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Joint effect of stock threshold level and production policy on an unreliable production environment

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ABSTRACT

In this article we develop an economic manufacturing quantity (EMQ) model subject to stochastic machine breakdown, repair and stock threshold level (STL). Instead of constant production rate, in this model production rate is considered as a decision variable. Since, the stress of the machine depends on the production rate, failure rate of the machine will be a function of the production rate. Again, in this article consideration of safety stock in all existing literature is replaced by the concept of stock threshold level (STL). Further, extra capacity of the machine is considered to buffer against the possible uncertainties of the production process where machine capacity is predetermined. The basic model is developed under general failure and general repair time distributions. Since, the assumption of variable production rate makes the objective function quite complex, so main emphasis is given on computational methodology to solve the present problem. We suggest two computational algorithms for the determination of production rate and stock threshold level which minimize the expected cost rate in the steady state. Finally, through numerical examples we illustrate the key insights of our model from managerial point of view.

1. Introduction

In this age of technology, modern manufacturing systems are becoming more and more complex. Even though they are more reliable than their predecessor, they are still subject to deterioration and failure with usage and age. Deterioration causes lower production rate and lower production quality. So, production, quality and maintenance are three important aspects in any manufacturing firm. Since global business competition is increasing day-by-day, managers in manufacturing industries are facing great challenges every day to produce better quality product and to provide better customer services than before. The classical EMQ model [1] usually does not take into account the process deterioration and machine breakdown during production run. But these models rarely meet the practical situations. Over the decades, numerous research efforts have been undertaken to fit closely to the real world situations by incorporating the imperfections of the production process (*i.e.*, quality and yield issues) and equipment (*i.e.*, machine breakdown and repair) in the classical lot sizing decisions. Mainly, imperfect EMQ models are extended in two parallel directions. In one direction it is assumed that the process may shift from 'in-control' state to 'out-of-control' state due to usage and age and as a consequence, begins to produce some percentage of defective items. Initial work was done simultaneously by Porteus [2] and Rosenblatt and Lee [3]. Porteus [2] developed a model where the process may shift from 'in-control' state to 'out-of-control' state with a given probability each

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Table 1
A comparison study of our present model with the related models.

Author	Model type	Production rate	Stock threshold level	Extra capacity of the machine	Failure rate pattern
Cheung and Hausman [20]	Imperfect EMQ model with maintenance and safety stocks	Fixed and production rate equals to demand rate	N	Y	Independent of production rate
Dohi et al. [21]	Extension of the above [20] model	Fixed and production rate equals to demand rate	N	Y	Independent of production rate
Our model	Imperfect EMQ model *i*th maintenance and stock threshold level	Variable and production rate greater than demand rate	Y	Y	Depends on production rate

time it produces an item. In a similar type of works, assuming exponential process shift distribution Rosenblatt and Lee [3] concluded that optimal production run length is shorter than that of the classical EMQ model. After that, several researchers [4–11] devoted their time to extend the classical EMQ model in this particular directions.

In the other direction of research for deteriorating production system, during production run the process or machine may breakdown at any random time. After machine failure, a repair action is carried out immediately. The time to repair the machine may be fixed or random. Shortages may arise due to longer repair time.[1] Shortages may be of lostsales[2] or backlogged[3] type after the resumption of the machine. Again, if there is no machine failure during the production run, then a maintenance action is carried out at the end of the production to bring back the system to initial working condition before the start of the next production run. Sometimes to increase the system reliability or to delay the occurrence of machine failure, maintenance actions are considered during the production run. The initial work was done by Groenevelt et al. [12] who studied the impact of machine breakdown and repair on the optimal lot sizing decisions. Assuming exponential inter failure time distribution and instantaneous repair time they showed that the optimal lot size is always larger than that of the classical EMQ model and always increases with the failure rate. They justified their conclusions by the argument that larger production lot size is in order to compensate the production loss due to machine breakdown. They [13] further extended their model [12] by incorporating the issue of safety stock required to meet the managerial prescribed service level. Kim et al. [14] reformulated Groenevelt et al. [12] model by assuming constant failure rate and concluded that the optimal lot size does not increase always with the increase of failure rate which contradicts Groenevelt et al.'s claim. Recently, Chakraborty et al. [15,16] developed models by integrating the joint effect of process deterioration and machine breakdown by assuming general process shift, machine breakdown, corrective repair and preventive maintenance time distributions.

