Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Topology and its Applications 153 (2006) 2735–2744

Topology and its Applications

www.elsevier.com/locate/topol

Maps and f-normal spaces

John J. Schommer [a,*], Biswajit Mitra [b,1]

[a] *Department of Mathematics and Statistics, The University of Tennessee at Martin, Martin, TN 38238, USA*
[b] *Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, West Bengal, India*

Received 17 May 2005; received in revised form 14 November 2005; accepted 15 November 2005

Abstract

It has long been known that hyper-real maps preserve realcompactness. In this paper we show that hyper-real maps preserve nearly realcompactness as well. We will also introduce the concepts of ε-perfect maps and f-normal spaces and explore them in a way that mirrors Rayburn's 1978 study of δ-perfect maps and h-normal spaces.

MSC: primary 54C10; secondary 54D60, 54D99

Keywords: Nearly realcompact space; f-normal space; Fast set; Hyper-real map; Perfect map; ε-perfect map; Zenor map; WZ-map

1. Introduction

There are three different approaches to the study of nearly realcompact spaces in the literature. Johnson and Mandelker referred to nearly realcompact spaces as η-compact spaces in [12] , characterizing them in terms of ideals of continuous functions. Blair and van Douwen produced quite a few results by characterizing nearly realcompact spaces in terms of their relatively pseudocompact cozero-sets in [2]. In this paper we continue the development of nearly realcompact spaces begun in [17], one which parallels Henriksen and Rayburn's approach to nearly pseudocompact spaces initiated in [8].

We will assume that all spaces are Tychonoff. The basic theories and properties of the Stone–Čech compactification βX and the Hewitt realcompactification υX will also be assumed. Furthermore we adopt the notation and terminology of [5] for the terms: zero-sets, cozero-sets, z-ultrafilters, real z-ultrafilters, C-embedded, and C*-embedded. A subspace A of X is said to be *z-embedded* in X if every zero-set of A is the restriction to A of some zero-set of X. A subspace A of X is said to be *relatively pseudocompact* in X if and only if every continuous function on X is bounded on A if and only if $\mathrm{cl}_{\beta X} A \subseteq \upsilon X$.

Hyper-real maps will also play an important role in this paper. We say that a map $f : X \to Y$ is *hyper-real* iff the Stone extension $f_\beta : \beta X \to \beta Y$ satisfies $f_\beta(\beta X - \upsilon X) \subseteq \beta Y - \upsilon Y$. (*Note*: By a map we will always mean a continuous surjection.)

* Corresponding author.
E-mail addresses: jschomme@utm.edu (J.J. Schommer), b1mitra@yahoo.co.in (B. Mitra).
[1] The second author's research grant is supported by CSIR, Human Resource Development group, New Delhi-110012, India.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2005.11.006

We say that a space X is *nearly realcompact* iff $\beta X - \upsilon X$ is dense in $\beta X - X$. The primary tool for the investigations of nearly realcompactness in this paper will be the so-called "fast set". We say that a subset F of X is *fast* in X if F is closed in $X \cup J_X$ where $J_X = \mathrm{cl}_{\beta X}(\beta X - \upsilon X)$. We write J instead of J_X when there is no danger of confusion about the ambient space. We will also need to refer frequently to the subspace εX of βX given by $\varepsilon X = \beta X - (J - X) = X \cup (\beta X - J)$.

We will have cause later to refer to a result of Blair and van Douwen concerning another generalization of realcompactness, c-realcompactness. A space is said to be *c-realcompact* iff for every $p \in \beta X - X$, there is a decreasing sequence $\{A_n\colon n \in \omega\}$ of regular closed sets in βX such that $p \in \bigcap_{n\in\omega} A_n \subseteq \beta X - X$ (cf. [6, 1.1]). Blair and van Douwen have shown that every c-realcompact space is nearly realcompact. Not published in their lifetimes, this result was first published posthumously as [16, 14.3]. We re-present the result here to make its proof more widely available. We begin with some lemmas.

Lemma 1.1. [2, 1.4] *If G is open in X, then G is relatively pseudocompact in X iff whenever $\{F_n\colon n \in \omega\}$ is a decreasing sequence of regular closed subsets of X with $F_n \cap G \neq \emptyset$ for all $n \in \omega$, then $\bigcap_{n\in\omega} F_n \neq \emptyset$.*

Lemma 1.2. [2, 1.7] *X is nearly realcompact iff every relatively pseudocompact cozero-set of X is σ-compact.*

Theorem 1.3 *(Blair and van Douwen). Every c-realcompact space is nearly realcompact.*

Proof. Let X be c-realcompact and let $P = \mathrm{coz}\, f$, where $f : X \to [0, 1]$. Suppose P is not σ-compact. By the last lemma, it suffices to show that P is not relatively pseudocompact. Since $P = \bigcup_{n\in\omega} f^{\leftarrow}[\frac{1}{n}, 1]$ and each $f^{\leftarrow}[\frac{1}{n}, 1]$ is a zero-set, P must contain one non-compact zero-set $Z = f^{\leftarrow}[\frac{1}{n_0}, 1]$. Now since Z is closed in X and not compact, X must not be compact, so there is a point $p \in (\beta X - X) \cap \mathrm{cl}_{\beta X} Z$. Since X is c-realcompact, there is a decreasing sequence $\{U_n\colon n \in \omega\}$ of open sets in X with $p \in \bigcap_{n\in\omega} \mathrm{cl}_{\beta X} U_n \subseteq \beta X - X$. Since $Z \subseteq P$, $P \cap \mathrm{cl}_X U_n \neq \emptyset$ for all $n \in \omega$. But $\bigcap_{n\in\omega} \mathrm{cl}_X U_n = \emptyset$. Thus by Lemma 1.1, P is not relatively pseudocompact. □

Finally, we will make frequent reference to the results contained in the following theorem [17]:

Theorem 1.4. *The following are true*:

