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ON A THEOREM OF RUZIEWICZ

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A classical theorem in measure theory due to Steinhaus [**4**] states that if A is a Lebesgue measurable subset of real line $\mathbb{R}$ and has positive measure, then the set $A - A = \{x - y : x, y \in A\}$ contains a non-empty interval. The category analogue of the above result due to Piccard [**2**] has the same conclusion when A is a linear set of second category having the property of Baire. In 1925, Ruziewicz [**3**] extended this result of Steinhaus [**4**] in the following manner.

THEOREM. *For any set of m positive real numbers* $k_1, k_2, \dots, k_m$ *and for any linear set A of positive Lebesgue measure there exist points* $x_1 < x_2 < \cdots < x_{m+1}$ *all in A such that* $x_{i+1} - x_i = d \cdot k_i, i = 1, 2, \dots, m$, *where the positive number d does not depend on i.*

In this paper we endeavour to give an alternative proof of the above theorem of Ruziewicz and our proof seems to be simple and short in length. We also establish a generalization of the category analogue of the same theorem while A is a set of second category possessing the property of Baire.

A set S is said to have the property of Baire if S can be expressed as $S = F \Delta P$, where F is an open set and P is a set of first category where Δ stands for symmetric difference.

For a linear set A and a real number ω, we define the set $A - \omega$ by $A - \omega = \{a - \omega : a \in A\}$. We use invariance of Lebesgue measure under translation in our proof.

We first give the alternative proof of the above theorem.

THEOREM 1. *For any m positive real numbers* $\beta_1, \beta_2, \dots, \beta_m$ *and any linear set A with positive Lebesgue measure, there exist* $(m + 1)$ *points* $x_0 < x_1 < \cdots < x_m$ $(x_i \in A, \forall i = 0, 1, \dots, m)$ *such that* $x_i - x_{i-1} = c \cdot \beta_i$ $(i = 1, 2, \dots, m)$, *where the positive number c is independent of i.*

Proof. Let us set $\alpha_0 = 0, \alpha_i = \sum_{k=1}^{i} \beta_k$ for $1 \leq i \leq m$. Then, $\alpha_0 < \alpha_1 < \cdots < \alpha_m$.

Since A has positive Lebesgue measure, by the open regularity of Lebesgue measure and the property of open sets in $\mathbb{R}$ (see the second case of Theorem 4.8 of

[1, p. 21]) there exists an interval $[\alpha, \beta]$ such that,

$$\mu(A \cap [\alpha, \beta]) > \{(2m+1)/2(m+1)\}(\beta - \alpha),$$

where μ denotes the Lebesgue measure.

Let us take a positive number c, small enough so that $c\alpha_m < (\beta - \alpha)/2m$. Let $E = A \cap [\alpha, \beta]$ and $J = [\alpha - c\alpha_m, \beta]$. Then, $E - c\alpha_i \subset J$, $\forall i = 0, 1, \ldots, m$. Now,

$$\bigcap_{i=0}^{m}(E - c\alpha_i) = J \backslash \bigcup_{i=0}^{m}[J \backslash (E - c\alpha_i)].$$

Hence,

$$\begin{aligned}
\mu\left(\bigcap_{i=0}^{m}(E - c\alpha_i)\right) &\geq \mu(J) - \sum_{i=0}^{m}\mu(J \backslash (E - c\alpha_i)) \\
&= (m+1)\cdot\mu(E) - m\cdot\mu(J) \\
&> \{(2m+1)/2\}(\beta - \alpha) - m(\beta - \alpha + c\alpha_m) \\
&= \tfrac{1}{2}(\beta - \alpha) - mc\alpha_m \\
&> \tfrac{1}{2}(\beta - \alpha) - m\{(\beta - \alpha)/2m\} \\
&= 0.
\end{aligned}$$

Therefore $\cap_{i=0}^{m}(E - c\alpha_i) \neq \emptyset$. So, there exist points $x_0, x_1, \ldots, x_m \in E \subseteq A$ such that

$$x_0 - c\alpha_0 = x_1 - c\alpha_1 = \cdots = x_m - c\alpha_m.$$

Thus, $x_i - x_{i-1} = c\cdot(\alpha_i - \alpha_{i-1}) = c\cdot\beta_i$, $i = 1, 2, \ldots, m$ and hence, $x_0 < x_1 < \cdots < x_m$. This completes the proof. □

We now generalize the category analogue of the above theorem.

