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New characterizations of proper interval bigraphs

Ashok Kumar Das*, Ritapa Chakraborty

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-700019, India

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Abstract

A proper interval bigraph is a bigraph where to each vertex we can assign a closed interval such that the intervals can be chosen to be inclusion free and vertices in the opposite partite sets are adjacent when the corresponding intervals intersect. In this paper, we introduce the notion of astral triple of edges and along the lines of characterization of interval graphs via the absence of asteroidal triple of vertices we characterize proper interval bigraphs via the absence of astral triple of edges. We also characterize proper interval bigraphs in terms of dominating pair of vertices as defined by Corneil et al. Tucker characterized proper circular arc graphs in terms of circularly compatible 1's of adjacency matrices. Sen and Sanyal characterized adjacency matrices of proper interval bigraphs in terms of monotone consecutive arrangement. We have shown an interrelation between these two concepts.

Keywords: Astral triple of edges; Monotone consecutive arrangement; Zero partitionable; Circularly compatible 1's; Dominating pair of vertices

1. Introduction

The class of *interval graphs* (the intersection graph of intervals) is one of the most well studied class of graphs. *Proper interval graphs* are the intersection graphs of intervals in which no interval properly contains another in the interval model. *Unit interval graphs* are the intersection graphs of intervals of unit length. Roberts [1] introduced and showed that these two subclasses of interval graphs are equivalent.

Analogously, interval digraphs and interval bigraphs were introduced in [2] and [3] respectively.

A digraph is an *interval digraph* if to each vertex v there corresponds an ordered pair (S_v, T_v) of closed intervals such that there is an edge from vertex u to vertex v if and only if S_u intersects T_v. The sets S_v and T_v are the *source set* and *sink set* for v. An *interval bigraph* is a bipartite graph representable by assigning each vertex v an interval so that vertices in opposite partite sets are adjacent if and only if their intervals intersect. The *biadjacency matrix* of a bigraph is the submatrix of the adjacency matrix consisting of the rows indexed by one partite set and the columns indexed by the other. In this paper we denote the biadjacency matrix of B by $A(B)$.

As observed in [4] and [5], the two concepts of interval digraphs and interval bigraphs are equivalent.

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* Corresponding author.
E-mail addresses: ashokdas.cu@gmail.com (A.K. Das), ritapa.chakroborty@gmail.com (R. Chakraborty).

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Suppose D is a digraph with $V(D) = \{v_1, \ldots, v_n\}$. Define $B(D)$ to be the bipartite graph with fixed partition (X, Y) where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ with $x_i y_j$ an edge in $B(D)$ if and only if there is a directed edge from v_i to v_j in D.

Conversely, an interval bigraph becomes an interval digraph by adding isolated vertices to equalize the partite sets and then arbitrarily combining pairs of vertices from the two partite sets; the sets for two paired vertices in the representation of the bigraph become the source set and sink set of the new combined vertex.

The point is that the adjacency matrix of an interval digraph is the bi-adjacency matrix of an interval bigraph and conversely the bi-adjacency matrix of an interval bigraph becomes the adjacency matrix of an interval digraph when rows or columns of 0s are added to make it a square.

Interval digraphs and bigraphs and their relations to other families of graphs have been extensively studied in [6–9,2].

Sen and Sanyal [10] introduced the concept of proper interval digraphs where no source interval properly contains another source interval and no sink interval properly contains another sink interval. Recently Lundgren and Brown [6] generalized this definition as follows. A bigraph $B = (X, Y, E)$ is a *proper interval bigraph* if its vertices can be represented by a family of intervals I_v, $v \in X \bigcup Y$, with the property that no interval properly contains another and, for $x \in X$ and $y \in Y$, x and y are adjacent in B if and only if I_x and I_y intersect.

A unit interval bigraph is an interval bigraph where all the intervals are of same length. Sen and Sanyal [10] showed that the concepts of proper interval bigraphs and unit interval bigraphs are equivalent.

