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Joint project of fishery and poultry – A bioeconomic model

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ABSTRACT

This paper considers the optimal joint harvest of prawns and poultry in a linked bioeconomic system. Through the cultivation process, poultry and prawns are reciprocal predators of one another. Prawns of non-marketable quality are fed to the birds, and birds which perish (in greater numbers in the face of increased density) are fed to prawns, along with a lot of other things that one does not usually consider prawns to eat (hogs, broken rice, etc.). The paper derives optimality conditions for the joint "effort" imposed in each of these industries, where effort is somehow analogous to the control variable in classical Gordon–Schaefer fishery problems. Growth of both species is governed by parameters as well as externally applied nutrients and the biomass of the other species available as supplemental nutrition. Analysis of the boundedness of this dynamical system is discussed. The conditions for local and global stability are derived. Finally, an optimal harvesting policy is discussed by applying Pontryagin's Maximal Principle. Due to linearity of the objective function with respect to the control variable, the solution is bang-bang in this control and the best policy is to reach the singular equilibrium as quickly as possible by switching to the singular control.

1. Introduction

The exploitation of biological resources and the harvest of population species are commonly practiced in fisheries, forestry and wild life management. Bioeconomic modeling is concerned with scientific management of exploitation of biological resources, taking into account economic loss or gain. Problems related to the exploitation of multispecies systems are not only interesting but also difficult as there are theoretical as well as practical difficulties in the determination of an optimal policy for the harvesting of multispecies systems.

The need for sustenance of the resources for future generation is the motivating factor for most of the studies in this area. The essential concepts required by researchers in this field have been presented by Clark [1], where he developed optimal strategies for both single and multispecies deterministic fishery models. Clark [1] also examined the effects of harvesting single species in the Gause's model [2]. Harvesting of two species in the Gause's model [2] was discussed by Chaudhuri [3]. Ragozin and Brown [4] investigated the optimal policy for harvesting the predator of predator–prey system while the prey has no market value by virtue of being uncatchable. Mesterton-Gibbons [5] extended the work of Ragozin and Brown [4] by discussing the optimal policy for combined harvesting of both predator and prey. Mesterton-Gibbons [6] described a technique to obtain the optimal harvesting policy for a Lotka–Volterra ecosystem of two interdependent populations when harvesting rate is proportional to harvest effort and either a single stock is selectively or both stocks are harvested together. Mesterton-Gibbons [5,6] solved for the singular control and described the approach to equilibrium for a generalized ecolog-

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ical interaction. Chaudhuri and Saha Ray [7] considered the problem of exploiting a prey-predator community in which the growth of both the prey and the predator obeys the logistic law of growth. Hoagland and Jin [8] partially solved the same generalized interacting-species case while one species had extractive value and the other existence value. Hoagland et al. [9] examined the interactions between marine (or estuarine) aquaculture and a wild harvest fishery. In this connection, they developed a framework to analyse the tradeoffs between a wild harvest fishery and aquaculture occuring in the same region and selling into the same market. Jin et al. [10] developed an economic-ecological model by merging an input–output model of coastal economy with a model of a marine food web. According to them, specific environmental and natural resource parameters may be calculated by the ecological model and then used as input parameters for the economic model. Herrera [11] analysed the nonselective harvest of two stocks with generalized ecological interaction and different persistent distributions across two spatial strata. In his model, an aggregate effort control, depending on harvest response was shown to partially dissipate rents relative to the case where the spatial distribution of effort could be specified. Purohit and Chaudhuri [12] further developed this model considering taxation as a control instrument and opted for more realistic catch rate function, where the dynamics of the multispecies is governed by Gompertz law of growth.

In a competitive market, the quality of a product in any business organization is a big issue. The conforming quality of the product is sold in the market to keep brand image of the enterprize, whereas the non-conforming quality (deteriorated) items are used in another purpose at less marketing value. Deterioration, in general, may be considered as the result of various effects on stock such as damage, decreasing usefulness in main purpose and many more. Goyal and Giri [13] presented a review of the inventory literature for deteriorating items since early 1990s. Other papers related to this area are Wee [14,15], Cardenas-Barron [16], Goyal and Cardenas-Barron [17], Khanra and Chaudhuri [18], Sana and Chaudhuri [19], Mukhopadhyay et al. [20,21], Ghosh and Chaudhuri [22], among others.

Effects of inflation and time value of money can no longer be ignored in the present economy because large scale of inflation and time value of money decline the purchasing power of money sharply. Buzacott [23] was the first who extended the economic order quantity model, incorporating inflation and time value of money. The other noteable paper in this direction, mention should be made of the works by Beirman and Thomas [24], Misra [25,26], Aggarwal [27], Chandra and Bahner [28], Ray and Chaudhuri [29], Sana [30], among others.

According to Singholka [31] and Michael [32], giants prawns feed chicken or ground fish flesh mixed with cooked broken rice, beef, hog, etc. Probably much of the supplemental food (broken rice, dead poultry, beef, hog, fish processing waste, prawn processing wastes, snails, etc.) added acts as a fertilizer and increases the biological productivity of the pond as a whole rather than acting as a true prawn feed. Certainly it seems, particularly in macrobrachium culture that much of the pelleted food added is eaten by small fishes, which themselves form a source of food for the giant prawns. Based on the above philosophy, the current study is to model a combined project of prawn fishery and poultry of birds. It is rational to argue that the deteriorated (non-conforming quality) prawn (mainly shrimp) is used as a nutrient of poultry. Conversely, the dead birds/living birds and excreta of birds are used as a nutrient of prawn fishery. Consequently, the nutrients of prawn fishery and poultry are interconnected. Quite often, these are controlled with outside supply of nutrients. The growth rates of prawn and birds in poultry are considered as functions of available nutrients and volume of on-hand biomass simultaneously.

2. Notation

The following notation are considered to develop the model:

Notation	
$X(t)$	biomass of prawn at time t
$Y(t)$	biomass of birds (broiler) at time t
$N_x(t)$	amount of nutrient at time t for prawn
κ_x	upper limit of $N_x(t)$
$N_y(t)$	amount of nutrient at time t for broiler
κ_y	upper limit of $N_y(t)$
σ_x	natural growth rate of X which is independent of supplied nutrient N_x
σ_y	natural growth rate of Y which is independent of supplied nutrient N_y
θ_x	deterioration rate which is a fraction of on-hand biomass of $X(t)$
θ_y	deterioration rate which is a fraction of on-hand biomass of $Y(t)$
$E(t)$	joint effort function to look after the project at time t
γ_x	harvesting coefficient of prawn due to effort
β_x	natural harvesting coefficient of prawn, independent of effort, which is a fraction of on-hand biomass of $X(t)$
γ_y	harvesting (available stock for selling) coefficient of broiler due to effort
β_y	natural harvesting (available stock for selling) coefficient of broiler, independent of effort, which is a fraction of on-hand biomass of $Y(t)$
(N_{01}, N_{02})	supplied nutrients, from outside, per unit biomass of prawn and broiler, respectively

(*continued on next page*)

(*continued*)

Notation	
(α_x, α_y)	conversion factors, both are positive, of nutrients from deteriorated prawn and broiler respectively
(τ_x, τ_y)	absorbed nutrients per unit biomass of (X, Y) respectively
(C_1, C_2)	cost of unit mass of supplied nutrients (N_{01}, N_{02}) respectively
(ξ_x, ξ_y)	treatment (effort) cost per unit biomass of (X, Y) respectively
$C(X, Y)$	cost per unit effort
(p_x, p_y)	selling price per unit mass of prawn and broiler respectively
$\delta = (r - i)$	r and i are rates of interest and inflation per unit currency

