# NEW SOLUTIONS OF YANG–MILLS EQUATIONS WITH STATIC EXTERNAL SOURCES

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Received 31 March 1981

New axially symmetric static solutions of the SU(2) Yang–Mills equations with external sources are presented. The chromoelectric fields are totally screened and the chromomagnetic fields have asymptotically a multipole-like behaviour. Their energies are compared with the corresponding Coulomb energies.

The nonabelian analogue of electrostatics has been attracting some attention over the last few years e.g. refs. [1–6]. The nonlinearity of the Yang–Mills equations adds a new twist to the conventional U(1) electrostatics and the same external source may support multiple finite-energy nonsingular solutions. The instability [1] of the abelian Coulomb-like solution (see below) is, in this connection, of special importance and a lot of effort has gone into finding other lower energy solutions. Here we present some new solutions which also share this interesting property.

The classical SU(2) Yang–Mills equations in the presence of an external source may be written as

$$D_\mu F_a^{\mu\nu} = j_a^\nu , \quad a = 1, 2, 3 . \tag{1}$$

This equation implies a consistency condition on the source

$$D_\mu j_a^\mu = 0 . \tag{2}$$

We restrict ourselves to physically static solutions, i.e. the Yang–Mills potentials $A_a^\mu$ are time independent. The external source is chosen to be static

$$j_a^\mu(\boldsymbol{r}, t) = \delta^{\mu 0} \rho_a(\boldsymbol{r}) . \tag{3}$$

A gauge-invariant total charge $Q$ is defined by

$$Q = \int \mathrm{d}^3 r \, [\rho_a(\boldsymbol{r}) \, \rho_a(\boldsymbol{r})]^{1/2} . \tag{4}$$

By local gauge transformations one can go to the *abelian gauge* in which

$$\rho_a(\boldsymbol{r}) = \delta_{a3} \rho(\boldsymbol{r}) . \tag{5}$$

The remaining gauge freedom consists of time-independent gauge transformations about the SU(2) $\hat{3}$ direction. In view of (3) and (5) from (2) we obtain

$$A_a^0(\boldsymbol{r}) = \delta_{a3} \phi(\boldsymbol{r}) .$$

In this gauge, the Coulomb solution corresponds to setting $\boldsymbol{A}_a = 0$ and $A_a^0 = \delta_{a3} \phi_\mathrm{c}(\boldsymbol{r})$. Then $\phi_\mathrm{c}(\boldsymbol{r})$ satisfies

$$\nabla^2 \phi_\mathrm{c}(\boldsymbol{r}) = -\rho(\boldsymbol{r}) , \tag{6a}$$

with energy

$$H_\mathrm{c} = \frac{1}{8\pi} \int \mathrm{d}^3 r \int \mathrm{d}^3 r' \, \frac{\rho(\boldsymbol{r}) \, \rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} . \tag{6b}$$

The axially symmetric solutions that we present are obtained from the ansatz

$$\phi(\boldsymbol{r}) = \phi(r, \theta) \tag{7a}$$

and

$$\sum_a A_a^i T_a = \epsilon_{i3j} \frac{r_j}{r \sin\theta} \bar{A}(r, \theta) . \tag{7b}$$

Here $T_a$ forms a matrix representation of the generators of SU(2) and on the rhs of (7b) $\bar{A}$ is a matrix.

This is the form of the ansatz employed by Sikivie and Weiss to obtain the magnetic dipole solutions [2]. The class of solutions we present here includes this solution as a special case. As already shown by Sikivie and Weiss [2] the ansatz (7b) when substituted into the equations of motion imply $A_3 = 0$. Furthermore, since $A_1$ and $A_2$ are parallel we can exploit our remaining gauge freedom to set $A_2 = 0$. Then (7b) reduces to

$$A_1^i = \epsilon_{i3j}(r_j/r \sin\theta)\,A(r,\theta)\,, \quad A_2 = 0\,, \quad A_3 = 0\,, \tag{7c}$$

where $A$ is now an ordinary function and not a matrix.