A *zero partition* of a binary matrix (i.e. a 0, 1 matrix) is a coloring of each zero with R or C in such a way that every R has only 0*s* colored R to its right and every C has only 0*s* colored C below it. A matrix that admits a zero partition after suitable row and column permutation is *zero partitionable*. A *circular arc graph* is the intersection graph of a set of circular arcs of a host circle. Circular arc graphs have been extensively studied by Tucker [11,12] (also see [13]). A circular arc graph is a *proper circular arc graph* if the circular arc representation can be chosen to be inclusion free. If the vertices of a graph G can be covered by two cliques, then we say that G is a *two-clique graph*. Two-clique circular arc graphs have arisen as an important subclass of circular arc graphs and have been characterized in several ways by Tucker [14], Trotter and Moore [15] and Spinard [16].

Following theorem characterizes interval bigraph in two ways.

Theorem 1. *For a bipartite graph B the following statements are equivalent:*

(a) *B is an interval bigraph*

(b) *A(B) is zero partitionable* [2]

(c) $\bar{B}$ *is a two clique circular arc graph with a representation in which no two arcs together cover the host circle* [9].

Lekkerkerker and Boland [17] defined *asteroidal triple of vertices* to be an independent set of three vertices such that each pair of vertices is joined by a path that avoids the neighborhood of the third. They also characterized the interval graphs as chordal graphs which are free from asteroidal triple of vertices.

A set of three edges in a bigraph is said to form an *asteroidal triple of edges* (ATE) [18,4] if for any two of them there is a path between two that avoids the neighborhood of the third. A bigraph that does not contain $2K_2$ as an induced subgraph is called a *Ferrers bigraph*, and its biadjacency matrix is called a *Ferrers matrix*. It is easy to see that a binary matrix containing only one zero is a Ferrers matrix. So every binary matrix can be expressed as the intersection of finite number of Ferrers matrices. The minimum number of Ferrers bigraphs whose intersection is a given bigraph B is called the Ferrers dimension of B. In [5] it was shown that the bigraphs of Ferrers dimension 2 are free from asteroidal triple of edges. Since the class of interval bigraphs is properly contained in the class of bigraphs of Ferrers dimension 2 [2], so we can state

Theorem 2. *If a bigraph B has an asteroidal triple of edges, then B is not an interval bigraph.*

As observed in [18] a bigraph may not contain an ATE but not of Ferrers dimension 2. Thus the converse of the above theorem is not true.

2. Preliminary results

Let $B = (X, Y, E)$ be a bigraph where $|X| = n$ and $|Y| = m$. Then the biadjacency matrix *A(B)* of B has a *monotone consecutive arrangement (MCA)* if and only if it has independent row and column permutations such that the 1's appear consecutively in each row and the values $\{a_i\}$ and $\{b_i\}$ denoting the initial column and final column of the interval of 1's in the ith row satisfy $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$.

Sen and Sanyal [10] characterized proper interval digraphs/bigraphs in the following way.

Theorem 3 (*[10]*). *For a bipartite graph B the following statements are equivalent:*
(a) *B is a proper interval bigraph*
(b) *A(B) has a MCA.*

To characterize proper interval graphs, Jackowski [19] introduced the notion of astral triple of vertices. Three vertices u, v, w in a graph G form an *astral triple of vertices* if between any two of them there exists a path P in G such that the third vertex of the triple does not belong to P and any two adjacent vertices in P are not both adjacent to the third vertex of the triple. Analogously in this paper we introduce the notion of astral triple of edges in a graph and characterize proper interval bigraphs in terms of this notion. A set of three edges e_1, e_2, e_3 in a graph G is an *astral triple of edges* if for any two there is a path connecting them which (1) contains no vertex of the third and (2) does not contain adjacent vertices which are neighbors of the third.