3. Formulation of the model

We consider a joint project of fishery and poultry (see Fig. 1). Quite often, the excreta (matters discharged from animal bodies) of broilers and deteriorated broilers are used as nutrient (food) of prawn. The deteriorated prawn and shrimp which have less market price are used as nutrient (food) of the broiler. The state of nutrients of (X, Y) at time t are as follows:

$$N_x(t) = N_{01}X(t) + \alpha_x\theta_yY(t) - \tau_xX(t) = N_1X(t) + \alpha_x\theta_yY(t), \quad \text{where } N_1 = N_{01} - \tau_x \tag{1}$$

and

$$N_y(t) = N_{02}Y(t) + \alpha_y\theta_xX(t) - \tau_yY(t) = N_2Y(t) + \alpha_y\theta_xX(t), \quad \text{where } N_2 = N_{02} - \tau_y. \tag{2}$$

Here N_{01} is supplied nutrient per unit biomass of X from outside, $\alpha_x\theta_yY(t)$ is converted nutrient from deteriorated broiler and excreta of broiler and τ_x is the absorbed nutrient per unit biomass of X. N_{02} is supplied nutrient per unit mass of broiler from outside, $\alpha_y\theta_xX$ is converted nutrient from deteriorated prawn and τ_y is absorbed nutrient per unit biomass of broiler. In this model, (N_{01}, N_{02}) is less than (τ_x, τ_y) because the partial amounts $(\alpha_x\theta_yY, \alpha_y\theta_xX)$ of nutrients for (N_x, N_y) are produced in the firm itself. The governing dynamical system of the species (X, Y) are as follows:

$$\dot{X}(t) = (\sigma_x + N_x/\kappa_x)X - h_x - \theta_xX \tag{3}$$

and

$$\dot{Y}(t) = (\sigma_y + N_y/\kappa_y)Y - h_y - \theta_yY, \tag{4}$$

where θ_x is the deterioration (decay) rate of the on-hand stock of biomass of X and θ_y is the deterioration (decay) rate of the on-hand stock of biomass of Y, $0 < (\theta_x, \theta_y) < 1$. $(N_x/\kappa_x)X$ is the growth rate of X that depends upon existing nutrient (N_x) and volume of biomass of X. $(N_y/\kappa_y)Y$ is the growth rate of Y that depends upon existing nutrient (N_y) and volume of biomass of Y. The harvesting rates (h_x, h_y) of the species (X, Y) are considered as follows:

$$h_x = \beta_xX + E\gamma_xX \tag{5}$$

and

$$h_y = \beta_yY + E\gamma_yY, \tag{6}$$

where $\beta_x(0 < \beta_x < 1)$ is a normal harvesting coefficient of X, β_xX is normal harvesting rate of X, γ_x is harvesting coefficient due to effort E and γ_xEX is harvesting rate by effort E. Similarly, β_yY is normal harvesting (available stock for selling) rate of Y and γ_yEY is harvesting (available stock for selling) rate due to effort E. Although the species are non-interacting directly, their growths are mutually dependent because of the common effort (E) and also their nutrients are interconnected. Here, the term 'harvesting' is used in a broad sense. Harvesting is to store the species (broiler and prawn) to meet the demand in the market. The fishes are harvested with nets, bedding or any other processes. As the nutrient of poultry is dependent on fishes, the availability of poultry birds is indirectly dependent on effort E. Consequently, harvesting rate of poultry is a function of common effort E. Moreover, E is the common effort to look after the overall performance of the project.

Now, substituting Eqs. (1), (2), (5) and (6) in Eqs. (3) and (4), we have

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} X(-A_x + (N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y) \\ Y(-A_y + (\alpha_y\theta_x/\kappa_y)X + (N_2/\kappa_y)Y) \end{pmatrix}, \tag{7}$$

where $A_x = \beta_x + E\gamma_x + \theta_x - \sigma_x$ and $A_y = \beta_y + E\gamma_y + \theta_y - \sigma_y$. Now, solving $-A_x + (N_1/\kappa_x)\overline{X} + (\alpha_x\theta_y/\kappa_x)\overline{Y} = 0$ and $-A_y + (\alpha_y\theta_x/\kappa_y)\overline{X} + (N_2/\kappa_y)\overline{Y} = 0$, we have

$$\begin{pmatrix} \overline{X} = \frac{A_x\kappa_xN_2 - A_y\kappa_y\alpha_x\theta_y}{N_1N_2 - \alpha_x\alpha_y\theta_x\theta_y} \\ \overline{Y} = \frac{A_y\kappa_yN_1 - A_x\kappa_x\alpha_y\theta_x}{N_1N_2 - \alpha_x\alpha_y\theta_x\theta_y} \end{pmatrix}. \tag{8}$$

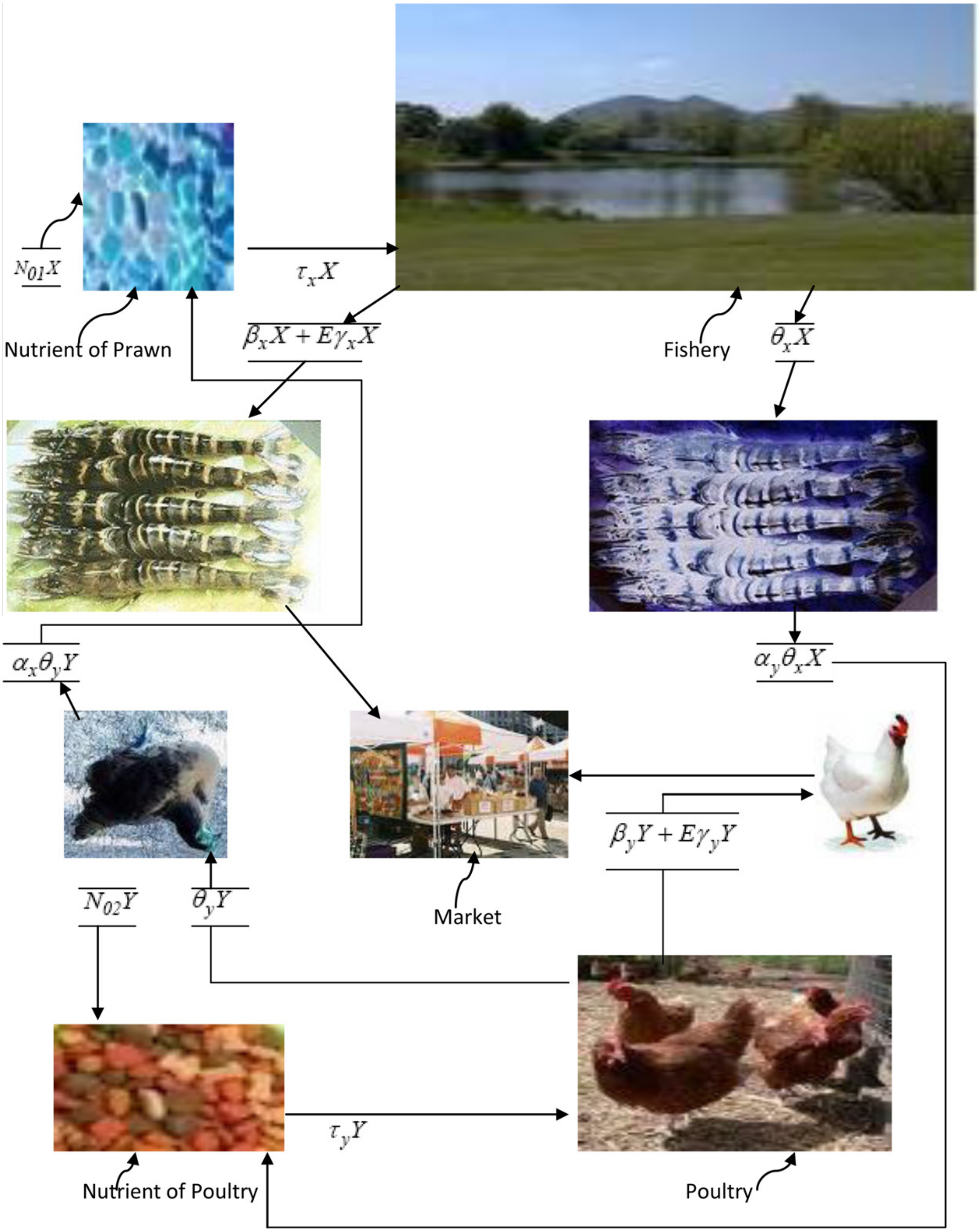

Fig. 1. Pictorial representation of the project.