(1) *F is fast in X iff $F = T \cap X$ where T is a compact subset of εX.*
(2) *A closed subset F is fast in X iff there exists a compact set T such that for every open set G containing T, $F - G$ is completely separated from the complement of some pseudocompact set in X.*
(3) *Every compact set in X is fast, but every fast set is compact iff $X = \varepsilon X$ iff X is nearly realcompact.*
(4) *$\varepsilon X = \bigcup\{\mathrm{cl}_{\beta X} F\colon F$ is fast in $X\}$.*
(5) *Every fast set of X is closed, but every closed set is fast iff X is pseudocompact.*
(6) *A fast set in X is relatively pseudocompact, and a regular fast set* (*i.e. a regular closed set which is fast in X*) *is pseudocompact.*
(7) *A closed subset of a fast set in X is fast in X.*
(8) *X is locally pseudocompact iff εX is locally compact.*

Practically every result in this paper will parallel an analogous result contained in [15]. In many cases, the proofs require only an appropriate modification to the proofs offered in Rayburn's paper. This does not always work smoothly though, and in some interesting cases more will be required of us.

2. ε-perfect maps

Recall that a map $f : X \to Y$ is called *perfect* if it is closed, onto, and $f^{\leftarrow}(y)$ is compact for every $y \in Y$. Indeed, it can be shown that a map f is perfect iff $f_\beta^{\leftarrow}(y) \subseteq X$ for every $y \in Y$, where $f_\beta : \beta X \to \beta Y$ is the Stone extension.

Definition 2.1. A map $f : X \to Y$ is *ε-perfect* if $f_\beta^{\leftarrow}(y) \subseteq \varepsilon X$ for every $y \in Y$.

Every perfect map is ε-perfect, but the converse need not be true. If Y is compact but X is merely pseudocompact, then there cannot be a perfect map from X to Y. But every map of X onto Y is ε-perfect, since X pseudocompact implies $\beta X = \upsilon X$, and consequently $\varepsilon X = \beta X$. It should be noted that ε-perfect is equivalent to perfect whenever X is nearly realcompact.

Lemma 2.2. *A hyper-real map $f : X \to Y$ is ε-perfect iff $f_\beta^{\leftarrow}[\varepsilon Y] \subseteq \varepsilon X$.*

Proof. ($\Rightarrow$) By hypothesis $f_\beta^{\leftarrow}[Y] \subseteq \varepsilon X$, so we need only show that $f_\beta^{\leftarrow}[\varepsilon Y - Y] \subseteq \varepsilon X$. Let $x \in f_\beta^{\leftarrow}[\varepsilon Y - Y]$. If $x \in X$ we are done, so assume $x \notin X$. Since $\varepsilon X = X \cup (\beta X - J_X)$, this leaves us to prove that $x \in \beta X - J_X$. Observe that

$$x \in f_\beta^{\leftarrow}[\varepsilon Y - Y] \subseteq f_\beta^{\leftarrow}(\beta Y - J_Y) = \beta X - f_\beta^{\leftarrow}(J_Y).$$

It remains then to demonstrate that $\beta X - f_\beta^{\leftarrow}(J_Y) \subseteq \beta X - J_X$. But this follows from the fact that f is hyper-real:

$$f_\beta(J_X) = f_\beta\big(\mathrm{cl}_{\beta X}(\beta X - \upsilon X)\big) \subseteq \mathrm{cl}_{\beta Y}\, f_\beta(\beta X - \upsilon X) \subseteq \mathrm{cl}_{\beta Y}(\beta Y - \upsilon Y) = J_Y.$$

Thus $J_X \subseteq f_\beta^{\leftarrow}(J_Y)$ and we are done.

($\Leftarrow$) This is immediate, not even requiring that f be hyper-real. □

Note 2.3. The hyper-real condition of Lemma 2.2 cannot be dropped. The counterexample will be more easily discussed in Note 2.11.

Now let us recall that a map $f : X \to Y$ is called a *Z-map* iff $f(Z)$ is closed in Y for all zero-sets Z of X, and f is called a *WZ-map* iff $f_\beta^{\leftarrow}(y) = \mathrm{cl}_{\beta X}\, f^{\leftarrow}(y)$ for all $y \in Y$. It is known that every Z-map is a WZ-map, but the converse is not true [11, 1.2,8.2]. Z-maps are also referred to in the literature as *z-closed* maps (cf. [20]).

Lemma 2.4. *A map $f : X \to Y$ is an ε-perfect WZ-map iff $f_\beta^{\leftarrow}(y) = cl_{\varepsilon X} f^{\leftarrow}(y)$ for all $y \in Y$.*

Proof. In general we have

$$\mathrm{cl}_{\varepsilon X}\, f^{\leftarrow}(y) \subseteq \mathrm{cl}_{\beta X}\, f^{\leftarrow}(y) \subseteq f_\beta^{\leftarrow}(y).$$

WZ makes an equation of the second of these containments, while ε-perfect equates the rest. □

Theorem 2.5. *Let f be a hyper-real map of X onto Y. Then each of the following implies the next*:

(1) *f is ε-perfect.*
(2) *f pulls fast sets back to fast sets.*
(3) *f pulls points back to fast sets.*

Moreover, if f is a WZ-map, all are equivalent.

Proof. Use Theorem 1.4 and Lemma 2.4, but otherwise follow Rayburn's technique for proving [15, 16]. □

Note 2.6. In our last theorem, the direct implication (1) $\Rightarrow$ (3) does not really require that f be hyper-real. If f is ε-perfect, then $f_\beta^{\leftarrow}(y)$ is a compact subset of εX for every $y \in Y$. But then $f^{\leftarrow}(y) = f_\beta^{\leftarrow}(y) \cap X$ and so $f^{\leftarrow}(y)$ is fast by Theorem 1.4(1). This fact about ε-perfect maps allows us to prove the following corollary.

