

Near-horizon geometry and the entropy of a minimally coupled scalar field in a Kerr black hole

Kaushik Ghosh*

Vivekananda College, University of Calcutta, 269 Diamond Harbour Road, Kolkata 700063, India

*E-mail: ghosh_kaushik06@yahoo.co.in

Received June 17, 2016; Revised July 25, 2016; Accepted July 25, 2016; Published September 30, 2016

.....
In this article we will discuss a Lorentzian sector calculation of the entropy of a minimally coupled scalar field in a Kerr black hole background. We will use the brick wall model of 't Hooft. In a Kerr black hole, complications arise due to the absence of a global timelike Killing field and the presence of the ergosphere. Nevertheless, it is possible to calculate the entropy of a thin shell of matter field in the near-horizon region using the brick wall model. The corresponding leading-order entropy of the nonsuperradiant modes is found to be proportional to the area of the horizon and is logarithmically divergent. Thus, the entropy of a three-dimensional system in the near-horizon region is proportional to the boundary surface. This aspect is also valid in the Schwarzschild black holes and is similar to that of the black hole entropy itself. The corresponding internal energy remains finite if the entropy is chosen to be of the order of the black hole entropy itself. For a fixed value of the brick wall cut-off, the leading order entropy in a Kerr black hole is found to be half of the corresponding term in a Schwarzschild black hole. This is due to rotation and is consistent with the preferential emission of particles in a Kerr black hole with azimuthal angular momentum in the same direction as that of the black hole itself. However, we can obtain the Schwarzschild case expression by including a subleading term and taking the appropriate slow rotation limit.
.....

Subject Index E76

1. Introduction

Since the four laws of black hole mechanics were formulated [1], there has been much effort to relate the laws of black hole mechanics to those of thermodynamics. The area theorem led Bekenstein to assign an entropy to a black hole [2]. He proposed a generalized second law of thermodynamics by considering the sum of a multiple of the area of horizon and the entropy of ordinary matter moving in the black hole background. A black hole can be assigned a temperature equal to a multiple of the surface gravity of the horizon. However, unless the black hole can emit particles, the second law can be violated by considering a black body radiation surrounding the black hole at a lower temperature. Hawking established the thermodynamical aspects of black holes by showing that a black hole can radiate like a hot body at a temperature equal to a multiple of the surface gravity of the horizon [3]. The entropy of a black hole, considered as a thermodynamical system, was found to be $\frac{A}{4}$. Here, A is the area of the horizon.

We have to consider quantum field theory in curved spaces in most of the works related to black hole thermodynamics. This is a nontrivial issue since, in a curved space, the vacuum and finite particle states are dependent on the observer. We do not face this problem for quantum field theory in flat space

due to Poincaré invariance. However, even in flat space, a uniformly accelerated observer detects a thermal spectrum when the field is in the Minkowski vacuum [4,5]. The temperature is dependent on the proper acceleration of the observer. A corresponding situation arises in a Schwarzschild black hole for a static observer outside the event horizon. The temperature is the same as that of the black hole if the static observer is at a large distance from the horizon. The corresponding vacuum is the Hartle–Hawking vacuum [6]. G. 't Hooft proposed a model to calculate the entropy of a scalar field in a Schwarzschild black hole in thermal equilibrium with the black hole [7]. He used the Wentzel–Kramers–Brillouin (WKB) approximation to count the states. We can define a thermal equilibrium between a black hole and a black body radiation surrounding the black hole at the same temperature as that of the black hole. The amount of radiation absorbed and emitted by the black hole is the same. This is an unstable equilibrium due to the negative specific heat of the black hole, which is characteristic of self-gravitating systems [8]. This is discussed for a few systems in Ref. [3]. A small perturbation in the background can give rise to a runaway situation. Divergences appear when we try to calculate the entropy of the scalar field [7]. This is associated with the continuous energy spectrum and unboundedness of the angular quantum numbers of the scalar wave equation. To regulate the divergence, 't Hooft proposed a boundary condition on the scalar field near the horizon. He assumed the scalar field to be vanishing at a small distance away from the horizon. This radial parameter is known as the brick wall cut-off parameter. This boundary condition is a good model since in thermodynamic equilibrium there is no net interchange of particles between the black hole and the surrounding matter. The boundary condition is similar to a perfectly reflecting mirror. An observer associated with the horizon detects no particle [6] and the brick wall boundary condition is reasonable in this respect. We can easily construct solutions of the scalar wave equation that vanish at the brick wall [9]. These are similar to the solutions having zero Cauchy data on the horizon and are considered in Hawking radiation. This will be discussed for the massless field in Sect. 2.

The brick wall boundary condition was also discussed in detail by 't Hooft [7]. The WKB quantization rule together with the brick wall boundary condition introduce the brick wall cut-off parameter as a regularizing parameter on the otherwise continuous energy spectrum of the scalar field wave equation. This is an advantage of the brick wall model where a mathematical cut-off can be directly expressed in terms of a physically relevant parameter. The expression of entropy obtained by 't Hooft is linearly divergent in the brick wall cut-off parameter. However, the metric component g_{rr} has a simple pole at the horizon. The WKB quantization rule indicates that the divergence is expected to be logarithmic. In a Schwarzschild black hole, we can also use the blueshift factor together with the form of the proper distance in the near-horizon region to understand this logarithmic divergence. This is also evident from the nature of the solutions in the near-horizon region and will be discussed in the later sections. A logarithmically divergent expression for the scalar field entropy in the Bañados–Teitelboim–Zanelli black hole is obtained in [10] using a different method. Similar behavior is obtained in the optical metric approach [11].