In all the above mentioned extended EMQ models, production rate is considered as predetermined and inflexible. Khouja & Meherez [17] first incorporated the concept of variable production rate into the EMQ literature. They formulated their model under exponential process shift distribution with mean as an increasing function of the production rate. As a random nature of the production system, safety stock play an important role to guarantee the continuous delivery of the product during the interruption of the production due to machine breakdown.Then Giri et al. [18,19] extended EMQ models with this assumption of variable production rate. Groenevelt et al. [13] first considered the impact of safety stock into the machine breakdown model. Later Cheung and Hausman [20] investigated the joint effect of preventive maintenance and safety stocks on unreliable production system considering extra capacity of the machine. In many manufacturing system extra capacity of the machine is maintained to buffer against various uncertainties of the production system. Dohi et al. [21] reconsidered the Cheung and Hausman [20]'s model from theoretical point of view. Recently, Chakraborty and Giri [22] developed an imperfect EMQ model and studied the combined effect of process deterioration, machine breakdown, corrective and preventive maintenance together with the impact of buffer stocks on the optimal decisions for an unreliable production system.

In the above existing literature where extra capacity of the machine and safety stock were considered, the production rate was assumed to be constant. Further, in contrast to the existing literature, in our present paper instead of maintaining the safety stock at the beginning of each production cycle we have considered a stock threshold level (STL). This STL is considered as a decision variable. Thus the purpose of the present work is to study the impact of the joint implementation of machine breakdown, maintenance and stock threshold level (STL) on a stochastic EMQ model where failure rate is assumed to be a function of the production rate. Here, we have considered the extra capacity of the machine. Instead of constant production rate, in this model the production rate is considered as a decision variable. A comparison study of our present model with the related existing literature is given in Table 1. We organize our paper in the following manner. The following section describes the assumptions and notation used through out the paper. The model description is given in Section 3. In Section 4, the model is formulated under general failure and general repair time distribution. Some solution approaches to obtain the optimal solution of our model are described in Section 5 through algorithms. Section 6 proposed some properties of the

[1] This case may arise for random repair time.

[2] In case of lostsales, shortages are not delivered after resumption of the production. They are totally lost. In this case some goodwill loss costs are involved into the expected cost of the system.

[3] In backlogged case after the resumption of the machine shortages are met first either totally or partially.

Table 2
Convergence of solution sequence in barrier method when $\gamma = 2$, $\epsilon = 0.00001$ at $c_h = 0.01$.

η	P	S	$C_\eta(P,S)$	$\eta B(P,S)$
10^5	1538.46	5974.74	4571.62	291.33
10^4	1551.25	7089.62	4307.80	29.82
10^3	1623.99	7637.95	4280.70	3.08
10^2	1743.03	7745.97	4277.92	0.32
10^1	1784.09	7758.49	4277.63	3.20×10^{-2}
10^0	1789.54	7759.78	4277.60	$3.21 \times 10)^{-3}$
10^{-1}	1790.10	7759.91	4277.60	3.21×10^{-4}
10^{-2}	1790.16	7759.92	4277.60	3.21×10^{-5}
10^{-3}	1790.17	7759.92	4277.60	3.21×10^{-6}
10^{-4}	1790.17	7759.92	4277.60	3.21×10^{-7}

model. Section 7 explains the model numerically from managerial point of view. Finally, the paper is concluded together with future research direction in Section 8.

2. Assumptions and notation

The following assumptions and notation are used throughout the paper.

2.1. Assumptions

(A1) Consider the production of a single item on a single-unit production system or machine. Production process starts with a variable production rate P ($\leqslant P_{\max}$, the capacity of the machine) to meet the constant demand rate D ($<P$) and it continues until the stock threshold level S is achieved.
(A2) The machine is subject to stochastic machine breakdown and the time to machine failure follows an arbitrary probability distribution.
(A3) The machine is subject to corrective repair after the immediate detection of machine failure. Corrective repair time follows an arbitrary probability distribution and after the completion of the corrective repair, the machine is restored back to the same initial working condition.
(A4) At most one failure can occur in a complete production cycle. Breakdown of the machine does not cause any quality differentiation of the product.
(A5) Shortages may occur due to longer corrective repair time. In that case, all unsatisfied demands are backlogged and penalty cost is incurred per item per unit basis.
(A6) As soon as the corrective repair is completed, the production process resumes at a fixed resumption cost (which is less than machine set up cost).
(A7) In the case of machine failure during a production cycle, the production resumes with a maximum rate $P_{\max}$ (the capacity of the machine, [20,21] after machine repair and the production of items stops when the stock level reaches the threshold level S. Here sufficient capacity of the machine is considered to allow rapid accumulation of the stock threshold level S so that the time to accomplish the threshold level is relatively small compared to the mean time between failures. Hence, atmost one failure can occur during a complete production cycle.
(A8) If there is no machine breakdown during the production run, a preventive maintenance is done in order to bring back the machine to the as-good-as new condition before the start of the next production cycle. Preventive maintenance time compared to corrective maintenance time is negligible.
(A9) Corrective and preventive maintenance times include machine setup time.
(A10) The time when the inventory starts from zero level and the time when it reaches to zero level after attaining the threshold level S is termed as one complete cycle.