For feasibility of our model, $\overline{X} \geqslant 0$ and $\overline{Y} \geqslant 0$ imply

$$\begin{pmatrix} A_x\kappa_x N_2 \geqslant A_y\kappa_y\alpha_x\theta_y \ \& \ A_y\kappa_y N_1 \geqslant A_x\kappa_x\alpha_y\theta_x & \text{when } N_1N_2 > \alpha_x\alpha_y\theta_x\theta_y \\ A_x\kappa_x N_2 \leqslant A_y\kappa_y\alpha_x\theta_y \ \& \ A_y\kappa_y N_1 \leqslant A_x\kappa_x\alpha_y\theta_x & \text{when } N_1N_2 < \alpha_x\alpha_y\theta_x\theta_y \end{pmatrix}. \tag{9}$$

3.1. Equilibria of the system

The possible steady states of the dynamical system (7) are: (i) $S_0 = (0,0)$, (ii) $S_1 = (A_x\kappa_x/N_1, 0)$, (iii) $S_2 = (0, A_y\kappa_y/N_2)$, (iv) $S_3 = (\overline{X}, \overline{Y})$ with conditions in Eq. (9).

3.2. Boundedness of the system

Lemma 1. *All solutions of the dynamical systems of Eq. (7) which starts in R_2^+ are uniformly bounded.*

Proof. See Appendix A □

3.3. Local stability analysis

We shall now investigate the local behavior of critical points of the dynamical system in Eq. (7). The variational matrix of the system of Eq. (7) is

$$V(X,Y) = \begin{pmatrix} -A_x + (2N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y & (\alpha_x\theta_y/\kappa_x)X \\ (\alpha_y\theta_x/\kappa_y)Y & -A_y + (\alpha_y\theta_x/\kappa_y)X + (2N_2/\kappa_y)Y \end{pmatrix}. \tag{10}$$

At the critical point $S_0(0,0)$, the variational matrix is

$$V(0,0) = \begin{pmatrix} -A_x & 0 \\ 0 & -A_y \end{pmatrix}.$$

Eigenvalues corresponding to S_0 are $-A_x$ and $-A_y$. Here, $S_0(0,0)$ is a stable node if $A_x > 0$ and $A_y > 0$ hold simultaneously. At the point $S_1(A_x\kappa_x/N_1, 0)$, the variational matrix is

$$V(A_x\kappa_x/N_1, 0) = \begin{pmatrix} A_x & \alpha_x\theta_y A_x/N_1 \\ 0 & -A_y + \alpha_y\theta_x A_x\kappa_x/(N_1\kappa_y) \end{pmatrix}.$$

The corresponding eigenvalues of the above are A_x and $-A_y + \alpha_y\theta_x A_x\kappa_x/(N_1\kappa_y)$ both of which are negative if $A_x < 0$ and $A_y > \alpha_y\theta_x A_x\kappa_x/(N_1\kappa_y)$ hold. These are the conditions for S_1 being a stable node. Similarly, at $S_2(0, A_y\kappa_y/N_2)$,

$$V(0, A_y\kappa_y/N_2) = \begin{pmatrix} -A_x + \alpha_x\theta_y\kappa_x A_y/(\kappa_x N_2) & 0 \\ \alpha_y\theta_x A_y/N_2 & A_y \end{pmatrix}.$$

The corresponding eigenvalues of S_2 are $-A_x + \alpha_x\theta_y\kappa_x A_y/(\kappa_x N_2)$ and A_y both of which are negative if $A_x > \alpha_x\theta_y\kappa_x A_y/(\kappa_x N_2)$ and $A_y < 0$ hold. These are the conditions of S_2 being a stable node. At the critical point $S_3(\overline{X}, \overline{Y})$, the variational matrix is

$$V(\overline{X}, \overline{Y}) = \begin{pmatrix} -A_x + (2N_1/\kappa_x)\overline{X} + (\alpha_x\theta_y/\kappa_x)\overline{Y} & (\alpha_x\theta_y/\kappa_x)\overline{X} \\ (\alpha_y\theta_x/\kappa_y)\overline{Y} & -A_y + (\alpha_y\theta_x/\kappa_y)\overline{X} + (2N_2/\kappa_y)\overline{Y} \end{pmatrix}.$$

The characteristic equation of $V(\overline{X}, \overline{Y})$ is

$$\lambda^2 - \lambda\left[-(A_x + A_y) + (\alpha_y\theta_x/\kappa_y + 2N_1/\kappa_x)\overline{X} + (\alpha_x\theta_y/\kappa_x + 2N_2/\kappa_y)\overline{Y}\right] + (-A_x + (2N_1/\kappa_x)\overline{X} + (\alpha_x\theta_y/\kappa_x)\overline{Y})$$
$$(-A_y + (\alpha_y\theta_x/\kappa_y)\overline{X} + (2N_2/\kappa_y)\overline{Y}) - (\alpha_x\alpha_y\theta_x\theta_y/(\kappa_x\kappa_y))\overline{XY} = 0.$$

Now, $S_3(\overline{X}, \overline{Y})$ will be a stable node if both the eigenvalues of the above is negative, i.e.,

$$(A_x + A_y) > (\alpha_y\theta_x/\kappa_y + 2N_1/\kappa_x)\overline{X} + (\alpha_x\theta_y/\kappa_x + 2N_2/\kappa_y)\overline{Y} \tag{11}$$

and

$$\left(\alpha_x\alpha_y\theta_x\theta_y/(\kappa_x\kappa_y)\right)\overline{XY} < (-A_x + (2N_1/\kappa_x)\overline{X} + (\alpha_x\theta_y/\kappa_x)\overline{Y})(-A_y + (\alpha_y\theta_x/\kappa_y)\overline{X} + (2N_2/\kappa_y)\overline{Y}) \tag{12}$$

are satisfied. Let us consider a numerical example as follows:

Example 1. We consider the values of the parameters in appropriate units as follows: $\kappa_x = 500$, $\kappa_y = 500$, $N_{01} = 2.0$, $N_{02} = 1.5$, $\sigma_x = 7.0$, $\sigma_y = 9.7$, $\gamma_x = 0.4$, $\gamma_y = 0.5$, $\beta_x = 0.5$, $\beta_y = 0.6$, $\theta_x = 0.4$, $\theta_y = 0.2$, $\alpha_x = 15.0$, $\alpha_y = 15.5$, $\tau_x = 6.5$, $\tau_y = 7.2$, $E = 15.0$. Then the critical point (338.30, 490.78) is a locally stable node because the eigenvalues are (−8.06, −0.58). Fig. 2 shows that (338.30, 490.78) is asymptotically stable.

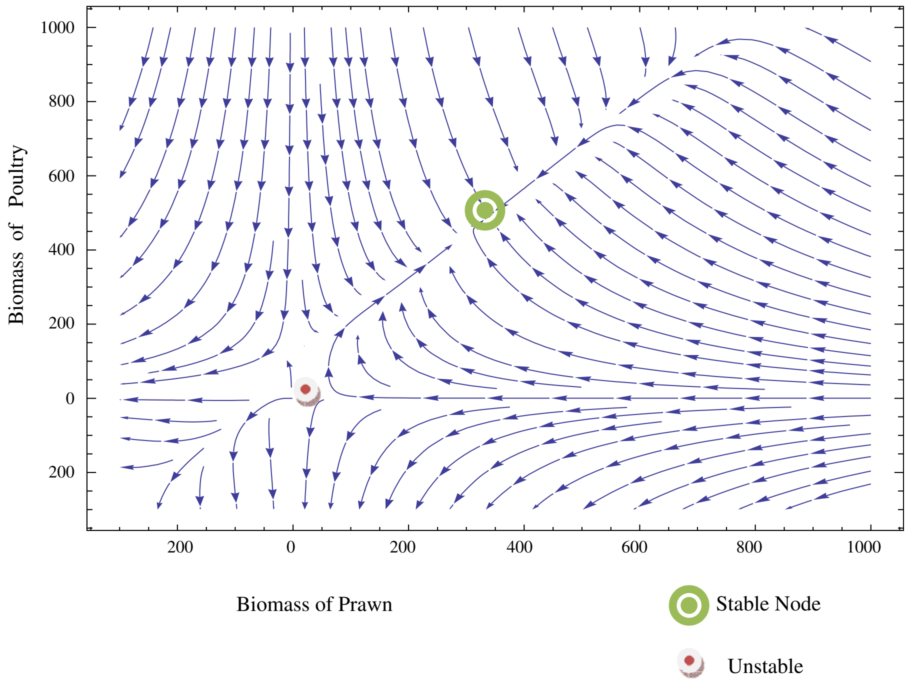

Fig. 2. Critical points of Example 1.