In a series of recently published articles [12–14], we have shown that an improved counting of the scalar field states leads to a logarithmically divergent expression for the scalar field entropy. We have found that the WKB approximation is particularly suitable for the scalar field solutions that are stationary in the near-horizon region, and again at a large distance away from the horizon. This approximately corresponds to a thin shell of scalar field confined in the near-horizon region due to the large width of the intermediate potential barrier [15]. We will find that the thickness of the thin shell takes care of the unbounded nature of the allowed angular quantum numbers. The divergences can be removed by considering the stress tensor, which is well behaved when the field is in a

Hartle–Hawking vacuum [16]. 't Hooft removed the divergence by equating the scalar field entropy to the black hole entropy. When we do so, the internal energy is found to be finite and the brick wall is almost coincident with the horizon [12]. This is unlike the case in [7], where the brick wall cut-off is of the order of the Planck length. The most interesting point to note is that the entropy of the thin shell does not contain a term proportional to the volume. Instead, we have the leading-order term proportional to the area of the horizon. Another interesting aspect of our work is that the proper distance thickness of the thin shell together with the brick wall cut-off parameter behave like a scale parameter similar to the Planck length in the gravitational entropy. Thus, the degrees of freedom of a three-dimensional thin shell are related to the boundary surface area. This is similar to the gravitational entropy of the black hole itself.

A few authors have used the brick wall model of 't Hooft to calculate the scalar field entropy for various purposes; the relevant references will be given in Sect. 3. A more complete set of references can be found in [12,17]. The present article is also significant in this respect. The scalar field entropy can be calculated from the Euclidean sector of the black hole [6,17–22]. One again considers the near-horizon region with boundary conditions similar to those considered by us. The scalar field entropy is again proportional to the horizon surface area. However, the scalar field entropy contains an undetermined parameter. This parameter can be fixed by comparing with the Lorentzian sector expression. Thus the logarithmically divergent expression of the Lorentzian sector entropy makes the Euclidean sector expressions only logarithmically divergent compared to the quadratically and quartically divergent expressions obtained earlier. These discussions are given in detail in [12].

In the present article we will consider the entropy of a minimally coupled scalar field in a Kerr black hole. We will consider again a thin shell of scalar field confined in the near horizon region. We are considering scalar field modes that are stationary in the near-horizon region and again at a far distance from the horizon. The amplitude at asymptotic infinity is negligible. A Kerr black hole is stationary and only axisymmetric. There does not exist a global timelike Killing vector field. The metric component g_{rr} has a simple pole at the outer horizon. Thus the brick wall model is expected to give a logarithmically divergent expression. This appears to be true, and the scalar field entropy is again proportional to the surface area of the horizon. The time translation generator $(\frac{\partial}{\partial t})^a$ is a Killing field, but unlike the Schwarzschild case it is not globally timelike. This Killing field becomes spacelike within the ergosphere, which is lying in the near-horizon region surrounding the black hole. The other Killing field $(\frac{\partial}{\partial \phi})^a$ is spacelike everywhere. Here, we have used the notation of Wald [8]. Since we are considering an observer at a large distance from the horizon, we can use the Killing field $(\frac{\partial}{\partial t})^a$ to define energy with respect to that observer. The solutions are taken to be stationary with respect to t, ϕ . In a Kerr black hole we have two sectors of solutions. The first sector contains the superradiant modes [23,24] and the second sector consists of the ordinary solutions. The argument in the exponential of the partition function becomes negative for a superradiant mode [3]. The superradiant modes give stimulated radiation. Their contribution to the thermal radiation and hence to the corresponding entropy is not very significant. The contribution of the ordinary modes is similar to the Schwarzschild black hole. The corresponding leading-order entropy is proportional to the area of the event horizon and is logarithmically divergent in the brick wall cut-off parameter. Although the Killing vector $(\frac{\partial}{\partial t})^a$ is not timelike inside the ergosphere, the ergosphere vanishes at the polar axis. The Killing vector $(\frac{\partial}{\partial t})^a$ remains timelike along the polar axis and is null on the horizon at $\theta = 0, \pi$. Thus we can consider a timelike geodesic starting at a suitable position at a large distance away from the horizon and intersecting the horizon at $\theta = 0$ or π without ever entering the ergosphere. We can now apply the blueshift argument to understand the logarithmic divergence.

There is an important but expected difference between the scalar field entropies in the Kerr black holes and the Schwarzschild black holes. For a fixed value of the brick wall cut-off, the leading-order entropy of a thin shell of scalar field confined in the near-horizon region in a Kerr black hole is half of the corresponding expression for the Schwarzschild case. This is expected due the preferential emission of particles in the Kerr black holes with azimuthal angular momentum in the same direction as that of the black hole itself. The thin shell in thermal equilibrium preferentially contains particles with azimuthal angular momentum in the same direction as that of the black hole. This is consistent with the frame dragging effect in the near-horizon region. However, we can obtain the Schwarzschild case expression by including a subleading term and taking the appropriate slow rotation limit. This is discussed in detail in Sect. 2. Another important point to note is that the proper radial thickness of the thin shell is dependent on the polar angle, although the radial coordinate thickness and the entropy is independent of the polar angle. We will discuss these issues in Sect. 2.

The Lorentzian sector calculation of the scalar field entropy in the Kerr black holes is more significant since the corresponding Euclidean sector literature is not well formulated [20,21]. The expressions obtained in this article are significant in the context of entanglement entropy and the holographic principles. We will discuss these aspects and the significance of the results obtained in this article in Sect. 3.

2. Entropy of a thin shell of scalar field in the Kerr black holes

In this section we will calculate the entropy of a minimally coupled scalar field in a Kerr black hole. We will discuss the case of a massless field, but the results can easily be extended for the massive case [14]. We will follow the method discussed in [12,13] and consider a thin shell of scalar field surrounding the horizon.

The spacetime metric of a Kerr black hole is given by the following expression:

$$ds^2 = -\frac{\Delta}{\rho^2}[dt - a(\sin^2\theta)d\phi]^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (1)$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2. \quad (2)$$

Here, M is the mass and a is the angular momentum per unit mass of the black hole.

The mass and angular momentum are defined with respect to an asymptotic observer. The position of the horizons are given by the zeros of Δ . There are two horizons, which are given by the following expressions:

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \quad (3)$$

The event horizon is at $r = r_+$.