THEOREM 2. *Let $\{\beta_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n\geq 1}\beta_n$ converges and let A be a linear set of second category having the property of Baire. Then there exists a sequence $\{x_n : n \geq 0\}$ in A such that $x_i < x_{i+1}$ $(i \geq 0)$ and*

$$x_i - x_{i-1} = c\cdot\beta_i \quad (i \geq 1),$$

where the positive number c does not depend on i.

Proof. Let $\alpha_0 = 0, \alpha_i = \sum_{k=1}^{i}\beta_k$ $\forall i \geq 1$. Then $\{\alpha_i\}_{i\geq 0}$ is a strictly increasing sequence which is convergent. Since A is a set of second category having the property of Baire, there exists an interval (α, β) such that, $(\alpha, \beta)\backslash A$ is a set of first category (see [**1**, p. 19]).

Let $E = A \cap (\alpha, \beta)$. Let c be a positive number such that $c\alpha_m < \frac{1}{2}(\beta - \alpha)$, $\forall m$. We show that

$$A \bigcap \bigcap_{i=1}^{\infty} (E - c\alpha_i) \neq \emptyset. \tag{1}$$

If possible let (1) be false. Then,

$$\left(\alpha, \frac{\alpha+\beta}{2}\right) = \left[\left(\alpha, \frac{\alpha+\beta}{2}\right)\backslash A\right] \bigcup \bigcup_{i=1}^{\infty} \left[\left(\alpha, \frac{\alpha+\beta}{2}\right)\backslash (E - c\alpha_i)\right]. \tag{2}$$

$(\alpha, (\alpha+\beta)/2)\backslash A$ is a subset of $(\alpha, \beta)\backslash A$ and hence a set of first category.

For any $i \geq 1$,

$$\begin{aligned}
\left(\left(\alpha, \frac{\alpha+\beta}{2}\right)\backslash (E - c\alpha_i)\right) + c\alpha_i &= \left(\left(\alpha, \frac{\alpha+\beta}{2}\right) + c\alpha_i\right)\backslash E \\
&\subseteq (\alpha, \beta)\backslash E \\
&= (\alpha, \beta)\backslash (A \cap (\alpha, \beta)) \\
&= (\alpha, \beta)\backslash A \quad \text{which is a set of first category.}
\end{aligned}$$

Hence for each $i \geq 1$, $((\alpha, (\alpha+\beta)/2)\backslash (E - c\alpha_i)) + c\alpha_i$ is a set of first category and hence $(\alpha, (\alpha+\beta)/2)\backslash (E - c\alpha_i)$ is a set of first category. Thus from (2), $(\alpha, (\alpha+\beta)/2)$ is an enumerable union of sets of first category and hence is itself a set of first category which is a contradiction, since $(\alpha, (\alpha+\beta)/2)$ is a set of second category.

Hence $A \cap \bigcap_{i=1}^{\infty} (E - c\alpha_i) \neq \emptyset$. So, there exists a sequence $\{x_n\}_{n\geq 0}$ of elements of A such that

$$x_0 = x_1 - c\alpha_1 = x_2 - c\alpha_2 = \cdots = x_n - c\alpha_n = \cdots.$$

Thus, $x_i - x_{i-1} = c\cdot(\alpha_i - \alpha_{i-1}) = c\cdot\beta_i$, $\forall i = 1, 2, \ldots$. This completes the proof. □

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