Corollary 2.7. *Let $f : X \to Y$ be hyper-real, ε-perfect map and let $g : Y \to Z$ be ε-perfect. Then $g \circ f : X \to Z$ is ε-perfect.*

Proof. Let $z \in Z$, $g_\beta^{\leftarrow}(z) \subseteq \varepsilon Y$. Since f is hyper-real and ε-perfect, $f_\beta^{\leftarrow}(g_\beta^{\leftarrow}(z)) \subseteq \varepsilon X$. That is, $(g_\beta \circ f_\beta)^{\leftarrow}(z) \subseteq \varepsilon X$. But $g_\beta \circ f_\beta = (g \circ f)_\beta$. So $(g \circ f)_\beta^{\leftarrow}(z) \subseteq \varepsilon X$ and thus $g \circ f$ is ε-perfect. □

The reader might recall that hyper-real maps preserve realcompactness [20, 17.17]. Hyper-real maps preserve nearly realcompactness too.

Theorem 2.8. *If X is nearly realcompact and $f : X \to Y$ is hyper-real, then Y is nearly realcompact.*

Proof. Since X is nearly realcompact, $X \cup J_X = \beta X$. We have seen in the proof of Lemma 2.2 that if f is hyper-real, then $f_\beta(J_X) \subseteq J_Y$. Observe then, that

$$Y \subseteq f_\beta(X \cup J_X) \subseteq f_\beta(X) \cup f_\beta(J_X) \subseteq Y \cup J_Y.$$

But $f_\beta(X \cup J_X)$ is a compact subset of βY containing Y as a dense subset. Thus $f_\beta(X \cup J_X) = \beta Y$, and so $Y \cup J_Y = \beta Y$. It follows that Y is nearly realcompact. □

Note 2.9. Hyper-real is not, strictly speaking, necessary to preserve near realcompactness. Blair and van Douwen have shown that the perfect irreducible image of a nearly realcompact space is nearly realcompact [2, 1.13]. Perfect alone, however, is not sufficient to guarantee this outcome, as the following example shows.

Example 2.10. Let X be the "fringed" plank obtained from the ordinary Tychonoff Plank by adding a convergent sequence $\{x_{j,n}\colon n \in \omega\}$ to each point (ω_1, j) on the right edge. Let the points on the right edge have their usual neighborhoods plus enough tails of those sequences to make a topology. All the added points are isolated. It follows from [18, 9], [18, 12], and Theorem 1.3 that X is nearly realcompact.

Now let Y be the Tychonoff Plank, and let $f : X \to Y$ be the function such that $f \restriction Y$ is the identity and f sends the added points from each sequence to the point to which the sequence converges. The map is perfect, but Y is not nearly realcompact.

Note 2.11. Example 2.10 also serves as the counterexample we promised in Note 2.3 to demonstrate the necessity of the hyper-real condition. The Tychonoff Plank is pseudocompact so $\varepsilon Y = \beta Y$, the one point compactification of Y. We have already noted that f is perfect (and consequently ε-perfect), but $f_\beta^{\leftarrow}[\varepsilon Y] \nsubseteq \varepsilon X = X$. Observe that f cannot be hyper-real because Y is not nearly realcompact.

Theorem 2.8 is an improvement of two theorems of Ikeda: that perfect open maps preserve nearly realcompactness [9, p. 3], and that hyper-real WN-maps preserve nearly realcompactness [10, 3.3]. In the first instance, perfect open maps are known to be hyper-real [20, p. 218]. In the second, we see that the property WN-map is not necessary.

The definition of WN-map involves a slight modification of the definition of WZ-map. A map is *WN* iff $f_\beta^{\leftarrow}(Z) = \mathrm{cl}_{\beta X} f^{\leftarrow}(Z)$ for every zero-set $Z \subseteq Y$. (The original definition of WZ-map is due to Isiwata [11, p. 457], and the original definition of WN-map is due to Woods [21, 1.1].)

Corollary 2.12. *Let X be compact. Then $X \times Y$ is nearly realcompact iff Y is nearly realcompact.*

Proof. ($\Rightarrow$) If X is compact, then the projection map $p_Y : X \times Y \to Y$ is perfect and open, and consequently hyper-real. It follows that Y is nearly realcompact.

($\Leftarrow$) This is [2, 2.8]. □

Rayburn was able to achieve some nice results by exploiting a special map originally due to Phillip Zenor. Zenor showed that if A is closed, then $\phi_A : X \to X/A$ is always a WZ-map, whenever X/A is topologized with the finest completely regular topology that renders ϕ_A continuous [22, p. 273]. Rayburn was able to put this map to immediate use; we on the other hand will first have to determine the conditions under which the Zenor map is hyper-real.

To motivate where we are headed, note that if X is not pseudocompact, it contains a copy of $\mathbb{N}$ which is closed in υX. Thus $\mathrm{cl}_{\beta X} \mathbb{N} \cap (\beta X - \upsilon X) \neq \emptyset$. Since $\phi_{\mathbb{N}}$ identifies $\mathbb{N}$ with a point in $X/\mathbb{N}$, say p, then $\phi_{\mathbb{N}}^{\beta}(\mathrm{cl}_{\beta X} \mathbb{N}) = p$ where $\phi_{\mathbb{N}}^{\beta}$ is the Stone extension of $\phi_{\mathbb{N}}$ from βX onto $\beta(X/\mathbb{N})$. It follows that $\phi_{\mathbb{N}}$ cannot be hyper-real. We will show that there is nothing really special about $\mathbb{N}$. If we take any closed subset of X which is not relatively pseudocompact, the corresponding Zenor map is not hyper-real. In the next theorem we show that the condition hyper-real is not only necessary, but sufficient too.