At this stage, it is interesting to recall that if $A_a$ can be written in the factorised form

$$A_a(r) = \eta_a(r)\,e(r)\,, \tag{8}$$

then such a non-coulombic solution always carries less energy than the corresponding Coulomb solution [6]. Since (7c) can be written in the form of (8) with $\eta_a = \delta_{a1}$, all the solutions that we find have this interesting energy behaviour.

In terms of $A$ and $\phi$ the Yang–Mills equations (1) become (the gauge coupling constant is scaled to unity)

$$-\nabla^2\phi + A^2\phi = \rho\,, \tag{9a}$$

$$\nabla^2 A - (r^2\sin^2\theta)^{-1}A + \phi^2 A = 0\,. \tag{9b}$$

The technique we employ to find solutions is the same as that used by Sikivie and Weiss [2] for the magnetic dipole solution. One makes a suitable starting guess for $A$ and then from (9b) calculates the corresponding $\phi$. Then eq. (9a) is used to calculate the charge distribution $\rho$. We look for solutions in which $\rho$ goes to zero exponentially fast as $r$ tends to infinity. From (9a) and (9b) we find that this requires $A$ to approach a solution of the equation

$$\nabla^2 A - (r^2\sin^2\theta)^{-1}A = 0\,,$$

exponentially fast for large $r$. We present two classes of solutions obtained by this method. The members of each class are parametrised by the integers $n$.

The solutions of the first class are

$$A = (c/a)[P_n^1(\cos\theta)/x^{n+1}]\,f(y)\,, \tag{10a}$$

where $P_n^1(\cos\theta)$ is the associated Legendre function and

$$\phi = [(2n+1)/a]\,x^{2n}\exp(-y/2)[f(y)]^{-1/2}\,, \tag{10b}$$

with

$$x = r/a\,, \quad y = x^{2n+1}\,, \quad f(y) = 1 - \exp(-y)\,. \tag{11,12}$$

Here $a$ is a parameter with the dimension of length which sets the scale for the problem. For a fixed value of $n$ the dimensionless parameter $c$ can be continuously varied to change the shape of the charge distribution $\rho$ as well as the total charge $Q$ (see below). $c = 0$ corresponds to the Coulomb solution so that there are non-abelian solutions within this ansatz arbitrarily close to the Coulomb solution. We will see that for a fixed value of $n$ there is always a critical total charge such that the solution does not exist below that charge. This minimal charge corresponds to the bifurcation point, $c = 0$. All these remarks are also valid for the second-class solutions.

The external charge density, $\rho$, corresponding to eq. (10) can be written in the form

$$\rho = a^{-3}h(y) + (c^2/a^3)\,k(y)\,[P_n^1(\cos\theta)]^2\,, \tag{13}$$

with

$$h(y) = -(2n+1)^2 x^{2n-2}\exp(-y/2)[f(y)]^{-5/2} \times \{2n[f(y)]^2 - (3n+1)\,yf(y) - \tfrac{1}{2}(2n+1)\,y^2 f(y) + \tfrac{3}{4}(2n+1)\,y^2\} \tag{14}$$

and

$$k(y) = (2n+1)\,x^{-2}\exp(-y/2)[f(y)]^{3/2}\,. \tag{15}$$

The nonvanishing colour electric fields can be easily calculated from the definition $E_a^i = F_a^{i0}$ and are found to be

$$E_1^i = 0\,,$$

$$E_2^i = [c(2n+1)/a^2]\,\epsilon_{i3j}(x_j/x\sin\theta) \times P_n^1(\cos\theta)\,x^{n-1}\exp(-y/2)[f(y)]^{1/2}\,,$$

$$E_3^i = [(2n+1)/2a^2](x_i/x)\,x^{2n-1}\exp(-y/2) \times \{(2n+1)\,y\,[f(y)]^{-3/2} - 4n[f(y)]^{-1/2}\}\,.$$

The electric fields are seen to be nonsingular and totally screened. The nonvanishing colour magnetic field is in the $\hat{1}$ direction in group space and asymptotically becomes

$$B_1 \underset{r \gg a}{\longrightarrow} (cn/a^2x^{n+2}) \times [\hat{r}(n+1)P_n(\cos\theta) - \hat{\boldsymbol{\theta}}P_n^1(\cos\theta)]\,, \tag{16}$$

the field due to a $(n, 0)$ magnetic multipole.