Example 1. The first two graphs G_1 and G_2 of Fig. 1 do not contain astral triple of edges. But for the graph G_3, $e_1 = v_1v_2$, $e_2 = v_3v_4$ and $e_3 = v_5v_6$ is an astral triple of edges because the path $v_3\ v_7\ v_5$ from e_2 to e_3 has no two adjacent vertices which are both neighbors of e_1. On the other hand in the first two graphs any path between e_2 and e_3 contains adjacent vertices which are neighbors of the third edge.

A bigraph is said to be *chordal bipartite* or *bichordal* if it does not contain C_{2n} $(n \ge 3)$ as an induced subgraph. It is easy to verify that any three non consecutive edges of C_{2n} $(n \ge 3)$ form an astral triple of edges. So we can conclude that if a bigraph has no astral triple of edges, then it is bichordal.

Example 2. The bigraph in Fig. 2 is an interval bigraph but it contains astral triple of edges as well as asteroidal triple of vertices.

Lin and West [20] characterized adjacency matrices of proper interval bigraphs in terms of forbidden submatrices. In particular, they proved

Theorem 4 (*[20]*). *A zero partitionable matrix has a MCA if and only if it does not contain any of the three* 3 *by* 4 *matrices listed below or their transposes as a submatrix.*

$$\mathbf{F_1} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{F_2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{F_3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The bigraphs corresponding to these matrices are respectively the following bigraphs.

It is easy to verify that the bigraphs of Fig. 3 contain astral triple of edges. Since the interval bigraphs must be bichordal, so we can conclude the following:

Proposition 1 (*[21,9]*). *If a bigraph B is bichordal and does not contain any bigraphs of Fig. 3 as an induced subgraph, then B is a proper interval bigraph.*

Hell and Huang [9] showed the equivalence between several classes of structured graphs (proper interval bigraphs, complements of proper circular arc graphs, asteroidal triple-free graphs, permutation graphs and co-comparability graphs). In a recent paper Brown and Lundgren [6] list 20 equivalent characterizations of proper interval bigraphs as presented in the literature by several researchers.

In this paper we give few additional characterizations of proper interval bigraphs.

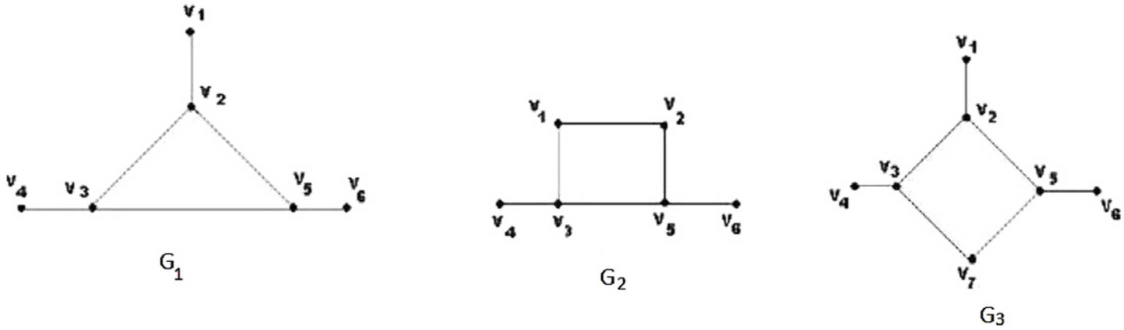

Fig. 1. Various examples and non examples of astral triple of edges.

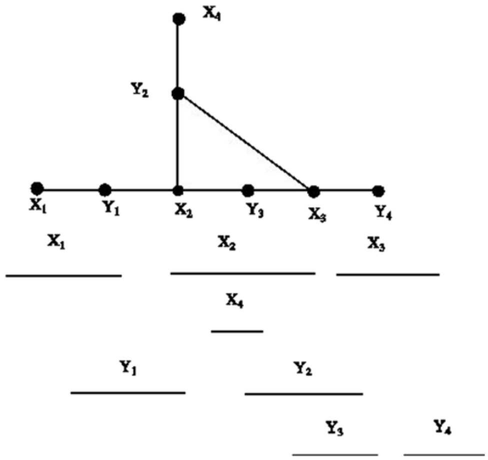

Fig. 2. An interval bigraph with its interval representation.