3.4. Global stability analysis

We shall study global stability of the system of Eq. (7) by considering a suitable Lyapunov function

$$F(X,Y) = \left[(X-\overline{X}) - \overline{X}\ln(X/\overline{X})\right] + h\left[(Y-\overline{Y}) - \overline{Y}\ln(Y/\overline{Y})\right],$$

where h is a suitable positive constant to be determined later. $F(\overline{X},\overline{Y})$ is zero at the equilibrium point $(\overline{X},\overline{Y})$ and is positive for all other values of $(X,Y)\epsilon R_2^+$. The time derivative of F along the trajectories of Eq. (7) is

$$\begin{aligned}\dot{F} &= \left(\frac{X-\overline{X}}{X}\right)\dot{X} + h\left(\frac{Y-\overline{Y}}{Y}\right)\dot{Y} = (X-\overline{X})\left(-A_x + (N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y\right) + h(Y-\overline{Y})\left(-A_y + (\alpha_y\theta_x/\kappa_y)X + (N_2/\kappa_y)Y\right)\\ &= (N_1/\kappa_x)(X-\overline{X})^2 + (\alpha_x\theta_y/\kappa_x + \alpha_y\theta_x/\kappa_y)(X-\overline{X})(Y-\overline{Y}) + (N_2/\kappa_y)(Y-\overline{Y})^2, \quad \text{where } h=1,\\ &= [X-\overline{X}, Y-\overline{Y}]^T P[X-\overline{X}, Y-\overline{Y}],\end{aligned}$$

where

$$P = \begin{pmatrix} (N_1/\kappa_x) & (1/2)(\alpha_x\theta_y/\kappa_x + \alpha_y\theta_x/\kappa_y) \\ (1/2)(\alpha_x\theta_y/\kappa_x + \alpha_y\theta_x/\kappa_y) & (N_2/\kappa_y) \end{pmatrix}.$$

The characteristic equation of the above matrix is

$$\lambda^2 - \lambda(N_1/\kappa_x + N_2/\kappa_y) + (N_1N_2)/(\kappa_x\kappa_y) - (1/4)(\alpha_x\theta_y/\kappa_x + \alpha_y\theta_x/\kappa_y)^2 = 0.$$

If the eigenvalues are negative, then the following inequalities must be satisfied so that $\dot{F}$ is negative. This means that

$$(N_1/\kappa_x) + (N_2/\kappa_y) < 0$$

and

$$(N_1N_2)/(\kappa_x\kappa_y) > (1/4)(\alpha_x\theta_y/\kappa_x + \alpha_y\theta_x/\kappa_y)^2$$

must hold simultaneously. Therefore, the interior equilibrium point $(\overline{X},\overline{Y})$ is globally asymptotically stable if the above inequalities hold simultaneously. The critical point (338.30, 490.78) of Example 1 is globally stable (see Fig. 3) as the above inequalities hold.

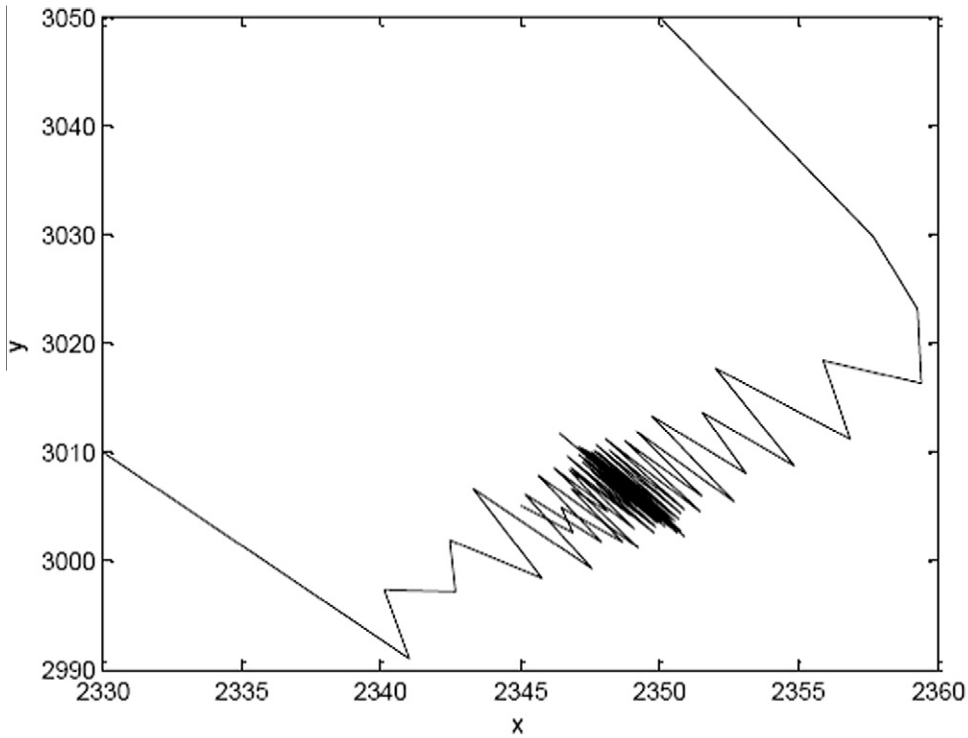

Fig. 3. Global attractor of Example 3. Here, X represents Prawn and Y represents Broiler.

3.5. Bionomic equilibrium

The biological equilibrium is given by Eq. (8) subject to the conditions (9). The economic equilibrium exists when total revenue earned by selling the harvested biomass equals the total system cost. Here, we consider that the reasonable cost per unit effort is

$$(X,Y) = \xi_x X + \xi_y Y,$$

where (ξ_x, ξ_y) cost per unit biomass of (X,Y) per unit effort like bedding and medicinal costs, etc. Then the revenue at any time is given by

$$\begin{aligned}\pi(X,Y,E) &= p_x h_x + p_y h_y - C_1N_{01}X - C_2N_{02}Y - (\xi_x X + \xi_y Y)E \\ &= p_x(\beta_x X + E\gamma_x X) + p_y(\beta_y Y + E\gamma_y Y) - C_1N_{01}X - C_2N_{02}Y - (\xi_x X + \xi_y Y)E \\ &= E[p_x\gamma_x X + p_y\gamma_y Y - \xi_x X - \xi_y Y] + [p_x\beta_x X + p_y\beta_y Y - C_1N_{01}X - C_2N_{02}Y] \\ &= E[(p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y] + [(p_x\beta_x - C_1N_{01})X + (p_y\beta_y - C_2N_{02})Y].\end{aligned}$$

For economic equilibrium, $\pi = 0$ implies

$$E = \frac{(p_x\beta_x - C_1N_{01})X + (p_y\beta_y - C_2N_{02})Y}{-(p_x\gamma_x - \xi_x)X - (p_y\gamma_y - \xi_y)Y}. \tag{13}$$