The energy carried by this solution can be obtained from the general formula

$$H_{\mathrm{na}} = \frac{1}{2}\sum_a \int \mathrm{d}^3r(|E_a|^2 + |B_a|^2) = W_1 + W_2 , \qquad (17)$$

where

$$W_1 = \frac{1}{2}\int \mathrm{d}^3r\, |\nabla\phi|^2 , \quad W_2 = \frac{1}{2}\int \mathrm{d}^3r\, 2\phi^2 A^2 . \quad (18,19)$$

We do not write down the explicit expressions for $W_1$ and $W_2$ as they are not very illuminating. The integral $W_1$ can be expressed in terms of the Riemann zeta-functions and the final results are

$$W_1 = \pi n/2a\,\Gamma(2n/(2n+1))$$
$$\times [(6n+2)(4n+1)\,\zeta((6n+2)/(2n+1))$$
$$- 2(4n^2+n-1)\,\zeta((4n+1)/(2n+1))] , \qquad (20)$$

and

$$W_2 = (2\pi/a)\,c^2(n+1)!/(n-1)! . \qquad (21)$$

It is interesting to compare explicitly the energy of this nonabelian solution with that of the Coulomb solution corresponding to the charge distribution (13). For this purpose it is helpful to use the expansion

$$[P_n'(\cos\theta)]^2 = \sum_{m=0}^{n} a_{2m}^n P_{2m}(\cos\theta) , \qquad (22)$$

where

$$a_m^n = \frac{(n+1)!}{(n-1)!}\,\frac{2m+1}{2n+1}\,C^m(n1, n1) , \qquad (23)$$

and

$$C^m(n1, n1) = \frac{2n+1}{2}\,\frac{(n-1)!}{(n+1)!}\int_{-1}^{+1} \mathrm{d}(\cos\theta)$$
$$\times P_m^0(\cos\theta)[P_n'(\cos\theta)]^2 . \qquad (24)$$

Some values of the $C^m(n1, n1)$ are available in the literature [7].

Using (6b), (13) and (22) we can perform the angular integrals and write

$$H_{\mathrm{c}} = V_1 + V_2 + V_3 , \qquad (25)$$

with

$$V_1 = \frac{4\pi}{a}\,\frac{1}{(2n+1)^2}\int_0^\infty \mathrm{d}y \int_0^y \mathrm{d}y'\, y^{-(2n-1)/(2n+1)}$$
$$\times y'^{-(2n-2)/(2n+1)}\, h(y)\, h(y') , \qquad (26)$$

$$V_2 = \frac{2\pi c^2}{a(2n+1)^2}\, a_0^n \int_0^\infty \mathrm{d}y \int_0^y \mathrm{d}y'\, y^{-(2n-1)/(2n+1)}$$
$$\times y'^{-(2n-2)/(2n+1)} [h(y)\, k(y') + h(y')\, k(y)] , \qquad (27)$$

and

$$V_3 = \frac{4\pi c^4}{a}\,\frac{1}{(2n+1)^2}\int_0^\infty \mathrm{d}y \int_0^y \mathrm{d}y' \sum_{m=0}^{n} (a_{2m}^n)^2\,\frac{1}{(4m+1)^2}$$
$$\times y^{-(2n+2m-1)/(2n+1)}\, y'^{-(2n-2m-2)/(2n+1)}$$
$$\times k(y)\, k(y') , \qquad (28)$$