It is obvious from the line element that the metric is not static but is only stationary and axisymmetric. There are two Killing vectors: $(\frac{\partial}{\partial t})^a$ and $(\frac{\partial}{\partial \phi})^a$. In terms of the metric in Eq. (1), they can be expressed as $\xi_t = (1, 0, 0, 0)$ and $\xi_\phi = (0, 0, 0, 1)$, respectively. The Killing field $(\frac{\partial}{\partial t})^a$ is time-like for large values of r but becomes spacelike within the ergosphere, whose position is given by $r_+ \leq r \leq M + \sqrt{(M^2 - a^2 \cos^2\theta)}$ and is situated in the near-horizon region. This indicates that we can expand the solutions of the scalar field wave equation in terms of the eigenfunctions of the

Killing vector fields. The eigenvalues of the Killing field $(\frac{\partial}{\partial t})^a$ can be interpreted as energy at a distance far away from the horizon.

The wave equation of a minimally coupled massless scalar field in a curved space is given by the following expression:

$$(-g)^{-\frac{1}{2}} \partial_\mu [(-g)^{\frac{1}{2}} g^{\mu\nu} \partial_\nu \psi] = 0. \tag{4}$$

The wave equation is separable. As discussed above, we can take the basis function to be of the following form [15]:

$$f(l, m, p|x) = N(p)(r^2 + a^2)^{-1/2} R_{lm}(p, a|r) S_{lm}(aE|\cos \theta) e^{im\phi} e^{-iEt}. \tag{5}$$

Here, $N(p)$ is a normalization constant, $p = E - m\Omega_H$, and $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the event horizon. E is the energy measured by the asymptotic observer. S_{lm} is a spheroidal harmonic satisfying the following equation:

$$\left[\frac{d}{d\xi} (1 - \xi^2) \frac{d}{d\xi} - \frac{m^2}{1 - \xi^2} + 2maE - (aE)^2 (1 - \xi^2) + \eta_{lm}(aE) \right] S_{lm}(aE|\xi) = 0. \tag{6}$$

The solutions can be expressed in terms of the oblate spheroidal harmonics [25] provided we set:

$$\eta_{lm}(aE) = (aE)^2 - 2maE + \lambda_{lm}(aE). \tag{7}$$

Here, $l = 0, 1, 2, \dots$ and $m = -l, -l+1, \dots, l-1, l$. The eigenvalues $\lambda_{lm}(aE)$ depend in a nontrivial way on l, m , and aE , and are given by the following expression:

$$\lambda_{lm}(aE) = \sum_{k=0}^{\infty} (-1)^k n_{2k} (aE)^{2k}. \tag{8}$$

Here, the sum is over $k = 0, 1, 2, \dots$. In the following, we will only need the leading-order terms in l . These are given by

$$n_0 = l(l+1); \quad n_2 = \frac{1}{2} \left[1 - \frac{(2m-1)(2m+1)}{(2l-1)(2l+3)} \right]. \tag{9}$$

The radial equation for $R_{lm}(p, a|r)$ is given by:

$$\left[\frac{d^2}{dr^{*2}} - V_{lm}(E, a|r) \right] R_{lm}(p, a|r) = 0. \tag{10}$$

In the following, we will be considering the near-horizon region of a macroscopic black hole and scalar waves with high values of l . In this limit, the function V_{lm} is given by the following expression:

$$V_{lm}(E, a|r) = -\left(E - m \frac{a}{r^2 + a^2}\right)^2 + \eta_{lm}(aE) \frac{\Delta}{(r^2 + a^2)^2}. \tag{11}$$

One can look at [12,15] for the details. Here, r^* is the Regge–Wheeler tortoise coordinates in the equatorial plane and is given by the following expression:

$$r^* = r + \frac{2M}{r_+ - r_-} \left(r_+ \ln \left\{ \frac{r - r_+}{r_+} \right\} - r_- \ln \left\{ \frac{r - r_-}{r_-} \right\} \right). \tag{12}$$

The functions V_{lm} take the values $-(E - m\Omega_H)^2$ in the limit $r \rightarrow r_+$ and $-E^2$ in the limit $r \rightarrow \infty$. This acts as a potential barrier in the intermediate region. We find that we can construct solutions of the form $\sin[pr^* - \delta(p)]$ with

$$p = E - m\Omega_H. \tag{13}$$

These solutions can be made to vanish at the brick wall in the near-horizon region with an appropriate choice of $\delta(p)$.

Without any boundary conditions, E has a continuous spectrum starting from zero and l takes all positive integral values starting from zero. This leads to divergences when we try to evaluate the free energy and the entropy. As mentioned in Sect. 1, when the scalar field is in thermal equilibrium with the black hole we can use the brick wall model condition to evaluate the scalar field entropy. In the near-horizon region, the radial wave function takes the following form:

$$R_{lm}(p, a|r) \sim \exp\left[i \int k(r) dr\right] = \exp\left[i \int K(r^*) dr^*\right]. \tag{14}$$

Here, the r -dependent radial wave vector is given by the following expression:

$$k(r) = \left(\frac{r^2 + a^2}{\Delta}\right) \left\{ \left(E - m\frac{a}{r^2 + a^2}\right)^2 - \eta_{lm}(aE) \frac{\Delta}{(r^2 + a^2)^2} \right\}^{\frac{1}{2}}. \tag{15}$$