For biological equilibrium, $\dot{X} = 0 = \dot{Y}$ is attained at $(\overline{X}, \overline{Y})$ of Eq. (8) under the constraints given in Eq. (9). Now, substituting $(\overline{X}, \overline{Y})$ from Eq. (8) in Eq. (15), we have

$$G_1E^2 + G_2E + G_3 = 0 \;\rightarrow\; E = [-G_2 \pm \sqrt{G_2^2 - 4G_1G_3}]/2G_1,$$

where

$$G_1 = (p_x\gamma_x - \xi_x)(\gamma_x\kappa_xN_2 - \gamma_y\kappa_y\alpha_x\theta_y) + (p_y\gamma_y - \xi_y)(\gamma_y\kappa_yN_1 - \gamma_x\kappa_x\alpha_y\theta_x),$$

$$\begin{aligned}G_2 = {}&(p_x\gamma_x - \xi_x)\{(\beta_x + \theta_x - \sigma_x)\kappa_xN_2 - (\beta_y + \theta_y - \sigma_y)\kappa_y\alpha_x\theta_y\} + (p_x\beta_x - C_1N_{01})(\gamma_x\kappa_xN_2 - \gamma_y\kappa_y\alpha_x\theta_y) + (p_y\gamma_y - \xi_y)\{(\beta_y \\ &+ \theta_y - \sigma_y)\kappa_yN_1 - (\beta_x + \theta_x - \sigma_x)\kappa_x\alpha_y\theta_x\} + (p_y\beta_y - C_2N_{02})(\gamma_y\kappa_yN_1 - \gamma_x\kappa_x\alpha_y\theta_x)\end{aligned}$$

and

$$\begin{aligned}G_3 = {}&(p_x\beta_x - C_1N_{01})\{(\beta_x + \theta_x - \sigma_x)\kappa_xN_2 - (\beta_y + \theta_y - \sigma_y)\kappa_y\alpha_x\theta_y\} + (p_y\beta_y - C_2N_{02})\{(\beta_y + \theta_y - \sigma_y)\kappa_yN_1 - (\beta_x + \theta_x \\ &- \sigma_x)\kappa_x\alpha_y\theta_x\}.\end{aligned}$$

Here, $[-G_2 \pm \sqrt{G_2^2 - 4G_1G_3}]/G_1 > 0$ must hold for feasibility of the model, i.e., $E > 0$. Let us consider a numerical example as follows:

Example 2. We consider the values of the parameters in appropriate units as follows: $\kappa_x = 500$, $\kappa_y = 500$, $N_{01} = 2.0$, $N_{02} = 1.5$, $\sigma_x = 7.0$, $\sigma_y = 9.7$, $\gamma_x = 0.4$, $\gamma_y = 0.5$, $\beta_x = 0.5$, $\beta_y = 0.6$, $\theta_x = 0.4$, $\theta_y = 0.2$, $\alpha_x = 15.0$, $\alpha_y = 15.5$, $\tau_x = 6.5$, $\tau_y = 7.2$, $C_1 = \$0.3$, $C_2 = \$0.2$, $\xi_x = \$3.5$, $\xi_y = \$2.0$, $p_x = \$7.0$, $p_y = \$4.0$. Then the critical point $(\overline{X} = 2214.89,\ \overline{Y} = 2839.01,\ \overline{E} = 8.0)$ is a locally as well as a globally stable node because the eigenvalues are $(-48.65, -3.64)$. The bionomic equilibrium of the system is attained at $(\overline{X} = 2214.89,\ \overline{Y} = 2839.01)$.

3.6. Optimal harvesting policy

The net profit of the project, including inflation and time value of money, is

$$J = \int_0^\infty \pi(X, Y, E)e^{-\delta t}dt, \tag{14}$$

where $\delta = r - i$, r is rate of inflation per unit dollar per unit time and i is the rate of interest per unit time. Now our objective is to maximize J subject to the state Eq. (7), using Pontryagin's maximum principle. The control variable $E(t)$ is subject to the constraint $0 \leqslant E(t) \leqslant E_{max}$, E_{max} being a feasible upper limit for the harvesting effort. The Hamiltonian of the problem is

$$\begin{aligned} H &= \left[p_x(E\gamma_x X + \beta_x X) + p_y(E\gamma_y Y + \beta_y Y) - C_1N_{01}X - C_2N_{02}Y - (\xi_x X + \xi_y Y)E\right]e^{-\delta t} \\ &\quad + \lambda_1(t)\left[-A_x X + (N_1/\kappa_x)X^2 + (\alpha_x\theta_y/\kappa_x)XY\right] + \lambda_2(t)B\left[-A_y Y + (\alpha_y\theta_x/\kappa_y)XY + (N_2/\kappa_y)Y^2\right] \\ &= [(p_x\beta_x - C_1N_{01})X + (p_y\beta_y - C_2N_{02})Y]e^{-\delta t} + \lambda_1(t)\left[-(\beta_x + \theta_x - \sigma_x)X + (N_1/\kappa_x)X^2 + (\alpha_x\theta_y/\kappa_x)XY\right] \\ &\quad + \lambda_2(t)\left[-(\beta_y + \theta_y - \sigma_y)Y + (\alpha_y\theta_x/\kappa_y)XY + (N_2/\kappa_y)Y^2\right] \\ &\quad + Max_{[0\leqslant E(t)\leqslant E_{max}]}\left[\{(p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y\}e^{-\delta t} - \gamma_x\lambda_1(t)X - \gamma_y\lambda_2(t)Y\right]E(t), \end{aligned} \tag{15}$$

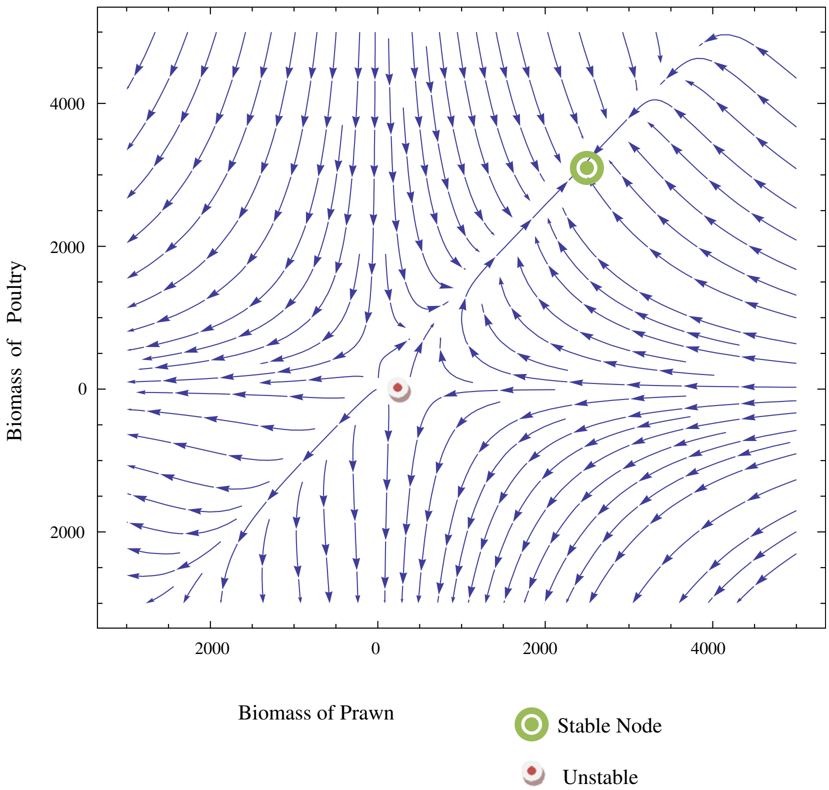

Fig. 4. Critical points of Example 3.