(14) and (15) have to be substituted into eqs. (26), (27) and (28) to evaluate $H_{\mathrm{c}}$. Actually some labour is saved because it can be shown in general [8]:

$$W_1 = V_1 , \quad W_2 = V_2 . \qquad (29)$$

Thus only $V_3$ needs to be calculated. We have performed these integrations numerically for different values of $c$. The results for $n = 1$, 2 and 3 are plotted in fig. 1 as a function of the gauge-invariant charge $Q$ [see eq. (4)]. For a discussion see below. The solutions of the second class correspond to

$$A = (c/a)[P_n^1(\cos\theta)/x^{n+1}]\ \tanh y , \qquad (30a)$$

$$\phi = [\sqrt{2}(2n+1)/a]\ x^{2n}\ \mathrm{sech}\, y , \qquad (30b)$$

where $x$ and $y$ are defined in (11) and (12). The $n = 1$ case is the Sikivie–Weiss magnetic dipole solution [2]. The source corresponding to this solution can be obtained using eq. (9a) and written in the form of eq. (13) with

$$h(y) = -(2n+1)^2\sqrt{2}\,\mathrm{sech}^3 y\; y^{-3/(2n+1)}$$
$$\times \{2[ny\cosh^2 y - (3n+1)y^2 \sinh y \cosh y]$$
$$+ (2n+1)\,y^3(\sinh^2 y - 1)\} , \qquad (31)$$

and

$$k(y) = \sqrt{2}(2n+1)\, y^{-2/(2n+1)} \tanh^2 y\ \mathrm{sech}\, y . \qquad (32)$$

The colour electric fields are again totally screened.

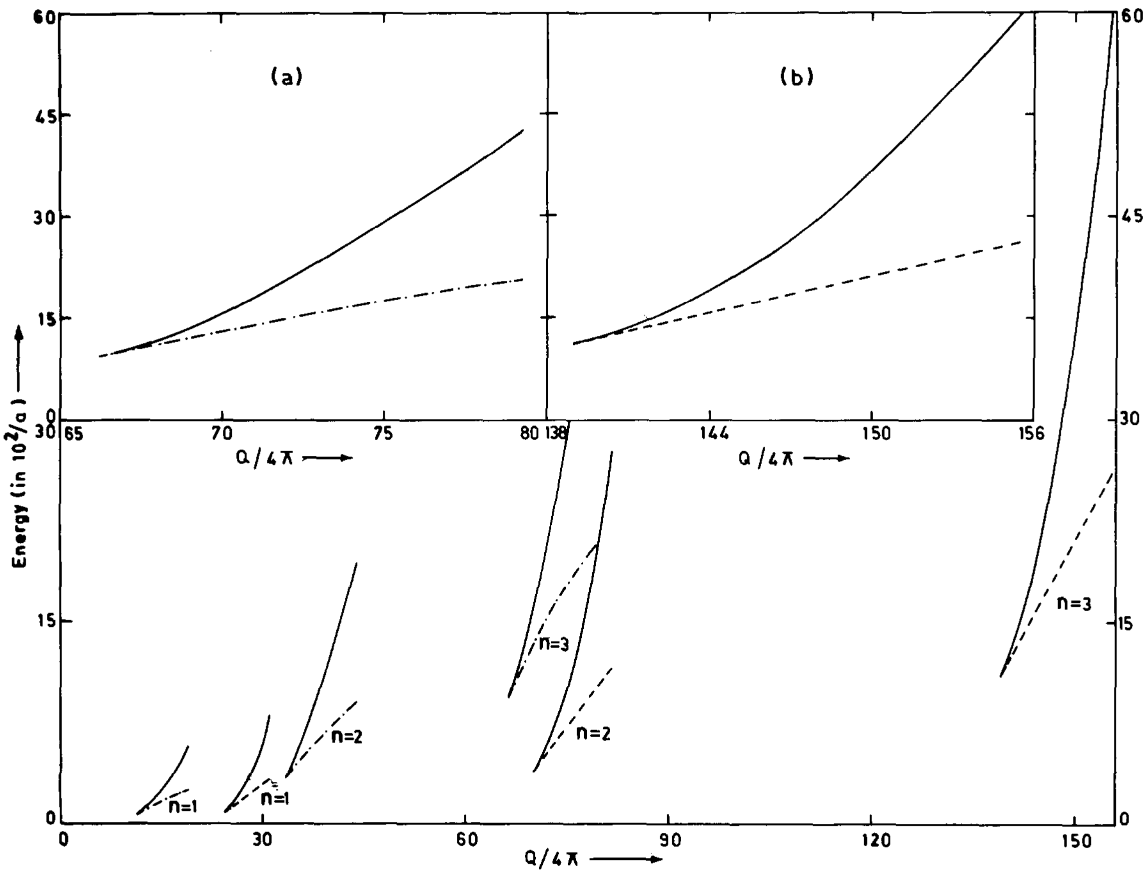


Fig. 1. The energy of the Coulomb solutions (solid lines) and the nonabelian solutions (dot–dashed lines for class one and dashed lines for class two) for $n$ = 1, 2 and 3 are plotted as functions of the gauge invariant charge. In the insets (a), (b) the $n$ = 3 solution of class one (two) is drawn in a larger scale.