This indicates that we can use the WKB approximation to the wave equation. We found earlier [12,13] that in spherical polar coordinates the WKB approximation is suitable for a thin shell of scalar field. Since the near-horizon region gives us interesting physics, we proceed to evaluate the entropy of a thin shell of scalar field confined in the near-horizon region. We impose the brick wall boundary condition:

$$\psi = 0, \quad \text{for } r \leq r_+ + h. \tag{16}$$

Thus, we can consider the scalar field to be confined in a half-infinite potential well in the near-horizon region. To implement the thin shell boundary condition, we consider solutions that vanish at the brick wall and are stationary up to a certain value d of the radial coordinate. We then apply the WKB quantization rule. The brick wall boundary condition can be implemented by using a linear combination of R_{lm} and its complex conjugate. We have the following condition:

$$h \ll d \ll r_+. \tag{17}$$

We note that we can replace the brick wall boundary condition by a finite value. We can consider a steeply but smoothly rising potential in place of the brick wall. The WKB quantization rule obtained from the smooth matching condition will reproduce the same expressions for the thermal variables considered below. We keep the brick wall boundary condition because there exist solutions that vanish at the horizon [9], and due to the reasons mentioned in the introduction. In the above, h, d are coordinate variables and we will replace them by the corresponding proper distances. We have to be careful since the proper radial distance depends on the polar angle θ . The corresponding solutions are again stationary at a large value of the radial coordinate. Details of the WKB approximation for the Schwarzschild black hole are discussed in [12].

The radial quantum number is given by the WKB quantization rule:

$$\pi n_d = \int_{r_++h}^{r_++d} dr k(r, l, E). \tag{18}$$

To evaluate n_d , we have to ensure that the wave number is real throughout the range of the radial integral for each value of E, l, m present in its argument. This imposes constraints on the allowed values of l . In the Schwarzschild case $k(r)$ is independent of the azimuthal quantum number m and we have the following expression for $k(r)$:

$$k_{\text{SCH}}(r) = \frac{1}{V(r)} \sqrt{E^2 - \frac{V(r)}{r^2} l(l+1)}; \quad K_{\text{SCH}}(r^*) = \sqrt{E^2 - \frac{V(r)}{r^2} l(l+1)}. \tag{19}$$

Here, $V(r)$ is the metric function of the Schwarzschild black hole. The reality of $k(r)$ gives the following upper limit on the allowed values of l [12,13]:

$$L(L+1) = \frac{E^2(2M+d+h)^3}{d+h} \approx \frac{8E^2M^3}{d}. \tag{20}$$

In the present case, we have the following expression of $k(r)$ in the near-horizon region:

$$k(r) = \left(\frac{r^2 + a^2}{\Delta} \right) \left\{ p^2 - \eta_{lm}(aE) \frac{\Delta}{(r^2 + a^2)^2} \right\}^{\frac{1}{2}}. \tag{21}$$

Here, p is given by Eq. (13). The r -dependent term in the second parentheses is zero at the horizon and is a monotonically increasing function of r near the horizon. Thus, we determine the maximum allowed value of l by making $k(r) = 0$ at $r = d$ for a fixed p . This ensures that $k(r)$ remains real in the intermediate region $(r_+ + h) \leq r \leq (r_+ + d)$ for all l less than the maximum value of l with a given p . The actual solutions, vanishing at the brick wall, are sinusoidal functions of r^* obtained from a linear combination of R_{lm} and its complex conjugate, with E determined from Eqs. (15) and (18).

In general it is difficult to determine the allowed values of l for a given p . We will use a few approximations which are appropriate for the WKB approximation in the near-horizon region. Since the WKB approximation holds for high values of the quantum numbers, we can consider leading-order terms in l, m . It is obvious that very near the horizon, $\Delta \approx 0$. In this case we can neglect the r -dependent term in Eq. (21) and the allowed values of l become free. It can be found easily that with a finite value of E , we can neglect terms with $k > 2$ in Eq. (8) [25]. On the other hand, high values of E are exponentially suppressed when we evaluate the partition function. We will consider the maximum possible value of $\eta_{lm}(aE)$ for a given large value of l , and this leads us to take $n_2 = 0$ in Eq. (8). We now have the following expression for $k(r)$:

$$k(r) = \left(\frac{r^2 + a^2}{\Delta(r)} \right) \left[p^2 - \frac{\Delta(r)\Omega_H^2}{a^2} \left\{ l(l+1) - 2ma(p + m\Omega_H) + a^2(p + m\Omega_H)^2 \right\} \right]^{\frac{1}{2}}. \tag{22}$$

Here we have taken $\Omega(r) = \Omega_H$. We want to consider the maximum value of the second term in the third parentheses with a given value of p for $(r_+ + h) \leq r \leq (r_+ + d)$. As mentioned above, the r -dependent term is maximum at $(r_+ + d)$. Since $a\Omega_H < \frac{1}{2}$ for a non-extreme black hole, it is appropriate to take $m = -l$ in the above expression and keep p fixed by choosing an appropriate E . However, we will consider the leading-order terms in l and we have the following constraint on the allowed values of l :

$$L^2 \leq \frac{a^2 p^2}{\alpha \Delta(d) \Omega_H^2}, \tag{23}$$

where

$$\alpha = [1 - (a\Omega_H)]^2. \tag{24}$$

For a Kerr black hole, $M \geq a$, $a\Omega_H \leq \frac{1}{2}$, and the right-hand side is never zero. Thus, compared to the Schwarzschild black holes the effect of rotation is to introduce the factor α in the leading-order maximum allowed values of l . We will find the situation to be different when we calculate the scalar field entropy.