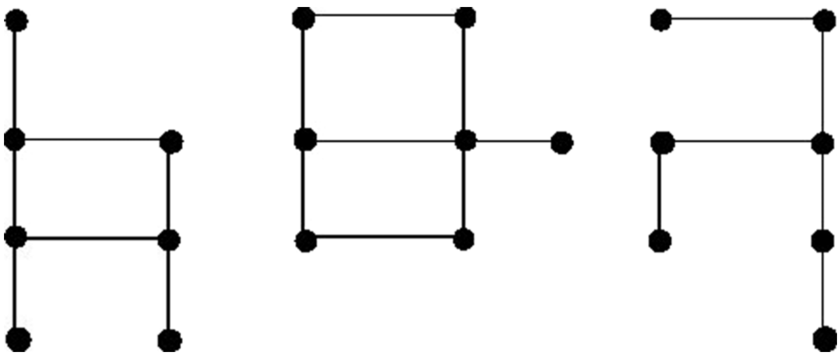

Fig. 3. Forbidden bichordal bigraphs of proper interval bigraphs.

3. Main results

In the next theorem we provide a characterization of proper interval bigraphs by replacing the prohibition against the chordless cycle of length at least six and the bigraphs in Fig. 3 with the prohibition against astral triple of edges.

Theorem 5. *Let B be a bipartite graph. Then B is a proper interval bigraph if and only if B has no astral triple of edges.*

Proof. Let $B = (X, Y, E)$ be a connected proper interval bigraph with $|X| = n$ and $|Y| = m$. Then rows and columns of the adjacency matrix *A(B)* of *B* can be permuted independently so that *A(B)* has a monotone consecutive

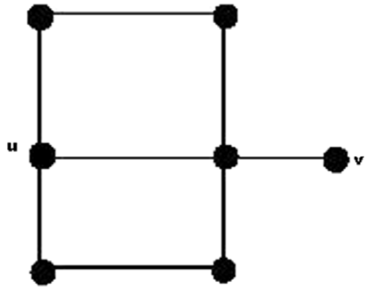

Fig. 4. The bigraph F.

arrangement. Note that monotone consecutive arrangement of the adjacency matrix has not a $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ decomposition because then B becomes a disconnected graph. Suppose $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ are respectively the ordering of rows and columns in the monotone consecutive arrangement of *A(B)*.

Now in *A(B)*, we consider disjoint maximal blocks (consisting of at least two rows and two columns) of 1. We also consider the maximal blocks in such a way that the blocks cover maximum number of 1's of $A(B)$. Also observe that the choice of the maximal blocks is not unique. Consider any two such blocks M' and M''. If y_i and y_j are respectively the first and last columns of the blocks M' and M'', then MCA of $A(B)$ and the maximality of the blocks guarantee that $i < j$. Also if x_r and x_s are respectively the first and last rows of M' and M'', then $r < s$. Thus we have a linear ordering of the blocks according to the increasing order of the suffix of the last columns or last rows of the blocks. It may be noted that these blocks may not cover all the 1's of $A(B)$.

Now we shall show that any three edges of B do not form an astral triple. We consider the following cases.

Case 1. Suppose $e_1 = x_i y_{i'}$, $e_2 = x_j y_{j'}$, $e_3 = x_k y_{k'}$ are respectively three edges i.e. three 1's (where i and i', j and j', k and k' may be equal) in any three blocks M_r, M_s, M_t where $r < s < t$.

Consider any path $P_{1,3}$ from e_1 to e_3, where $P_{1,3} : v_i v_l \cdots v_m v_n \cdots v_k$, where v_i and v_k are respectively one end vertex of e_1 and e_3. Then MCA of *A(B)* guarantees that this path must contain a row (column) vertex followed by a column (row) vertex of M_s. Thus $P_{1,3}$ contains two consecutive vertices which are neighbors of e_2. Hence e_1, e_2, e_3 do not form an astral triple.