$$E_1^i = 0 ,$$
$$E_2^i = [c\sqrt{2}(2n+1)/a^2]\ \epsilon_{i3j}(x_j/x \sin\theta) \times P_n^1(\cos\theta)\, x^{n-1} \tanh y\ \mathrm{sech}\, y ,$$
$$E_3^i = -[\sqrt{2}(2n+1)/a^2]\,(x_i/x)\, x^{2n-1} \times \mathrm{sech}\, y\,[2n-(2n+1)\,y\tanh y] .$$

Since asymptotically ($r \gg a$) (10a) and (30a) tend to the same limit the asymptotic form of the magnetic field is the same as (16).

The energy of this nonabelian solution can again be written in the form (17) with

$$W_1 = \frac{4\pi(2n+1)}{a} \int_0^\infty \mathrm{d}y\ y^{2n/(2n+1)}\ \mathrm{sech}^2 y \times [4n^2 - 4n(2n+1)\,y \tanh y + (2n+1)^2\, y^2 \tanh^2 y] , \tag{33}$$
$$W_2 = (8\pi c^2/3a)(n+1)!/(n-1)! . \tag{34}$$

(33) has been evaluated numerically for the first few values of $n$. The energy of the Coulomb solution can again be written in the form of eqs. (25)–(28) with $h(y)$ and $k(y)$ now given by eqs. (31), (32). The general relation (29) is again valid and only (28) has to be numerically calculated. We have plotted both $H_c$ and $H_{na}$ as a function of the gauge invariant charge, $Q$, in fig. 1.

As already discussed below eq. (8) it is known that for any $Q, H_c > H_{na}$. The bifurcating structure of the solutions of both classes resembles a similar feature of some of the solutions found by Jackiw et al. [3]. However, unlike their solutions here the bifurcation is between the *Coulomb solution* and a nonabelian solution. Since the Coulomb solution is the higher-energy branch it can be expected to be unstable [1].

It is to be noted that $H_c$ as a function of $Q$ is not a parabola passing through the origin, as it seems to have been suggested in some published literature. This comes about since changing the parameter $c$ not only changes the charge $Q$ but also the shape of the charge distribution.

We conclude with a few remarks. It is seen that the critical charge increases with increasing $n$ for both classes of solutions. It may be noteworthy that in this general axially symmetric framework it is impossible to find solutions with zero critical charge [8]. In this letter we have restricted ourselves to the gauge group SU(2). It can be shown that for the group SU($n$) with $n > 2$ any solution of this type is an embedding of an SU(2) solution in this group [8].

In summary, we have presented new axially symmetric static solutions of the SU(2) Yang–Mills equations with external sources. We find that the solutions are always of lower energy than the corresponding Coulomb solutions. The colour electric fields have a total screening property while the colour magnetic field has a multipole-like behaviour. There is a critical charge below which the solutions do not exist and the energy exhibits a bifurcating behaviour as a function of the total charge $Q$.

U.S. wishes to thank Dr. A.K. Das of Calcutta University and Dr. P.K. Roy of The B.E.S. College for encouragement. A.R. thanks Professor A.K. Raychaudhuri of Presidency College for helpful discussions. Some of this work was carried out while A.R. was attending the Winter School in Particle Physics, 1981 at Kalpakkam and he is grateful to the organisers for their generous hospitality.

*Note added:* After this paper was submitted for publication we received a preprint by Rosy Teh et al. [9] in which some other axially symmetric solutions with multipole-like asymptotic magnetic fields are discussed.

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