We now turn to the entropy calculation. We will follow the method developed in [7,12]. The stationarity spread of ψ increases with the radial quantum number n . For the present purpose we consider the maximum value of n given by Eq. (18), multiply it by the corresponding angular degeneracy factor, and integrate over m, l, r , and E respectively to obtain the free energy. This amounts to taking the same degeneracy factor for all the lower values of $n < n_d$. This method gives a good estimate of the entropy. This is evident from the nature of the near-horizon solutions and the final expression of the entropy. The details of the method are given in [12]. Note that n increases with d while L^2 varies as $\frac{p^2}{\Delta(d)}$. The radial parameter d may be taken as the thickness of a thin shell in the near-horizon region. We will fix the value of d later. The free energy is given by the following expression:

$$\pi\beta F = -\int_0^\infty \frac{\beta dE}{[\exp\{\beta(E - m\Omega_H)\} - 1]} \int_{r_+^h}^{r_+^d} dr \left(\frac{r_+^2 + a^2}{\Delta(r)}\right) \int_0^L dl \int_{-l}^{+l} dm \left\{ (E - m\Omega_H)^2 - \eta_{lm}(aE) \frac{\Delta(r)}{(r_+^2 + a^2)^2} \right\}^{\frac{1}{2}}. \tag{25}$$

As usual for the Kerr black holes, we have superradiant modes corresponding to $p = (E - m\Omega_H) < 0$. Thus in the above integral we replace E by p with the range of the p -integral from some large negative value to a high positive value of p . One should check that the constraint given by Eq. (23) is still consistent. We thus have

$$\pi\beta F = -\int_{-p_0}^\infty \frac{\beta dp}{[\exp\{\beta p\} - 1]} \int_{r_+^h}^{r_+^d} dr \left(\frac{r_+^2 + a^2}{\Delta(r)}\right) \int_0^L dl \int_{-l}^{+l} dm \left\{ p^2 - \eta_{lm}(p) \frac{\Delta(r)}{(r_+^2 + a^2)^2} \right\}^{\frac{1}{2}}. \tag{26}$$

Note that the Jacobian for the $(E, m) \rightarrow (p, m)$ transformation is 1. We will discuss the superradiant modes and the lower limit later. We now concentrate on the m -integral. We consider only the quadratic terms in m in the radical. This holds good even for $l \sim 10$. The m -integral then reduces to the following expression:

$$\begin{aligned} \int_{-l}^{+l} dm \sqrt{A^2 m^2 + B^2} &= l\sqrt{A^2 l^2 + B^2} + \frac{B^2}{2A} \ln \left| \frac{\sqrt{A^2 l^2 + B^2} + Al}{\sqrt{A^2 l^2 + B^2} - Al} \right|, \quad B^2 \geq 0 \\ &= l\sqrt{A^2 l^2 - B'^2} - \frac{B'^2}{2A} \ln \left| \frac{\sqrt{A^2 l^2 - B'^2} + Al}{\sqrt{A^2 l^2 - B'^2} - Al} \right|, \quad B^2 < 0. \end{aligned} \tag{27}$$

Here, $A^2 = a\Omega_H(2 - a\Omega_H)\frac{\Delta\Omega_H^2}{a^2}$ and $B^2 = -B'^2 = p^2 - \frac{\Delta\Omega_H^2}{a^2}l^2$. Note that with L given by Eq. (23), B^2 can be negative. We first consider the $a \rightarrow 0$ Schwarzschild limit. In this case, α in

Eq. (24) can be taken to be one and B^2 can be taken to be positive. The right-hand side then reduces to the Schwarzschild case expression $2l\sqrt{E^2 - \frac{V(r)}{r^2}l^2}$, and the corresponding entropy will give the Schwarzschild case expression.

We now consider the Kerr black holes with finite a . It is evident from Eq. (25) that the contribution of particles with negative values of the azimuthal quantum is exponentially small compared to that for particles with positive values of the same. We should also consider the small value of the Hawking temperature for a macroscopic black hole. This makes the contribution of the negative values of m negligible for a positive value of Ω_H . This is consistent with the classical frame dragging in the near-horizon region. In the quantum case, the thin shell confined in the near-horizon region and in thermal equilibrium with the black hole preferentially contains particles whose azimuthal angular momentum is in the same direction as that of the horizon. This aspect is apparently lost in Eq. (26), where everything is expressed in terms of the parameter p . However, it is evident from Eq. (27) that the m -integral leads to the following expression in the leading order:

$$I(L) = \int_0^L l dl \sqrt{p^2 - \frac{\alpha \Delta(r) \Omega_H^2}{a^2} l^2}. \tag{28}$$

It is useful to note that the radicals in Eq. (27) are small compared to l for high values of l . When we compare this expression with the Schwarzschild case, we find that the factor $(2l + 1)$ is replaced by l . Thus, the final expression of the entropy will be half of the corresponding expression in a Schwarzschild black hole. This is consistent with the preferential emission of scalar particles with positive azimuthal quantum number. Note that the lower limit of the l -integral is taken to be zero. Lower-order terms neglected in Eq. (11) become important in this limit [15]. This will give a negligible contribution since the upper limit is too high.

We now break the energy integral into two parts. The first part corresponds to the ordinary modes and has $p \geq 0$. The second part contains the superradiant modes with $p < 0$. We first evaluate the free energy for the ordinary modes. The calculation remains similar to the Schwarzschild case with an additional half factor. We have the following expression for the free energy:

$$\pi\beta F = - \int_0^\infty \frac{\beta dp}{[\exp\{\beta p\} - 1]} \int_{r_+ + h}^{r_+ + d} dr \left(\frac{r_+^2 + a^2}{\Delta(r)} \right) \frac{1}{3} \left[p^3 - \left\{ p^2 - \frac{s\alpha \Delta(r) \Omega_H^2}{a^2} L^2 \right\}^{3/2} \right]. \tag{29}$$