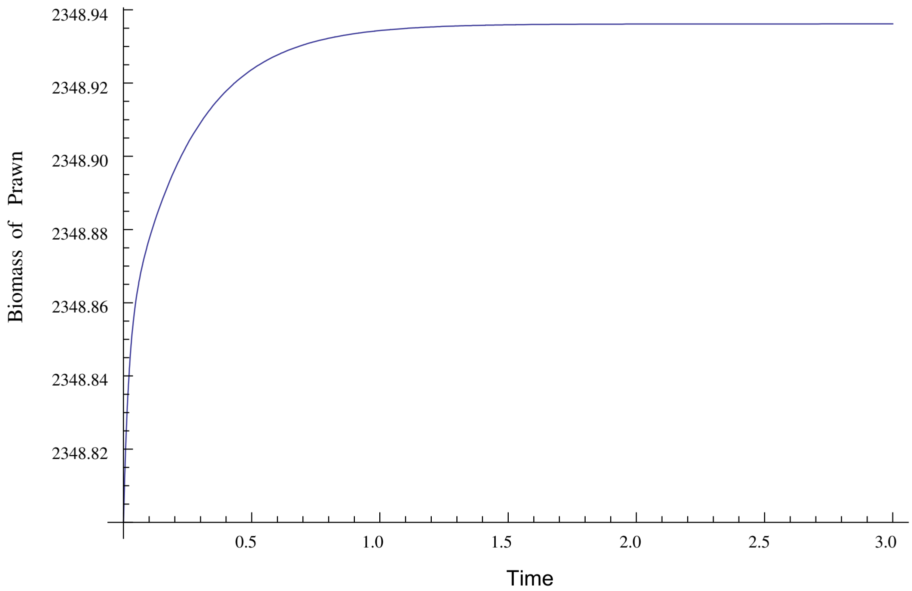

Fig. 5. Biomass of Prawn versus time.

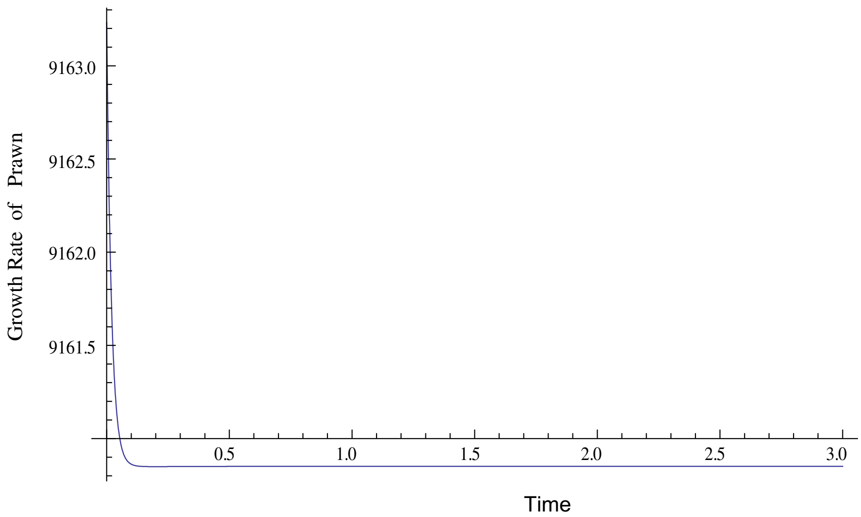

Fig. 6. Growth rate of Prawn versus time.

where $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables. The control variable $E(t)$ is linear in Hamiltonian function H. Therefore, optimal control will be a combination of bang-bang control and singular control. The optimal control $E(t)$ which maximizes H must satisfy the following conditions:

$$E = \begin{pmatrix} E_{max}, & \text{while } (\gamma_x\lambda_1 X + \gamma_y\lambda_2 Y)e^{\delta t} < (p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y \\ 0, & \text{while } (\gamma_x\lambda_1 X + \gamma_y\lambda_2 Y)e^{\delta t} > (p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y \\ \textit{see below}, & \text{while } (\gamma_x\lambda_1 X + \gamma_y\lambda_2 Y)e^{\delta t} = (p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y \end{pmatrix},$$

For singular control, we have $\partial H/\partial E = 0$. This gives

$$\left[(p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y\right]e^{-\delta t} = \lambda_1\gamma_x X + \lambda_2\gamma_y Y. \tag{16}$$

The adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial X} \tag{17}$$

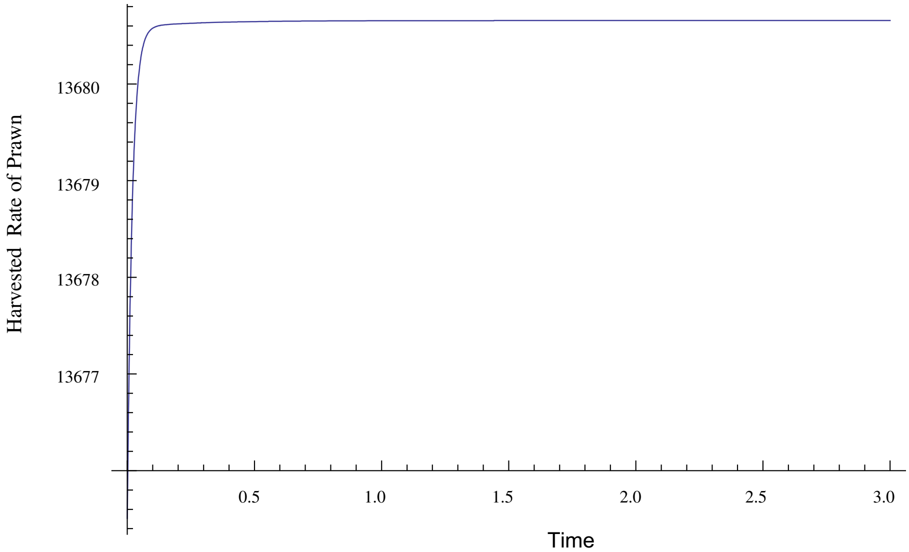

Fig. 7. Harvest rate of Prawn versus time.

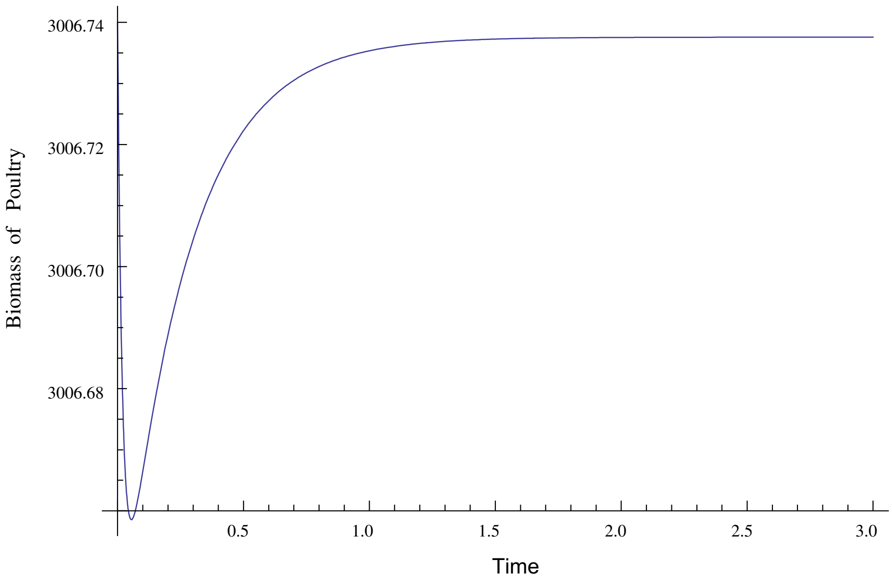

Fig. 8. Biomass of Poultry versus time.

and

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial Y}. \tag{18}$$

Our aim is to determine *optimal equilibrium solution* of the system so that we have

$$E = \left[(N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y - \beta_x - \theta_x + \sigma_x\right]/\gamma_x = \left[(\alpha_y\theta_x/\kappa_y)X + (N_2/\kappa_y)Y - \beta_y - \theta_y + \sigma_y\right]/\gamma_y. \tag{19}$$