Theorem 2.13. *Let A be a closed subset of X. Then the Zenor map $\phi_A : X \to X/A$ is hyper-real if and only if A is relatively pseudocompact.*

Proof. ($\Rightarrow$) Let $\phi_A : X \to X/A$ be hyper-real and suppose A is not relatively pseudocompact. Then $\exists p \in \mathrm{cl}_{\beta X} A \cap (\beta X - \upsilon X)$. But then $\phi_A^\beta(p) = \phi_A(A) \in X/A$, contradicting the fact ϕ_A is hyper-real.

($\Leftarrow$) Let A be a relatively pseudocompact subset of X and let $p \in \beta X - \upsilon X$. Then $p \notin \mathrm{cl}_{\beta X} A$ and $\exists f$ and $g \in C^*(X)$, such that $f > 0$ on X, $f^\beta(p) = 0$, $g^\beta(\mathrm{cl}_{\beta X} A) = 1$, $g^\beta(p) = 0$, and $0 \leqslant g \leqslant 1$. Define $h \in C^*(X)$ by $h = [0 \vee (f + g)] \wedge 1$. Then $h^\beta(p) = 0$ and $h^\beta(\mathrm{cl}_{\beta X} A) = 1$ with $h \neq 0$ on X. Now define $l : X/A \to \mathbb{R}$ by $l(x) = h(x)$ if $x \neq \phi_A(A)$ and 1 if $x = \phi_A(A)$. Then it is easy to check that $l \circ \phi_A = h$. But $h \in C^*(X)$, so it follows that $l \in C^*(X/A)$. It is also clear that $h^\beta = l^\beta \circ \phi_A^\beta$.

Now suppose $\phi_A^\beta(p) \in \upsilon(X/A)$. Then $l^\beta(\phi_A^\beta(p)) = h^\beta(p) = 0$ and therefore $Z(l^\beta) \cap \upsilon(X/A) \neq \emptyset$, which implies that there exists $t \in X/A$ such that $l^\beta(t) = 0$. By construction of l, $t \neq \phi_A(A) \Longrightarrow \phi_A^\leftarrow(t) \notin A$. Now $h^\beta(\phi_A^\leftarrow(t)) = l(t) = 0$ and thus there exists a point $\phi_A^\leftarrow(t)$ in X at which h vanishes. This contradicts the fact that $h \neq 0$ on X. So $\phi_A^\beta(p) \in \beta(X/A) - \upsilon(X/A)$ and thus ϕ_A is hyper-real. □

We know that every fast set is relatively pseudocompact but the converse is not true. For example, the subset $[0, \omega_0)$ is relatively pseudocompact in $[0, \omega_1)$ but not fast in $[0, \omega_1)$. Our next corollary offers a description of when closed relatively pseudocompact subsets of X are fast in X.

Corollary 2.14. *A closed relatively pseudocompact subset A of X is fast in X if and only if ϕ_A is ε-perfect.*

Proof. Use Rayburn's argument for [15, 19] along with Theorems 2.5 and 2.13. □

Corollary 2.15. *X is nearly realcompact if and only if every ε-perfect map is perfect.*

Corollary 2.16. *A space X is pseudocompact iff every map on X is hyper-real and ε-perfect.*

Definition 2.17. A map $f : X \to Y$ is an *F-map* if the image of each fast set is closed in Y. If f carries fast sets to fast sets, we call f a *fast map*.

Note 2.18. We have the following immediate observations regarding F-maps and fast maps.

(1) Closed maps and fast maps are F-maps.
(2) Every F-map on a pseudocompact space is closed.
(3) If X is nearly realcompact, every map on X is a fast map.
(4) An F-map need not be WZ. Isiwata has shown that every WZ-map on a normal space is closed [11, 1.3]. Therefore let f be any non-closed map on a normal realcompact space. Note that f is fast and consequently an F-map, but not WZ.

Theorem 2.19. *If $f : X \to Y$ is an ε-perfect F-map, then f is a WZ-map.*

Proof. This proof is analogous to Rayburn's proof of [15, 24]. □

Theorem 2.20. *$f : X \to Y$ is a fast map iff f is an F-map and $\varepsilon X \subseteq f_\beta^\leftarrow(\varepsilon Y)$.*

Proof. Use Theorem 1.4 and Rayburn's combined arguments for [15, 25] and [15, 26]. □

Corollary 2.21. *Let $f : X \to Y$ be an ε-perfect hyper-real F-map. Then f is a fast map iff $\varepsilon X = f_\beta^\leftarrow(\varepsilon Y)$.*

Corollary 2.22. *Let X be pseudocompact and $f : X \to Y$ a closed map. Then Y is pseudocompact iff f is a fast map.*

3. f-normal spaces

We have seen that every map on a nearly realcompact space is a fast map. We now wish to generalize the notion of nearly realcompact and examine how fast maps and F-maps behave on this new class of spaces.

In Rayburn's paper [15], he introduced the notion of a T-normal space, where $X \subseteq T \subseteq \beta X$. Rayburn defined a subset H of X to be *T-hard* if H is closed in $X \cup \mathrm{cl}_{\beta X}(T - X)$. A space X is said to be *T-normal* if disjoint T-hard sets of X are completely separated in X. It follows immediately from this definition that every normal space is T-normal. Rayburn then defined *h-normal* as the special case $T = \upsilon X$; in this case T-hard sets are nothing but what Rayburn referred to in his earlier work more simply as "hard sets".

In this paper we wish to study another special case: the case when $T = X \cup (\beta X - \upsilon X)$. By strictly following Rayburn's lead in terminology we risk causing some confusion, but with this particular T, Rayburn's T-hard sets now become what we have been calling "fast sets". In this case we will call T-normal spaces *f-normal*. Note that every nearly realcompact space is f-normal; f-normal proves to be the generalization of nearly realcompactness that we wish to pursue.