Here, s is a parameter ≤ 1 . We can now do a binomial expansion in the radial integrand in terms of s . We will consider only the linear-order term in s since $\Delta(r)$ is small in the near-horizon region. The higher-order terms for the Schwarzschild black holes are given by Eq. (35) of [12]. The trial solution given by Eq. (14) will satisfy Eq. (10) provided we can neglect the $\frac{dK(r^*)}{dr^*}$ term. Note that the actual solutions are the sinusoidal functions discussed below Eq. (21). In the near-horizon region, $\frac{dK(r^*)}{dr^*}$ is proportional to $(1 - \frac{2M}{r})$ for the Schwarzschild black holes and we can neglect this term compared to V_{lm} given by Eq. (11). A similar aspect will remain valid for the Kerr black holes with $(1 - \frac{2M}{r})$ replaced by $\frac{\Delta(r)}{(r_+^2 + a^2)}$. Thus, there is no additional constraint on the allowed values of l apart from Eq. (20), and we can put $s = 1$ in the final expressions for both the Schwarzschild and Kerr black holes. This will change the numerical estimate for the brick wall cut-off given in [12] by an insignificant amount. We had taken $s = 10^{-2}$ in [12]. We will mention the corresponding values in Sect. 3. However, it is helpful to keep s as an expansion parameter in the intermediate steps. This

helps us to identify terms of different orders in $\Delta(r)$ easily. We previously used s to implement the WKB condition $|\nabla[S(r)]|^2 \gg |\nabla^2[S(r)]|$ on the solutions of the form $\rho(r)\exp[iS(r)]$, which are stationary throughout the manifold. This gives the constraint $s \ll 1$ for the globally stationary solutions in the Schwarzschild black holes [12]. This can be found from Eqs. (10), (14), and (19) with $r = 3M$. The significance of the point $r = 3M$ can be found from [8,12].

The free energy is given by the following expression:

$$F = -\frac{1}{\beta^4} \frac{\pi^3}{30} \frac{(r_+^2 + a^2)^3}{(r_+ - r_-)^2} \frac{1}{d} \ln\left(\frac{d}{h}\right). \tag{30}$$

As expected, the free energy is proportional to the fourth power of the temperature. We now introduce the covariant cut-off parameter defined by

$$\epsilon_b = \int_{r_+}^{r_+ + b} \frac{\rho}{\sqrt{\Delta(r)}} dr. \tag{31}$$

It is obvious that the proper radial distance depends on θ . We have the following relation between the proper radial distance at the point where the ergosphere vanishes (here $\theta = 0$ or π) and at an arbitrary value θ of the polar angle:

$$\epsilon_\theta^2 = (1 - a\Omega_H \sin^2\theta)\epsilon^2. \tag{32}$$

Here, ϵ is the proper radial distance along $\theta = 0$ or π . With $a\Omega_H < \frac{1}{2}$, different choices of θ lead to a minor numerical factor. However, the expression of the free energy given by Eq. (30) is independent of θ . Note that the ergosphere is covariantly defined in terms of two invariant quantities: the norm of the Killing field $(\frac{\partial}{\partial t})^a + \Omega_H(\frac{\partial}{\partial \phi})^a$ and the norm of the Killing field $(\frac{\partial}{\partial t})^a$. We choose the radial brick wall cut-off and the radial thickness in such a way that they yield definite values for the corresponding proper variables along the directions where the ergosphere vanishes. Thus we may consider that we have a thin shell in the near-horizon region whose boundaries have θ -dependent proper radii as measured from the horizon. The free energy and the entropy are expressed in terms of the proper radial variables evaluated along the directions along which the ergosphere vanishes. In the familiar Kerr metric given by Eq. (1), the polar axis represents the points where the ergosphere vanishes. *In all the subsequent discussions, ϵ will refer to the proper radial distance determined in this way.* We will discuss the numerical values of the proper variables in Sect. 3.

The entropy can be obtained from the relation $S = \beta^2 \frac{dF}{d\beta}$. In terms of the covariant parameters, we have

$$S = \frac{16\pi^3}{15} \frac{1}{\beta^3} \frac{(r_+^2 + a^2)^4}{(r_+ - r_-)^3} \frac{1}{\epsilon_d^2} \ln\left(\frac{\epsilon_d}{\epsilon_h}\right). \tag{33}$$

For the temperature, we take the value measured by the asymptotic static observer at infinity:

$$\beta = \frac{4\pi(r_+^2 + a^2)}{(r_+ - r_-)}. \tag{34}$$

The entropy of a scalar field confined within the near-horizon region and predicted by an observer at infinity is then given by

$$S = \frac{1}{60} \ln\left(\frac{\epsilon_d}{\epsilon_h}\right) \frac{A}{4\pi\epsilon_d^2}. \tag{35}$$

The corresponding expression in a Schwarzschild black hole is given by the following expression:

$$S' = \frac{1}{30} \ln \left(\frac{\epsilon_d}{\epsilon_h} \right) \frac{A}{4\pi \epsilon_d^2}. \quad (36)$$

As mentioned earlier, the entropy of the scalar field in a Kerr black hole is half of the corresponding value in a Schwarzschild black hole.

We now turn to the superradiant modes. The corresponding free energy is given by the following expression:

$$\pi\beta F' = - \int_{-P_0}^0 \frac{\beta dp}{[\exp\{\beta p\} - 1]} \int_{r_++h}^{r_++d} dr \left(\frac{r_+^2 + a^2}{\Delta(r)} \right) \int_0^L dl \int_{-l}^{+l} dm \left\{ p^2 - \eta_{lm}(p) \frac{\Delta(r)}{(r_+^2 + a^2)^2} \right\}^{\frac{1}{2}}. \quad (37)$$

The constraint given by Eq. (23) and $E \rightarrow \infty$ indicate that $P_0 \rightarrow \infty$. Note that we always have $E \geq 0$. However, with p being negative, the thermal factor becomes approximately -1 for all p and the integral diverges badly. We can keep P_0 finite and take the limit $P_0 \rightarrow \infty$ at the end. In this case, the free energy becomes independent of the temperature and the entropy vanishes. This is not unexpected, since for the superradiant modes the stimulated emission dominates and the thermal radiation is not very significant. Thus we exclude the superradiant modes from our consideration and regard Eq. (35) to be the leading-order entropy of a minimally coupled scalar field confined in the near-horizon region and in thermal equilibrium with the Kerr black hole.