Solving Eqs. (17) and (18) (see Appendix B), we have

$$\lambda_1(t) = \frac{Q_1}{\delta^2 + \psi_1\delta + \psi_2}e^{-\delta t} \tag{20}$$

and

$$\lambda_2(t) = \frac{Q_2}{\delta^2 + \psi_1\delta + \psi_x} e^{-\delta t}, \tag{21}$$

where

$$\psi_1 = (A_x + A_y) - (2N_1/\kappa_x + \alpha_y\theta_x/\kappa_y)X - (2N_2/\kappa_y + \alpha_x\theta_y/\kappa_x)Y > 0, \quad \text{by virtue of Eq. (11);}$$

$$\psi_2 = -[(\alpha_x\alpha_y\theta_x\theta_y)/(\kappa_x\kappa_y)]XY + (-A_x + (2N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y) \times (-A_y + (\alpha_y\theta_x/\kappa_y)X + (2N_2/\kappa_y)Y) > 0$$
$$\text{by virtue of Eq. (12),}$$

$$Q_1 = (\alpha_y\theta_x/\kappa_y)Y\{(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)\} - \{-A_y + (\alpha_y\theta_x/\kappa_y)X + (2N_2/\kappa_y)Y\} \times \{(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)\}$$
$$+ \delta\{(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)\}$$

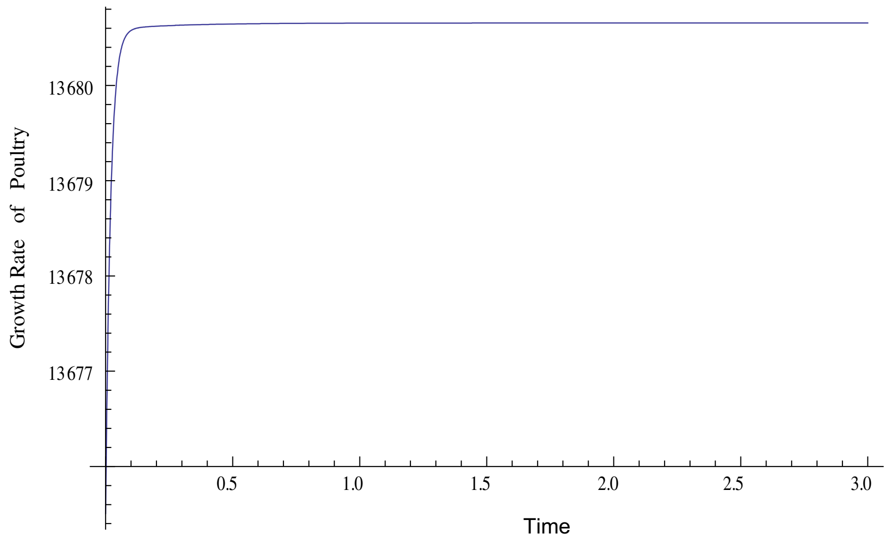

Fig. 9. Growth rate of Poultry versus time.

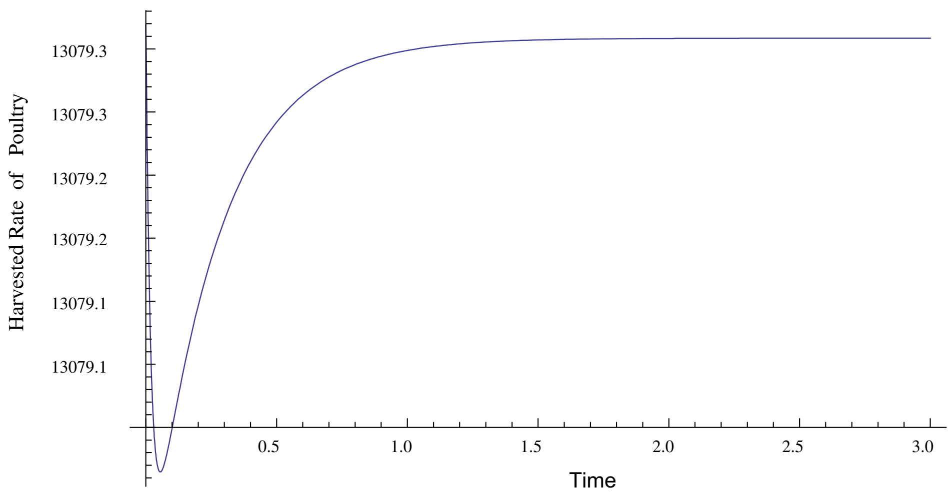

Fig. 10. Harvested rate of Poultry versus time.

and

$$Q_2 = (\alpha_x\theta_y/\kappa_x)X\{(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)\} - \{-A_x + (\alpha_x\theta_y/\kappa_x)Y + (2N_1/\kappa_x)X\} \times \{(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)\} + \delta\{(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)\}.$$

Substituting $\lambda_1(t)$ and $\lambda_2(t)$ in Eq. (18), we have

$$[(p_x\gamma_x - \xi_x)X + (p_y\gamma_y - \xi_y)Y] = \left[\frac{Q_1\gamma_x X}{\delta^2 + \psi_1\delta + \psi_2}\right] + \left[\frac{Q_2\gamma_y Y}{\delta^2 + \psi_1\delta + \psi_2}\right]. \tag{22}$$

Now, solving Eqs. (19) and (22), we have the optimal equilibrium solution $(\overline{X}, \overline{Y}, \overline{E})$. Let us consider a numerical example as follows.

Example 3. We consider the values of the parameters in appropriate units as follows: $\kappa_x = 500$, $\kappa_y = 500$, $N_{01} = 2.0$, $N_{02} = 1.5$, $\sigma_x = 7.0$, $\sigma_y = 9.7$, $\gamma_x = 0.4$, $\gamma_y = 0.5$, $\beta_x = 0.5$, $\beta_y = 0.6$, $\theta_x = 0.4$, $\theta_y = 0.2$, $\alpha_x = 15.0$, $\alpha_y = 15.5$, $\tau_x = 6.5$, $\tau_y = 7.2$, $C_1 = \$0.3$, $C_2 = \$0.2$, $\xi_x = \$0.5$, $\xi_y = \$1.0$, $p_x = \$10.0$, $p_y = \$5.0$, $r = 16\%$, $i = 11\%$, $\delta = 0.05$. Then the optimal critical point $(\overline{X} = 2348.94,\ \overline{Y} = 3006.74,\ \overline{E} = 7.5)$ is locally as well as globally stable node because the eigenvalues are $(-51.55, -3.86)$. Fig. 3 and 4 show that the above critical point is a globally stable node. The on-hand stock (see Fig. 5) growth rate (see Fig. 6) and harvesting rate (see Fig. 7) of prawn are asymptotically stable. Similarly, The on-hand stock (see Fig. 8), growth rate (see Fig. 9) and harvesting rate (see Fig. 10) of poultry birds are asymptotically stable.

4. Discussion

The present model is a combined project of prawn fishery and poultry of birds. The motivation behind the concept is the use of deteriorated prawn (mainly shrimp) as the nutrient for poultry and after conversion of poultry litter (*a brood of annimals*), the excreta of birds and dead birds can be used as nutrient of fishery.

From the analysis of the model, the following factors can be summerized as follows.

1. The growth rates of prawn and birds in poultry are considered as functions of available nutrients and volume of on-hand biomasses simultaneously.
2. The existence of local stability and global stability validates the model and the phase portrait also shows that it is a node. It confirms the fact that the nutrients of fishery and poultry are interconnected.
3. The bionomic (biological as well as ecconomic) equilibria of the exploited system have been estblished through a numerical example.
4. The optimal harvest policy is analysed invoking Pontryagin's maximal principle, subject to the state equations and the control constraints. The optimal equilibrium solution is obtained by a suitable and realistic numerical example. It is found that the shadow prices remain constant over time in optimal equilibrium when they satisfy the transversality condition.

The present model is unique in many ways. The unique features of the model are outlined below:

(i) The relationship between fishery and poultry through nutrients is established.
(ii) Introduction of deterioration concept to the dynamical system is made here.
(iii) Concept of inflation and time value of money is introduced.

This model can be investigated further considering the effect of time delay. The age structure of both the species without and with time delays, considering mature populations of harvesting, is also a good phenomenon to be considered further.

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Appendix A

Lemma 1. *All solutions of the dynamical system of Eq. (7) which start in R_2^+ are uniformly bounded.*

Proof. Let us consider the function

$$U(X, Y) = X + \frac{1}{\ell}Y, \tag{23}$$

where ℓ is a positive constant.