It is immediate that a pseudocompact space is f-normal if and only if it is normal. As for examples of f-normal spaces which are not normal, we may turn to a theorem of Blair and van Douwen. Blair and van Douwen have shown that X is nearly realcompact and nowhere locally compact iff the product of X and any space is nearly realcompact [2, 1.11]. One such example of an f-normal non-normal space then is $\mathbb{Q} \times T$, where T is the Tychonoff Plank. (A necessary condition for a product to be normal is that each factor be normal [4, VII 3.3].)

As for an example of an f-normal space which is not nearly realcompact, consider $\mathbb{N} \oplus [0, \omega_1)$. It is normal since both the summands are. Note though that $\upsilon(\mathbb{N} \oplus [0, \omega_1)) = \mathbb{N} \oplus [0, \omega_1]$ is locally compact. It follows from [12, 6.5] that $\mathbb{N} \oplus [0, \omega_1)$ cannot be nearly realcompact.

In any case, we will find the following theorem useful. It is a direct application of Rayburn's more general result [15, 34] to our present circumstances.

Theorem 3.1. *The following are equivalent*:

(1) *X is f-normal.*
(2) *There is a Y, $X \subseteq Y \subseteq X \cup (\beta X - \upsilon X)$ and Y is f-normal.*
(3) *$X \cup \mathrm{cl}_{\beta X}(\beta X - \upsilon X)$ is normal.*
(4) *Each closed subset of X is completely separated from every disjoint fast set.*

Corollary 3.2. *X is f-normal if and only if $X \cup J$ is normal.*

Let X be non-pseudocompact but locally pseudocompact space. Then J is a non-empty compact set disjoint from X. In [17, 3.5] it was shown that X has a one-point pseudocompactification $\widehat{X}$. It is easy to observe that any subset A of X is closed in $\widehat{X}$ if and only if A is fast in X. By the construction of $\widehat{X}$, it can be shown that $\widehat{X}$ is the quotient space of $X \cup J$ under the quotient map $p : X \cup J \to \widehat{X}$ given by

$$p(x) = \begin{cases} x & \text{if } x \in X, \\ \omega & \text{if } x \in J, \end{cases}$$

for some fixed $\omega \in J$.

Corollary 3.3. *If X is locally pseudocompact but non-pseudocompact, then X is f-normal if and only if $\widehat{X}$ is normal.*

Proof. ($\Rightarrow$) Since X is f-normal, $X \cup J$ is normal. It now follows by [4, VII 3.5] that $\widehat{X}$ is normal.

($\Leftarrow$) If $\widehat{X}$ is normal, then any two disjoint fast subsets of X are disjoint closed sets in $\widehat{X}$. Hence they can be completely separated by an $f \in C(\widehat{X})$. Our result follows. □

Corollary 3.4. *If $\alpha X = X \cup (\beta X - \upsilon X)$ is normal, then X is f-normal. Moreover for any space X in which $\beta X - \upsilon X$ is closed in $\beta X - X$, αX is normal if and only if X is f-normal.*

αX does not appear to have been studied extensively in the literature. It is known to be a pseudocompactification of X [1, Theorem 1]. In any case, the converse of the first statement in our corollary is not true. The referee has graciously provided an example of a normal space X for which αX is not normal. We will postpone the presentation of that example until the last section of this paper.

We might also note that, though the property "$\beta X - \upsilon X$ is closed in $\beta X - X$" is trivially true of all pseudocompact and realcompact spaces, we are not aware that this property has any internal topological characterization. It does, however, have an algebraic one. In [7], a special class of real maximal ideal has been introduced called a Strong Real Maximal (SRM) ideal. A real maximal ideal M^p, $p \in \beta X$, is said to be an SRM ideal if there exists a fast zero set Z in $Z(M^p)$ such that $\mathrm{cl}_{\beta X} Z$ is a neighborhood of p. From an algebraic standpoint, it is proved in [7] that a real maximal ideal M is an SRM ideal if and only if $\exists f \in C^*(X) - M$ such that $\forall g \in C(X)$, $fg \in C^*(X)$. Thus using the notion of SRM ideal we have the following characterization:

Theorem 3.5. *For any space X, $\beta X - \upsilon X$ is closed in $\beta X - X$ if and only if every free real maximal ideal is SRM.*

Proof. By [7], $\{M^p\colon p \in \beta X - J\}$ is precisely the collection of all SRM ideals. Now $\beta X - \upsilon X$ is closed in $\beta X - X$ if and only if $J \cap (\beta X - X) = \beta X - \upsilon X$ if and only if every free real maximal ideal is SRM. □

Corollary 3.6. *If αX is normal, then every closed C-embedded subset of X is f-normal.*

Proof. Let A be a closed C-embedded subspace of X. Then $\beta A - \upsilon A = \mathrm{cl}_{\beta X} A \cap (\beta X - \upsilon X)$. Since A is closed in X, αA is a closed subspace of αX. Hence αA is normal. By Corollary 3.4, A is f-normal. □

Note 3.7. The condition "closed" cannot be dropped. For example, let X be the free union of countably many copies of T^*, i.e. $X = \mathbb{N} \times T^*$, where T^* denotes the one-point compactification of the Tychonoff plank T. Since $\mathbb{N} \times T^*$ is realcompact, $\alpha X = \beta X$ and thus αX is normal. Now fix any $n \in \mathbb{N}$. Then $\{n\} \times T$ is a C-embedded subset of $\mathbb{N} \times T^*$ which is not closed. But $\{n\} \times T$ – being a closed copy of the non-normal pseudocompact space T – is not normal and hence not f-normal.

Note 3.8. The product of a normal space and a compact space need not be f-normal. The pseudocompact space $[0, \omega_1) \times [0, \omega_1]$ is not normal [14, 2.2], and hence not f-normal by Corollary 3.4.