3. Discussion

Equations (35) and (36) give the leading-order expressions for the entropy of the scalar field in the Kerr and Schwarzschild black holes respectively. It is easy to calculate the higher-order terms. In a Schwarzschild black hole the higher-order terms are again proportional to the area, but are negligible compared to the expression given by Eq. (36). This is given by Eq. (35) of [12]. It is important to note that there is no term proportional to the volume of the thin shell. Hence, the degrees of freedom of a three-dimensional thin shell in the near-horizon region are related to the inner boundary, which is almost coincident with the horizon. A similar result is also valid for the Kerr black holes. This is similar to the black hole entropy itself. This is an important aspect of the present method. A related field is to determine the expressions for the entanglement entropy of a scalar field in a given spacetime. The entanglement entropy of the matter fields are found from the reduced density matrices obtained by summing over the degrees of freedom confined within a given spatial region of the flat spacetime [26–28]. This is significant for a scalar field in the near-horizon region of both the Schwarzschild and the Kerr black holes. In terms of the tortoise coordinates, the wave equation is similar to that of a one-dimensional free particle in the flat space when we consider the near-horizon region of both the black holes. In the Kerr black holes we have to consider the classical frame dragging and its consequences for the quantum case. This is obvious from Sect. 2. We will later discuss this issue. In this context, it will be interesting to consider the entanglement entropy for a thin shell of scalar field surrounding the horizon and compare the result with the expression obtained in this article. This may be important to explain the black hole entropy in terms of the near-horizon physics.

The expression of the scalar field entropy given by Eq. (35) contains two free parameters, the proper thickness of the thin shell and the proper brick wall cut-off parameter as defined in Sect. 2. We first fix the proper radial thickness of the thin shell. In a Schwarzschild black hole, we took the

thickness of the thin shell to be of the order of atomic lengths [12]. Here we choose the same. The other undetermined parameter is the proper brick wall cut-off. It has been shown by Candelas that the stress tensor of the scalar field in the Hartle–Hawking vacuum is finite [16]. 't Hooft wanted to explain the black hole entropy in terms of the near-horizon part of the matter field entropy and equated the two [7]. If we do so, the internal energy turns out to be finite: $U = \frac{3}{8}M$. This also determines the proper brick wall cut-off parameter. In the present case we have:

$$\epsilon_h = \epsilon_d \exp \left[-60\pi \left(\frac{\epsilon_d}{l_p} \right)^2 \right], \tag{38}$$

where l_p is the Planck length. Note that this is different from the Schwarzschild value [12]:

$$\epsilon_h = \epsilon_d \exp \left[-30\pi \left(\frac{\epsilon_d}{l_p} \right)^2 \right]. \tag{39}$$

If we take $\epsilon_d = 10^{-10} \text{ cm}$, ϵ_h is given by the following expression in terms of the Planck length:

$$\epsilon_h \sim [10^{24}] \exp(-10^{50}). \tag{40}$$

Thus, the brick wall is almost coincident with the horizon, although the free energy is finite. This is expected since there exist solutions vanishing on the horizon. Note that the power 50 in the above equation is replaced by $\frac{50}{s}$ for $s < 1$. For any $s < 1$, this lowers the value of ϵ_h further. We now consider the internal energy of the thin shell. When ϵ_h is given by Eq. (38), the internal energy is given by $U = \frac{3}{8} \sqrt{M^2 - a^2}$. Alternatively, we can keep the value of the proper brick wall cut-off the same as that for the Schwarzschild black holes. In this case the entropy of the thin shell will be half that of the black hole and the internal energy is given by $U = \frac{3}{16} \sqrt{M^2 - a^2}$. In both cases the internal energy is finite and we may disregard the back reaction problem up to a first approximation.

In passing, we note that the WKB approximation is a naive approach. It is not expected to give a satisfactory result for the complete set of solutions of the wave equation in a curved space. It gives the expected result for a thin shell of matter field. We considered the almost vanishing transmission coefficient for the modes which are only stationary in the near-horizon region. They are again stationary at a large distance away from the horizon with a vanishingly small amplitude. These aspects lead us to consider the entropy of a thin shell of matter field confined in the near-horizon region. We obtain expected results for the Kerr–Newman family of black holes. This is consistent with the nature of the scalar wave equation in terms of the tortoise coordinates. In the near-horizon region, this is similar to a one-dimensional free wave equation in the flat space, to a first approximation. Lastly, we note that the expression of the entropy obtained in Ref. [7] is given by:

$$\tilde{S} = \frac{1}{90\pi\epsilon^2} \frac{A}{4}. \tag{41}$$

The quadratic divergence in the proper radial brick wall cut-off parameter is unexpected. A detailed comparison between the result obtained by us and in [7] can be found in [12].

The brick wall model approach is based on the Lorentzian sector of the black hole spacetime and is expected to give the most robust expression. It is convenient to use the zeta function regularization scheme in the Euclidean sector calculation for the partition function of a scalar field in a black hole background. This is considered in [18,29,30] for the Schwarzschild black holes with boundary

conditions similar to those used in this article. In the zeta function regularization scheme to evaluate the partition function, the partition function is given by the Eq. (3.2) of [18]:

$$\ln[Z] = \frac{1}{2} \left[\zeta'(0) + \ln \left(\frac{1}{4} \pi \mu^2 \right) \zeta(0) \right]. \quad (42)$$

Here, μ is a normalization factor and $\zeta(s)$ is the zeta function. The evaluation of the scalar field entropy is considered by many authors. One can look at [17] for a review. The WKB approximation is also used here [15]. The expression of entropy contains an undetermined parameter. The expression of the entropy obtained by us can be used to determine the undetermined parameter in terms of the physical brick wall parameter. The logarithmic divergence obtained by us is significant in this respect. It renders quadratically and quartically divergent terms in the Euclidean sector expression to logarithmically divergent. The Euclidean sector calculation of the scalar field entropy in the Kerr black holes is not well formulated [20,21]. The expression obtained by us in this article is thus illuminating in this context.