Case 2. Suppose $e_1 = x_i y_{i'}$, $e_2 = x_j y_{j'}$, $e_3 = x_k y_{k'}$ are three edges that do not belong to any of the blocks (where $i < j < k$ and $i' < j' < k'$).

Now for any path $P_{1,3}$ from e_1 to e_3, MCA of *A(B)* guarantees that e_2 must be adjacent some y_l ($x_{l'}$) and then followed by some x_r ($y_{r'}$) in this path. So e_2 is adjacent to two consecutive vertices of $P_{1,3}$ and hence e_1, e_2, e_3 do not form an astral triple of edges.

Similarly in other cases (e.g., if two edges belong to two different blocks but one edge does not belong to any block) we can prove that B does not have an astral triple of edges.

Conversely, suppose B does not contain an astral triple of edges. Then as observed in Section 2, B must be bichordal. Also B does not contain any of the graphs of Fig. 3 as an induced subgraph. Therefore by Proposition 1, B is a proper interval bigraph. ■

A set S of vertices of a graph G is said to be *dominating* if every vertex outside S is adjacent to at least one vertex of S. A path joining vertices u and v is termed as u, v-path. Corneil et al. [22] defined a pair (u, v) of vertices of G as *dominating pair* if all u, v-paths are dominating. In a bigraph B we define a pair (u, v) of vertices as a *dominating pair* where u and v are non adjacent and all u, v-paths are dominating.

In this paper we show that the proper interval bigraphs also contain a dominating pair of vertices. It is worthwhile to note that an interval bigraph may not contain dominating pair of vertices (Example 2).

It is also interesting to observe that the middle graph of Fig. 3, which is not a proper interval bigraph has (u, v) as a dominating pair of vertices (see Fig. 4).

Now in the following theorem we give a new characterization of proper interval bigraphs in terms of dominating pair of vertices.

Theorem 6. *A bipartite graph B is a proper interval bigraph if and only if B is bichordal, does not contain the graph F as an induced subgraph and has a dominating pair of vertices.*

$\mathbf{A}' =$

	X	Y
X	1	A^c
Y	$(A^c)^T$	1

Fig. 5(a). The biadjacency matrix of A'.

Proof. Let B be a proper interval bigraph. Then as observed before B must be bichordal and does not contain the graph F as an induced subgraph. As in the previous theorem suppose $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ are respectively the ordering of rows and columns in the monotone consecutive arrangement of $A(B)$. Now we shall show that (x_1, y_m) is a dominating pair of vertices of B. We consider a path $x_1 y_1 x_i y_j x_k y_l \cdots y_m$ in B where in the 1st column ith row has the last 1 and in the ith row jth column has the last 1. In the jth column kth row has the 1 and so on. Now this path is a dominating path since the vertices $x_2, \ldots, x_{i-1}$ are adjacent to y_1; the vertices $y_2, \ldots, y_{j-1}$ are adjacent to x_i; the vertices $x_{i+1}, \ldots, x_{k-1}$ are adjacent to y_j. Similarly it can be shown that every vertex of B which is outside of this path is adjacent to one vertex of this path. Similarly if we consider any other path P from x_1 to y_m, then MCA of $A(B)$ guarantees that this path is a dominating path. Thus B has a dominating pair of vertices.

Conversely, suppose $B = (X, Y, E)$ be chordal bipartite, does not contain F as an induced subgraph and has a dominating pair (u, v) of vertices where $u, v \in X \bigcup Y$. Since the first and the third graph in Fig. 3 does not contain a dominating pair of vertices, the bigraph B is bichordal and does not contain the graphs in Fig. 3. Thus by Proposition 1, B is a proper interval bigraph. ■

In the sequel of the above two theorems we prove that chordal bipartite graph $B = (X, Y, E)$, does not contain F as an induced subgraph and has a dominating pair (u, v) of vertices where $u, v \in X \bigcup Y$, B contains no astral triple of edges. On the contrary suppose e_1, e_2 and e_3 form an astral triple of edges of B. Let $e_k = x_k y_k$ where $k = 1, 2, 3$.