The time derivative of Eq. (23) is

$$\dot{U} = \dot{X} + \frac{1}{\ell}\dot{Y} = X(-A_x + (N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y) + \frac{1}{\ell}Y(-A_y + (\alpha_y\theta_x/\kappa_y)X + (N_2/\kappa_y)Y).$$

For each $s > 0$, we obtain

$$\begin{aligned}\dot{U} + sU &= X(-A_x + (N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y) + \frac{1}{\ell}Y(-A_y + (\alpha_y\theta_x/\kappa_y)X + (N_2/\kappa_y)Y) + sX + \frac{s}{\ell}Y \\ &= (N_1/\kappa_x)X^2 + (N_2/(\ell\kappa_y))Y^2 + (\alpha_x\theta_y/\kappa_x + \alpha_y\theta_x/(\ell\kappa_y))XY + (-A_x + s)X + (-A_y + s)Y/\ell \\ &= \tilde{a}w_1^2 + \tilde{b}w_2^2 + \hat{c}w_1 + \hat{d}w_2,\end{aligned}$$

where

$$\begin{aligned}&(w_1, w_2) \equiv (Xcos\phi - Ysin\phi, Xsin\phi + Ycos\phi),\\ &\phi = \frac{1}{2}tan^{-1}\left(\frac{\kappa_x N_2 - \ell\kappa_y N_1}{\ell\alpha_x\theta_y\kappa_y + \alpha_y\theta_x\kappa_x}\right),\\ &\hat{a} = [(\ell\kappa_y N_1 + N_2\kappa_x) + (\ell N_1\kappa_y - N_2\kappa_x)cos2\phi - (\ell\kappa_y\alpha_x\theta_y + \alpha_y\kappa_x\theta_x)sin2\phi]/(2\ell\kappa_x\kappa_y),\\ &\hat{b} = [(\ell\kappa_y N_1 + N_2\kappa_x) + (N_2\kappa_x - \ell N_1\kappa_y)cos2\phi + (\ell\kappa_y\alpha_x\theta_y + \alpha_y\kappa_x\theta_x)sin2\phi]/(2\ell\kappa_x\kappa_y),\\ &\hat{c} = (s - A_x)cos\phi - \frac{1}{\ell}(s - A_y)sin\phi,\\ &\hat{d} = (s - A_x)sin\phi + \frac{1}{\ell}(s - A_y)cos\phi.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{U} + sU &< |\hat{a}|\left(w_1 + \frac{|\hat{c}|}{2|\hat{a}|}\right)^2 + |\hat{b}|\left(w_2 + \frac{|\hat{d}|}{2|\hat{b}|}\right)^2 = |\hat{a}|\left(Xcos\phi - Ysin\phi + \frac{|\hat{c}|}{2|\hat{a}|}\right)^2 + |\hat{b}|\left(Xsin\phi + Ycos\phi + \frac{|\hat{d}|}{2|\hat{b}|}\right)^2 \\ &< |\hat{a}|\left(X + Y + \frac{|\hat{c}|}{2|\hat{a}|}\right)^2 + |\hat{b}|\left(X + Y + \frac{|\hat{d}|}{2|\hat{b}|}\right)^2 < |\hat{a}|\left(L + \frac{|\hat{c}|}{2|\hat{a}|}\right)^2 + |\hat{b}|\left(L + \frac{|\hat{d}|}{2|\hat{b}|}\right)^2, \\ &\text{where } L = L_1 + L_2 \quad \text{so that } 0 \leqslant X \leqslant L_1 \text{ and } 0 \leqslant Y \leqslant L_2, = \left(\sqrt{|\hat{a}|}L + \frac{|\hat{c}|}{2\sqrt{|\hat{a}|}}\right)^2 + \left(\sqrt{|\hat{b}|}L + \frac{|\hat{d}|}{2\sqrt{|\hat{b}|}}\right)^2 \quad < K,\end{aligned}$$

where $\sqrt{K/2} = Max\left[\left(\sqrt{|\hat{a}|}L + \frac{|\hat{c}|}{2\sqrt{|\hat{a}|}}\right), \left(\sqrt{|\hat{b}|}L + \frac{|\hat{d}|}{2\sqrt{|\hat{b}|}}\right)\right]$.

Applying the theory of differential inequality (Birkhoff and Rota [33]), we have

$$0 < U(X, Y) < (K/s)(1 - e^{-st}) + U(X(0), Y(0))e^{-st}. \tag{24}$$

When $t \to \infty$, the above yields $0 < U < K/s$. Therefore, all the solutions of Eq. (7) that start in R_2^+ are also confined to the region R where

$$R = \{(X, Y)\epsilon R_2^+ : U = (K/s) + \epsilon, \quad \text{for any } \epsilon > 0\}.$$

Hence the proof. □

Appendix B

From Eqs. (17) and (18), we have

$$-\dot{\lambda}_1 = [(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)]e^{-\delta t} + \lambda_1[-A_x + (2N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y] + \lambda_2(\alpha_y\theta_x/\kappa_y)Y \tag{25}$$

and

$$-\dot{\lambda}_2 = [(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)]e^{-\delta t} + \lambda_2[-A_y + (2N_2/\kappa_y)Y + (\alpha_y\theta_x/\kappa_y)X] + \lambda_1(\alpha_x\theta_y/\kappa_x)X. \tag{26}$$

Eliminating λ_2 from Eqs. (25) and (26), we have

$$\ddot{\lambda}_1 - \psi_1\dot{\lambda}_1 + \psi_2\lambda_1 = Q_1e^{-\delta t}, \tag{27}$$

where

$$\psi_1 = (A_x + A_y) - (2N_1/\kappa_x + \alpha_y\theta_x/\kappa_y)X - (2N_2/\kappa_y + \alpha_x\theta_y/\kappa_x)Y > 0, \quad \text{by virtue of Eq. (11);}$$

$$\psi_2 = -[(\alpha_x\alpha_y\theta_x\theta_y)/(\kappa_x\kappa_y)]XY + (-A_x + (2N_1/\kappa_x)X + (\alpha_x\theta_y/\kappa_x)Y) \times (-A_y + (\alpha_y\theta_x/\kappa_y)X + (2N_2/\kappa_y)Y) > 0 \quad \text{by virtue of Eq. (12)}$$

and

$$Q_1 = (\alpha_y\theta_x/\kappa_y)Y\{(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)\} - \{-A_y + (\alpha_y\theta_x/\kappa_y)X + (2N_2/\kappa_y)Y\} \times \{(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)\} + \delta\{(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)\}.$$

The auxiliary equation of Eq. (27) is

$$\mu^2 - \psi_1\mu + \psi_2 = 0. \tag{28}$$

The roots (μ_1, μ_2) of Eq. (28) are positive because ψ_1 and ψ_2 are positive. Therefore, the solution of $\lambda_1(t)$ is

$$\lambda_1(t) = Ae^{\mu_1 t} + Be^{\mu_2 t} + \frac{Q_1}{\delta^2 + \psi_1\delta + \psi_2}e^{-\delta t}.$$

The shadow price $\lambda_1(t)e^{\delta t}$ remains bounded as $t \to \infty$ if and only if $A = 0 = B$ and then

$$\lambda_1(t) = \frac{Q_1}{\delta^2 + \psi_1\delta + \psi_2}e^{-\delta t}. \tag{29}$$

Similarly, we have

$$\lambda_2(t) = \frac{Q_2}{\delta^2 + \psi_1\delta + \psi_2}e^{-\delta t}, \tag{30}$$

where

$$Q_2 = (\alpha_x\theta_y/\kappa_x)X\{(p_x\beta_x - C_1N_{01}) + E(p_x\gamma_x - \xi_x)\} - \{-A_x + (\alpha_x\theta_y/\kappa_x)Y + (2N_1/\kappa_x)X\} \times \{(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)\} + \delta\{(p_y\beta_y - C_2N_{02}) + E(p_y\gamma_y - \xi_y)\}.$$

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