Note 3.9. At the beginning of this section we used the examples $X = \mathbb{Q} \times T$ and $Y = \mathbb{N} \oplus [0, \omega_1)$ to demonstrate that an f-normal space need be neither normal nor nearly realcompact. The topological sum of these spaces provides a single example of an f-normal space which is neither normal nor nearly realcompact. Note that $X \oplus Y$ is not nearly realcompact: Y is a cozero-set of $X \oplus Y$ but not nearly realcompact [2, 1.8]. Neither is $X \oplus Y$ normal: X is a closed subset of $X \oplus Y$ but not normal. It remains to show that $X \oplus Y$ is f-normal. This follows immediately from Theorem 3.11 below.

Lemma 3.10. *A set H is fast in $X \oplus Y$ iff $H = F \cup G$ where F is fast in X and G is fast in Y.*

Proof. ($\Rightarrow$) Let H be fast in $X \oplus Y$ and let $F = H \cap X$ and $G = H \cap Y$. Since F is a closed subset of a fast set in $X \oplus Y$, it too must be fast in $X \oplus Y$ (Theorem 1.4). Observe that X is not only closed, but C-embedded in $X \oplus Y$. It now follows from [17, 4.6] that F is fast in X. The same argument proves that G is fast in Y.

($\Leftarrow$) Now assume $H = F \cup G$ where F is fast in X and G is fast in Y. Since F is fast in X, it follows from Theorem 1.4 that there is a compact set T_1 in X such that for every neighborhood V of T_1, there is a pseudocompact subset P_1 of X such that $F - V$ is completely separated from $X - P_1$. Similarly, since G is fast in Y, there is a compact set T_2 in Y such that for every neighborhood W of T_2, there is a pseudocompact subset P_2 of Y such that $G - W$ is completely separated from $Y - P_2$. Let $T = T_1 \cup T_2$ and $P = P_1 \cup P_2$. Note that T is compact and P is pseudocompact. Now let U be any neighborhood of T in $X \oplus Y$, and let $V = U \cap X$ and $W = U \cap Y$. Then $T_1 \subseteq V$, $T_2 \subseteq W$, $F - V$ is completely separated from $X - P_1$, and $G - W$ is completely separated from $X - P_2$. It remains to note that since $X \cap Y = \emptyset$,

$$X \oplus Y - P = (X - P_1) \cup (Y - P_2),$$
$$F \cup G - U = (F - V) \cup (G - W)$$

and so we may paste together a function that completely separates $F \cup G - U$ and $X \oplus Y - P$. Thus $F \cup G$ is fast in $X \oplus Y$. □

Theorem 3.11. *If X and Y are f-normal, then $X \oplus Y$ is f-normal.*

Proof. Let F be fast in $X \oplus Y$ and let H be closed in $X \oplus Y$ with $F \cap H = \emptyset$. By our lemma, $F \cap X$ is fast in X. Moreover $H \cap X$ is closed in X and disjoint from $F \cap X$. Since X is f-normal, it follows from Theorem 3.1 that $F \cap X$ is completely separated from $H \cap X$ in X. Similarly we may show that $F \cap Y$ is completely separated from $H \cap Y$ in Y. We may then paste together a continuous function that separates F and H in $X \oplus Y$, and so $X \oplus Y$ must be f-normal. □

As Rayburn has noted, Zenor makes the following remarks about the map ϕ_A in [22]:

(1) X is normal if and only if ϕ_A is a quotient map for each closed set A in X.
(2) Each closed set is completely separated from every disjoint zero set in X if and only if ϕ_A is a quotient map for each zero set Z in X.

We establish these facts more generally.

Definition 3.12. Let P be a topological property. We call a closed subset of X *P-closed* if it satisfies the property P, and we call a space *P-normal* if each closed set can be completely separated from every disjoint P-closed set in X.

Theorem 3.13. *X is P-normal if and only if ϕ_A is quotient map for every P-closed set in X.*

Proof. (⇒) Suppose X is P-normal. Let U be a non-empty subset of X/A such that $\phi_A^{\leftarrow}(U)$ is open in X, where A is P-closed in X. We need to show that U is open in X. Let $y \in U$; if $y \neq \phi_A(A)$ then $\phi_A^{\leftarrow}(y) = y \in \phi_A^{\leftarrow}(U)$. Moreover there exists an open set V in X such that $y \in V \subseteq \phi_A^{\leftarrow}(U)$ and $V \cap A = \emptyset$. Now there exists $g \in C(X)$ such that $g(y) = 0$ and $g(X - V) = \{1\}$. Note that $A \subseteq X - V$. Define $l : X/A \to \mathbb{R}$ by $l(x) = g(x)$ if $x \neq \phi_A(A)$ and 1 if $x = \phi_A(A)$. Then $l \circ \phi_A(A) = g \in C(X)$ which implies that $l \in C(X/A)$. Then $y \in \{x \in X/A\colon l(x) < \frac{1}{2}\} \subseteq U$. So y is an interior point of U whenever $y \neq \phi_A(A)$.

Now suppose $y = \phi_A(A) \in U$. Then $A \subseteq \phi_A^{\leftarrow}(U)$ and $X - \phi_A^{\leftarrow}(U)$ is a closed set in X disjoint from P-closed set A. Since X is P-normal, $\exists f \in C(X)$ such that $f(A) = \{0\}$ and $f(X - \phi_A^{\leftarrow}(U)) = \{1\}$. Define l analogously. Then we have $\phi_A(A) \in \{x \in X/A\colon l(x) < \frac{1}{2}\} \subseteq U$. So $\phi_A(A)$ is an interior point of U and thus U is open in X.

(⇐) Immediate. □

Corollary 3.14. *X is h-normal if and only if ϕ_A is a quotient map for each hard set A in X.*

Corollary 3.15. *X is f-normal if and only if ϕ_A is a quotient map for each fast set A in X.*

We now close this section with a theorem and corollary that are direct analogs to the closing results of Rayburn's paper. Their proofs are accomplished similarly.