Now consider a u, v-path P. Obviously all the three edges e_1, e_2, e_3 do not belong to P as e_1, e_2, e_3 form an astral triple of edges. So at least one edge say e_1 does not belong to P. Since P is a dominating path x_1 and y_1 are adjacent to P. Again as B is bichordal, there exist two vertices $x', y' \in$ P such that $x_1 y', x' y_1 \in$ E and $x' y' \in$ E. Since e_1, e_2, e_3 form an astral triple of edges at most one edge of e_1, e_2 and e_3 may belong to P. For the same reason if one edge e_k (say e_3) belongs to P, then one end vertex of e_3 should be either u or v. Again if e_2 does not belong to P, then as before e_2 is adjacent to two consecutive vertices of P. Now if all the three edges e_1, e_2, e_3 have different neighbors in P, then the three edges e_1, e_2, e_3 do not form an astral triple of edges. Similarly if e_2, e_3 have only one common neighbor in P, then e_1, e_2, e_3 also do not form an astral triple of edges. Next if two edges e_2, e_3 have at least two common neighbors in P, then a careful scrutiny shows that either B is not bichordal or contains F as an induced subgraph of B. Thus in either case we arrive at a contradiction. Thus B contains no astral triple of edges.

To characterize proper circular arc graphs Tucker [11] introduced the notion of *circularly compatible 1's*. A symmetric (0, 1) matrix is said to have circularly compatible 1's if the 1's in each column are circular and if, after inverting and/or circularly permuting the order of the rows and (corresponding) columns, the last 1 (in cyclicly descending order) of the circular set in the second column is always at least as low as the last 1 of the circular set in the first column unless one of these columns is all 1's or all 0's.

The next theorem shows the relationship between MCA and circularly compatible 1's property of a binary matrix (via the appropriate transformation).

Theorem 7. *For a bipartite graph $B = (X, Y, E)$ the following statements are equivalent:*
(i) *B is a proper interval bigraph;*
(ii) *$J - A(B)$ has circularly compatible 1's (J is a suitably sized matrix of all 1's).*

Proof. (i) $\Longrightarrow$ (ii) For brevity, let us denote the biadjacency matrix *A(B)* of B by A and the matrix $J - A(B)$ by A'. Since B is a proper interval bigraph, there exist independent row and column permutations of A such that A has a monotone consecutive arrangement. Consider the matrix A'. It has the following structure, (see Fig. 5(a)) where A^c denotes the complement of the biadjacency matrix A (interchanging the 0s and 1s of A) and $(A^c)^T$ denotes the transpose of A^c, and we arrange the vertices of X- partite and Y- partite set so that A exhibits a monotone consecutive arrangement. Then in A^c and $(A^c)^T$ the 1's are circular in each column and accordingly 1s are also circular in each column of A'. Also in any column of A' the last 1 (in cyclicly descending order) of the circular set is always at least as low as the last 1 of the circular set of the previous column. This implies that the matrix A' has circularly compatible 1's.

(ii) $\Longrightarrow$ (i) Conversely, let the matrix A' have circularly compatible 1's with the arrangement of rows and columns as in Fig. 5(b). Now the circular compatibility of 1's implies that 0's are consecutive in each row of M and moreover

$$\mathbf{A}' = \begin{array}{c|c|c|} & X & Y \\ \hline X & 1 & M \\ \hline Y & M^T & 1 \\ \hline \end{array}$$

Fig. 5(b). A matrix with circularly compatible 1's.

these 0's have monotone consecutive arrangement. Therefore A has monotone consecutive arrangement and consequently B is a proper interval bigraph. ■

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