The scalar field entropy has been interpreted by some authors as a quantum correction to the black hole entropy, thus giving infinite renormalization to the gravitational constant G_B . A few authors used the brick wall model for this purpose [31–33]. The one-loop effective action of a scalar field in a curved spacetime can be found by using the DeWitt–Schwinger proper time representation. The divergent parts are given by Eq. (6.44) in [34]. These terms are divergent only when the spacetime dimension is four and may be interpreted as giving infinite renormalizations to the different coupling constants present in the Einstein–Hilbert action for the gravitational field itself. The divergent term is of the form $\frac{1}{(n-4)}$, where n is the spacetime dimension. From Eqs. (35) and (36) it is obvious that the scalar field entropies in the Schwarzschild and Kerr black holes are logarithmically divergent. A similar aspect also remains valid in the Reissner–Nordstrom black holes. Thus we may not fix the brick wall cut-off parameter as we have done before. We add the scalar field entropy to the black hole entropy with G replaced by G_B and absorb the divergence to give a renormalized value. The structure of this quantum correction is different from the quadratically divergent expression obtained before. Moreover, this scheme can work if we find that the scalar field entropies in different black holes are of a similar form as obtained for the asymptotically flat black holes considered here. Thus, they should be proportional to the gravitational entropy with a logarithmic divergence. In this context it will be interesting to use the methods developed by us to calculate the entropy of a minimally coupled scalar field in nontrivial black holes like the Taub–NUT spacetime and constant curvature black holes.

4. Conclusion

To conclude, in this paper we have found the entropy of a minimally coupled scalar field in the Kerr black hole backgrounds. We have used the Lorentzian sector brick wall model of 't Hooft. We have used an improved method to calculate the entropy previously developed by us. We have considered a thin shell of scalar field confined in the near-horizon region and in thermal equilibrium with the black hole. The corresponding entropy is proportional to the area of the event horizon and is logarithmically divergent in a covariant radial parameter known as the proper radial brick wall cut-off parameter. This is an improvement compared to the earlier results, which are quadratically divergent in the proper radial brick wall cut-off parameter. Logarithmic divergence is expected on physical grounds. The method used by us is significantly different from that used by 't Hooft. The scalar field entropy is proportional to the area of the horizon and does not contain any term proportional

to the volume of the thin shell. For a fixed value of the brick wall cut-off, the scalar field entropy in a Kerr black hole is found to be half of the corresponding expression in a Schwarzschild black hole background. This is expected for the thermal radiation in a Kerr black hole due to preferential emission of particles with azimuthal angular momentum in the same direction as that of the black hole. The contribution of the particles in the thin shell with opposite values of the azimuthal angular momentum is exponentially suppressed. This is due to rotation of the black hole. However, we can obtain the Schwarzschild case expression by including a subleading term and taking the appropriate limit. We have briefly discussed the significances of the results in Sect. 3. Further significance with regard to the entanglement approach to explain the black hole entropy and the holographic principles will be discussed later.

Acknowledgements

I am thankful to the journal for waiving the article processing charge.

Funding

Open access funding: RCUK.

References

- [1] J. M. Bardeen, B. Carter, and S. W. Hawking, *Commun. Math. Phys.* **31**, 161 (1973).
- [2] J. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
- [3] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
- [4] S. A. Fulling, *Phys. Rev. D* **7**, 2850 (1973).
- [5] W. G. Unruh, *Phys. Rev. D* **14**, 870 (1976).
- [6] J. B. Hartle and S. W. Hawking, *Phys. Rev. D* **13**, 2188 (1976).
- [7] G. 't Hooft, *Nucl. Phys. B* **256**, 727 (1985).
- [8] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago and London, 1984).
- [9] D. G. Boulware, *Phys. Rev. D* **11**, 1404 (1975).
- [10] I. Ichinose and Y. Satoh, *Nucl. Phys. B* **447**, 34 (1995).
- [11] S. P. de Alwis and N. Ohta, *Phys. Rev. D* **52**, 3529 (1995).
- [12] K. Ghosh, *J. Phys. Soc. Jpn.* **85**, 014101 (2016) [arXiv:0902.1601 [gr-qc]] [Search INSPIRE].
- [13] K. Ghosh, *J. Phys. Conf. Ser.* **410**, 012137 (2013).
- [14] K. Ghosh, *Nucl. Phys. B* **814**, 212 (2009).
- [15] B. S. DeWitt, *Phys. Rep.* **19**, 295 (1975).
- [16] P. Candelas, *Phys. Rev. D* **21**, 2185 (1980).
- [17] S. N. Solodukhin, *Living Rev. Relat.* **14**, 8 (2011).
- [18] S. W. Hawking, *Commun. Math. Phys.* **55**, 133 (1977).
- [19] G. W. Gibbons and S. W. Hawking, *Commun. Math. Phys.* **66**, 291 (1979).
- [20] V. Frolov and D. Fursaev, *Class. Quant. Grav.* **15**, 2041 (1998).
- [21] D. Fursaev and S. Solodukhin, *Phys. Rev. D* **52**, 2133 (1995).
- [22] G. Cognola, L. Vanzo, and S. Zerbini, *Class. Quant. Grav.* **12**, 1927 (1995).
- [23] A. A. Starobinsky, *Sov. Phys. JETP* **37**, 28 (1973).
- [24] W. G. Unruh, *Phys. Rev. D* **10**, 3194 (1974).
- [25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1964).
- [26] L. Bombelli, R. K. Kaul, J. Lee, and R. Sorkin, *Phys. Rev. D* **34**, 373 (1986).
- [27] C. Callan and F. Wilczek, *Phys. Lett. B* **333**, 55 (1994).
- [28] M. Saravani, R. D. Sorkin, and Y. K. Yazdi, *Class. Quant. Grav.* **31**, 214006 (2014).
- [29] D. B. Ray and I. M. Singer, *Adv. in Math.* **7**, 145 (1971).
- [30] P. B. Gilkey, *Adv. in Math.* **15**, 334 (1975).

- [31] J. G. Demers, R. Lafrance, and R. C. Myers, Phys. Rev. D **52**, 2245 (1995).
- [32] L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994).
- [33] A. Ghosh and P. Mitra, Phys. Lett. B **357**, 295 (1995).
- [34] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).