}$  ( $n > 1$ ) with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution is conformally symmetric if and only if the manifold is of constant curvature  $-1$ .*

Since  $R \cdot R = 0$  implies  $R \cdot C = 0$ , we obtain the following:

**Corollary 4.2.** *An almost Kenmotsu manifold  $M^{2n+1}$  ( $n > 1$ ) with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution is semisymmetric if and only if the manifold is of constant curvature  $-1$ .*

The above corollary has been proved by Wang and Liu [13].

### 5. EXAMPLE OF A 5-DIMENSIONAL ALMOST KENMOTSU MANIFOLD

In this section, we construct an example of an almost Kenmotsu manifold such that  $\xi$  belongs to the  $(k, \mu)'$ -nullity distribution and  $h' \neq 0$ , which is an Einstein manifold. We consider 5-dimensional manifold  $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ , where  $(x, y, z, u, v)$  are the standard coordinates in  $\mathbb{R}^5$ . Let  $\xi, e_2, e_3, e_4, e_5$  be five vector fields in  $\mathbb{R}^5$  which satisfies [4]

$$\begin{aligned} [\xi, e_2] &= -2e_2, & [\xi, e_3] &= -2e_3, & [\xi, e_4] &= 0, & [\xi, e_5] &= 0, \\ [e_i, e_j] &= 0, & \text{where } i, j &= 2, 3, 4, 5. \end{aligned}$$

Let  $g$  be the Riemannian metric defined by

$$\begin{aligned} g(\xi, \xi) &= g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1 \\ \text{and } g(\xi, e_i) &= g(e_i, e_j) = 0 \quad \text{for } i \neq j; i, j = 2, 3, 4, 5. \end{aligned}$$

Let  $\eta$  be the 1-form defined by

$$\eta(Z) = g(Z, \xi),$$

for any  $Z \in \chi(M)$ . Let  $\phi$  be the  $(1, 1)$ -tensor field defined by

$$\phi(\xi) = 0, \quad \phi(e_2) = e_4, \quad \phi(e_3) = e_5, \quad \phi(e_4) = -e_2, \quad \phi(e_5) = -e_3.$$

Using the linearity of  $\phi$  and  $g$  we have

$$\eta(\xi) = 1, \quad \phi^2 Z = -Z + \eta(Z)\xi$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any  $Z, U \in \chi(M)$ . Moreover,

$$h'\xi = 0, \quad h'e_2 = e_2, \quad h'e_3 = e_3, \quad h'e_4 = -e_4, \quad h'e_5 = -e_5.$$

The Levi-Civita connection  $\nabla$  of the metric tensor  $g$  is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\begin{aligned} \nabla_\xi \xi &= 0, & \nabla_\xi e_2 &= 0, & \nabla_\xi e_3 &= 0, & \nabla_\xi e_4 &= 0, & \nabla_\xi e_5 &= \xi, \\ \nabla_{e_2} \xi &= 2e_2, & \nabla_{e_2} e_2 &= -2\xi, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= 0, & \nabla_{e_2} e_5 &= 0, \\ \nabla_{e_3} \xi &= 2e_3, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= -2\xi, & \nabla_{e_3} e_4 &= 0, & \nabla_{e_3} e_5 &= 0, \\ \nabla_{e_4} \xi &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= 0, & \nabla_{e_4} e_5 &= 0, \\ \nabla_{e_5} \xi &= 0, & \nabla_{e_5} e_2 &= 0, & \nabla_{e_5} e_3 &= 0, & \nabla_{e_5} e_4 &= 0, & \nabla_{e_5} e_5 &= 0. \end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X,$$

for any  $X \in \chi(M)$ . Therefore, the structure  $(\phi, \xi, \eta, g)$  is an almost contact metric structure such that  $d\eta = 0$  and  $d\Phi = 2\eta \wedge \Phi$ , so that  $M$  is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor  $R$  as follows:

$$\begin{aligned} R(\xi, e_2)\xi &= 4e_2, & R(\xi, e_2)e_2 &= -4\xi, & R(\xi, e_3)\xi &= 4e_3, \\ R(\xi, e_3)e_3 &= -4\xi, \\ R(\xi, e_4)\xi &= R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0, \\ R(e_2, e_3)e_2 &= 4e_3, & R(e_2, e_3)e_3 &= -4e_2, & R(e_2, e_4)e_2 &= R(e_2, e_4)e_4 = 0, \\ R(e_2, e_5)e_2 &= R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0, \\ R(e_3, e_5)e_3 &= R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0. \end{aligned}$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field  $\xi$  belongs to the  $(k, \mu)'$ -nullity distribution, with  $k = -2$  and  $\mu = -2$ .

Using the expressions of the curvature tensor we find the values of the Ricci tensor  $S$  as follows:

$$S(\xi, \xi) = S(e_2, e_2) = S(e_3, e_3) = -8, \quad S(e_4, e_4) = S(e_5, e_5) = 0.$$

Since  $\{\xi, e_2, e_3, e_4, e_5\}$  forms a basis, any vector field  $X, Y \in \chi(M)$  can be written as

$$X = a_1\xi + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$$

and

$$Y = b_1\xi + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5,$$

where  $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 \in \mathbb{R} \setminus \{0\}$  such that  $a_4b_4 + a_5b_5 = 0$ . Hence,

$$g(X, Y) = a_1b_1 + a_2b_2 + a_3b_3$$

and

$$S(X, Y) = -8(a_1b_1 + a_2b_2 + a_3b_3).$$

Therefore, we see that  $S(X, Y) = -8g(X, Y)$ , that is, the manifold  $M$  is an Einstein manifold.

Thus, [Theorem 3.2](#) is verified.

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