Theorem 3.16. *For any space X, the following are equivalent*:

(1) *X is f-normal.*
(2) *Every WZ-map on X is an F-map.*
(3) *Every ε-perfect WZ-map on X is closed.*

Corollary 3.17. *Let X be an f-normal space and $f : X \to Y$ be a hyper-real, ε-perfect, WZ-map. Then Y is f-normal if and only if for every ε-perfect WZ-map g on Y, $g \circ f$ is a WZ-map.*

4. An example

Example 4.1. As promised, we now present the referee's example of a normal space X for which αX is not normal.

Let $p \in \mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ such that $\mathbb{N}^*\backslash\{p\}$ is not normal. That many such points exist (assuming only ZFC) was established in [3]. (This paper is not readily available; the reader may find [19, Theorem 4] more accessible.) The space $X = \omega_1 \times (\mathbb{N} \cup \{p\})$ has the properties we desire.

X is normal. Let $Y = (\omega_1 + 1) \times (\mathbb{N} \cup \{p\})$. It is easy to see that Y is normal: Y is σ-compact, and so Lindelöf and hence normal. If we can show that disjoint closed sets of X have disjoint closures in Y, we will be done.

Let A and B be two disjoint closed sets in X. Let L_n denote the "horizontal line" $\omega_1 \times \{n\}$, let $S(\alpha, K) = (\alpha, \omega_1] \times (K \cup \{p\})$ where K is a subset of $\mathbb{N}$ and $\alpha \in [0, \omega_1)$, and let $\widehat{S}(\alpha, K) = (\alpha, \omega_1] \times K$. Since $A \cap L_n$ and $B \cap L_n$ are two disjoint closed sets of L_n, where $n \in \mathbb{N} \cup \{p\}$ and L_n is a copy of ω_1, one of $A \cap L_n$ and $B \cap L_n$ should be bounded in the sense that there exists $\alpha < \omega_1$ such that either $\widehat{S}(\alpha, \{n\}) \cap A$ or $\widehat{S}(\alpha, \{n\}) \cap B$ is void. Since $\widehat{S}(\alpha, \{n\})$ is an open neighborhood of (ω_1, n), for each $n \in \mathbb{N}$, $(\omega_1, n) \notin \mathrm{cl}_Y A \cap \mathrm{cl}_Y B$.

Next we show that $(\omega_1, p) \notin \mathrm{cl}_Y A \cap \mathrm{cl}_Y B$. Let $N_A = \{n \in \mathbb{N}\colon\ A \cap L_n$ is not bounded$\}$ and $N_B = \{n \in \mathbb{N}\colon\ B \cap L_n$ is not bounded$\}$. Then $N_A \cap N_B = \emptyset$. Let $N = \mathbb{N}\backslash N_A \cup N_B$. Thus $\{N, N_A, N_B\}$ forms a partition of $\mathbb{N}$. Now note that for $K \subset \mathbb{N}$, $K \cup \{p\}$ is open in $\mathbb{N} \cup \{p\}$ if and only if $p \in \mathrm{cl}_{\beta\mathbb{N}} K$. We have two cases to consider.

Case 1. Suppose $N \cup \{p\}$ is open. Now on N, both A and B are bounded, and on the top edge L_p either A or B is bounded. Thus there exists $\beta < \omega_1$ such that $S(\beta, N) \cap B = \emptyset$ or $S(\beta, N) \cap A = \emptyset$. Since $S(\beta, N)$ is a neighborhood of (ω_1, p), $(\omega_1, p) \notin \mathrm{cl}_Y A \cap \mathrm{cl}_Y B$.

Case 2. Suppose $N \cup \{p\}$ is not open. Then either $N_A \cup \{p\}$ or $N_B \cup \{p\}$ is open. Without loss of generality let us assume that $N_A \cup \{p\}$ is open. Then $A \cap L_p$ is not bounded on L_p. In fact choose any $(\alpha, p) \in L_p$. Then we have an increasing sequence of ordinals $\{t_i\}$ such that $t_i > \alpha$ and $(t_i, n_i) \in A$ for each $n_i \in N_A$. Take $t = \sup_i\{t_i\}$. Then $(t, p) \in A$ as $p \in \mathrm{cl}_{\beta\mathbb{N}} A$ and $t > \alpha$. Since for each $x \in N_A \cup \{p\}$, $B \cap L_x$ is bounded, there exists $\alpha < \omega_1$ such that $S(\alpha, N_A) \cap B = \emptyset$ and hence $(\omega_1, p) \notin \mathrm{cl}_Y B$. A similar argument proceeds if $N_B \cup \{p\}$ is open.

Thus A and B have disjoint closures in Y and hence X is normal.

αX is not normal. Note that Y is realcompact and that $Y = \upsilon X$. Indeed Y is the product of a compact and realcompact space, and X is C-embedded in Y. To see that X is C-embedded in Y, note that every continuous function on X is constant on a tail of each horizontal line L_x, where $x \in \mathbb{N} \cup \{p\}$. Thus there exists $\alpha < \omega_1$ such that $f(S_1(\alpha, \{n\})$ is constant and equal to r_n (say) for each $n \in \mathbb{N} \cup \{p\}$. Extend f to Y by assigning the value r_n at (ω_1, n) for each $n \in \mathbb{N} \cup \{p\}$. Let us denote the extended function by f^Y. Clearly f^Y is continuous at (ω_1, n), for each $n \in \mathbb{N}$. To show that f^Y is continuous at (ω_1, p), choose any point (β, p) at the top edge L_p such that $\beta \gneqq \alpha$. Then $f^Y(\beta, p) = r_p$ and by continuity of f, for any neighborhood W of r_p, there exists a neighborhood U of α and V of p with the property that $U \times \{p\} \subsetneq S_1(\alpha, \{p\})$ and $f(U \times V) \subseteq W$. Then it can be shown that $f^Y(S(\alpha, V - \{p\})) \subseteq W$. Since $S(\alpha, V - \{p\})$ is an open neighborhood of $(\omega_1, \{p\})$, f^Y is also continuous at $(\omega_1, \{p\})$. Thus X is C-embedded in Y and $Y = \upsilon X$ as claimed.

Now let $F : \beta X \to \omega_1 + 1 \times \beta\mathbb{N}$ be the Stone-extension of the identity map on X. Then from the fact that $F|_Y$ is the identity map on Y, it is easy to verify that

$$F(\beta X - \upsilon X) = (\omega_1 + 1 \times \beta\mathbb{N})\backslash Y. \qquad (*)$$

Let $Z = F^{\leftarrow}(\{\omega_1\} \times \beta\mathbb{N})$. Then Z is a compact subset of βX and moreover $Z \cap X = \emptyset$. Clearly $F|_Z : Z \to \{\omega_1\} \times \beta\mathbb{N}$ is a perfect map and hence by [13, 6.5(c)], there exists a closed subset K of Z such that $F|_K : K \to \{\omega_1\} \times \beta\mathbb{N}$ is an irreducible map.

Claim. *$F|_K : K \to \{\omega_1\} \times \beta\mathbb{N}$ is a homeomorphism.*

Proof. Let $A = \{x_n\colon\ x_n \in F^{\leftarrow}(n),\ \forall n \in \mathbb{N}\}$. Now A is a copy of $\mathbb{N}$ and since $F|_K$ is irreducible, $\mathrm{cl}_K A = K$, otherwise $\{\omega_1\} \times \mathbb{N} \subset F(\mathrm{cl}_K A) \subseteq \{\omega_1\} \times \beta\mathbb{N}$. Since $\mathrm{cl}_K A$ is compact, $F(\mathrm{cl}_K A) = \{\omega_1\} \times \beta\mathbb{N}$, a contradiction. Thus K is a compactification of $\mathbb{N}$ and $\beta\mathbb{N}$ is a continuous image of K as $\{\omega_1\} \times \beta\mathbb{N}$ is homeomorphic with $\beta\mathbb{N}$. Thus K is larger than $\beta\mathbb{N}$ in the lattice of compactifications of $\mathbb{N}$ and hence K must be homeomorphic with $\beta\mathbb{N}$.

Now let $B = K \setminus F^{\leftarrow}(\mathbb{N} \cup \{p\})$. Since $F^{\leftarrow}(\mathbb{N} \cup \{p\})$ is contained in Y, it follows from $(*)$ that $B = K \cap (\beta X - \upsilon X)$. Thus B is a closed subset of αX because $K \cap X = \emptyset$ and thus every limit point of B in βX must lie in $\upsilon X - X$ which has been truncated from βX to obtain αX. It is easy to observe that B is a homeomorphic copy of $\mathbb{N}^* \setminus \{p\}$ which is not normal. Thus αX is not normal. □

Many thanks to the referee for this last example and his other helpful suggestions.

References

[1] C. Aull, J. Sawyer, The pseudocompact extension αX, Proc. Amer. Math. Soc. 102 (4) (1988) 1057–1064.
[2] R.L. Blair, E. van Douwen, Nearly realcompact spaces, Topology Appl. 47 (3) (1992) 209–221.
[3] A. Błaszczyk, A. Szymanski, Some non-normal subspaces of the Čech–Stone compactification of a discrete space, in: Proceedings of the 8th Winter School on Abstract Analysis, Ŝpindlerovy Mlyn, 1980, pp. 35–39.
[4] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.
[5] L. Gillman, M. Jerison, Rings of Continuous Functions, University Series in Higher Mathematics, Van Nostrand, Princeton, NJ, 1960.
[6] K. Hardy, R.G. Woods, On c-realcompact spaces and locally bounded normal functions, Pacific J. Math. 43 (1972) 647–656.
[7] M. Henriksen, B. Mitra, $C(X)$ can sometimes determine X without X being realcompact, Comm. Math. Univ. Carolin., in press.
[8] M. Henriksen, M. Rayburn, On nearly pseudocompact spaces, Topology Appl. 11 (1980) 161–172.
[9] Y. Ikeda, Mapping and μ-, η-, ψ-compact spaces, Bull. Tokyo Gakugei Univ. Ser. IV 27 (1975) 1–4.
[10] Y. Ikeda, On mapping theorems of the spaces determined by ideals in rings of continuous functions, Bull. Tokyo Gakugei Univ. Ser. IV 36 (1984) 17–22.
[11] T. Isiwata, Mappings and spaces, Pacific J. Math. 20 (1967) 455–480.
[12] D. Johnson, M. Mandelker, Functions with pseudocompact support, Gen. Topology Appl. 3 (1973) 331–338.
[13] J. Porter, R.G. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer, New York, 1988.
[14] T. Przymusiński, Products of normal spaces, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 781–826.
[15] M. Rayburn, Maps and h-normal spaces, Pacific J. Math. 79 (2) (1978) 549–561.
[16] J. Schommer, Nearly realcompact and nearly pseudocompact spaces, PhD dissertation, Ohio Univ., Athens, Ohio, USA, 1992.
[17] J. Schommer, Fast sets and nearly realcompact spaces, Houston J. Math. 20 (1) (1994) 161–174.
[18] J. Schommer, M.A. Swardson, Almost* realcompactness, Comm. Math. Univ. Carolin. 42 (2) (2001) 385–394.
[19] A. Szymanski, On b-points in special compact spaces, Preprint.
[20] M. Weir, Hewitt–Nachbin Spaces, North-Holland Math. Studies, American Elsevier, New York, 1975.
[21] R.G. Woods, Maps that characterize normality properties and pseudocompactness, J. London Math. Soc. 7 (1973) 453–461.
[22] P. Zenor, A note on Z-mappings and WZ-mappings, Proc. Amer. Math. Soc. 23 (1969) 273–275.