2.2. Notation

The following notations are adopted to develop the proposed model.

D	constant demand rate
$P(>D)$	variable production rate
$P_{\max}(\geqslant P)$	the maximum production rate
X	non-negative random variable denoting the time to machine breakdown
$f(\cdot)$, $F(\cdot)$	probability density function, cumulative distribution function of X

(*continued on next page*)

l	random variable denoting the corrective repair time
$g(\cdot)$, $G(\cdot)$	probability density function and cumulative distribution function of l
c_0	setup cost for each production run
c_h	inventory holding cost per unit product per unit time
$c_m(<c_0)$	fixed resumption cost
c_r	corrective repair cost per unit time
M	fixed preventive maintenance cost
S	the threshold level
c_s	shortage cost per unit product
$C(P)$	per unit production cost, a function of P

3. Model description

In this paper we consider the production of single item on a single production system or single machine which is deteriorating in nature and subject to stochastic machine breakdown. Here we consider the extra capacity of the machine (In this model ($P_{\max}-P$) is the extra capacity of the machine) to buffer against the possible uncertainties of the production process where machine capacity $P_{\max}$ is predetermined. Again, stock threshold level (STL) is considered to buffer against possible stockout due to stochastic machine breakdown and repair. At the beginning of the production process *i.e.*, at time $t=0$, the production starts at a variable production rate $P(\leqslant P_{\max})$ to satisfy the constant demand rate D. At the beginning the inventory remains at zero level and it increases at a rate ($P-D$). The production is continued until the threshold level S is achieved. If during the time period $\frac{S}{P-D}$, machine failure does not take place then at the end of the production a preventive maintenance is carried out at negligible time so that the system is restored at initial working condition *i.e.*, the production process remains at 'as-good-as-new' condition at the start of the next production cycle. A new production cycle starts only when all the accumulated inventory is depleted after achieving the stock threshold level. On the other hand, if machine failure occurs during the time period $\frac{S}{P-D}$, corrective repair action starts immediately. Corrective repair time is random and hence shortages may occur due to longer corrective repair time. Shortages are backlogged after the resumption of the machine. As soon as corrective repair is completed, the production resumes at maximum rate $P_{\max}$ to allow the rapid accumulation of the threshold level to avoid the further machine failure. Because of the short accumulation time, machine can be assumed to be almost 'as-good-as-new' at the beginning of the next production cycle. In many manufacturing industries, extra capacity of the machine is maintained as a buffer against various uncertainties (in the present model the extra capacity is indicated by ($P_{\max}-D$)) for emergency purposes. Figs. 1–3 present the schematic diagram of the model.

4. Model formulation

The expected total cost consists of setup cost, holding cost,corrective repair cost, preventive maintenance cost, resumption cost, production cost and penalty cost due to shortage. Next, we shall consider the different cost components which are involved in this model.

Lemma 1. *The expected holding cost per production cycle is*

$$HC = c_h\int_0^{S/(P-D)}\left\{\int_0^{(P-D)t/D}\left[\frac{1}{2}(P-D)t^2+\frac{1}{2}(2(P-D)t-lD)l+\frac{(S-(P-D)t+lD)(S+(P-D)t-lD)}{2(P_{\max}-D)}+\frac{1}{2}\frac{S^2}{D}\right]dG(l)\right.$$
$$\left.+\int_{(P-D)t/D}^{\infty}\left[\frac{(P-D)Pt^2}{2D}+\frac{S^2P_{\max}}{2D(P_{\max}-D)}\right]dG(l)\right\}dF_X(t)+c_h\int_{S/(P-D)}^{\infty}\frac{1}{2}\frac{PS^2}{D(P-D)}dF_X(t).$$

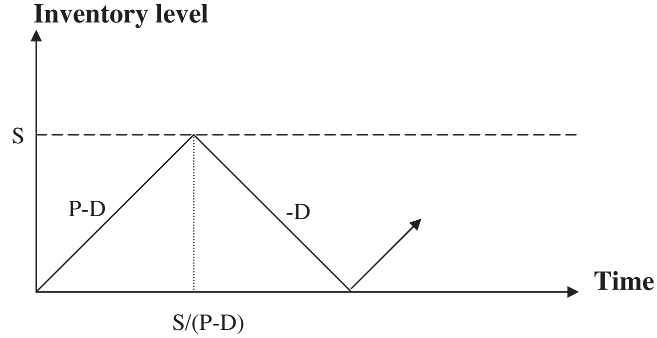

Fig. 1. Schematic diagram of the production-inventory model with no machine breakdown case.

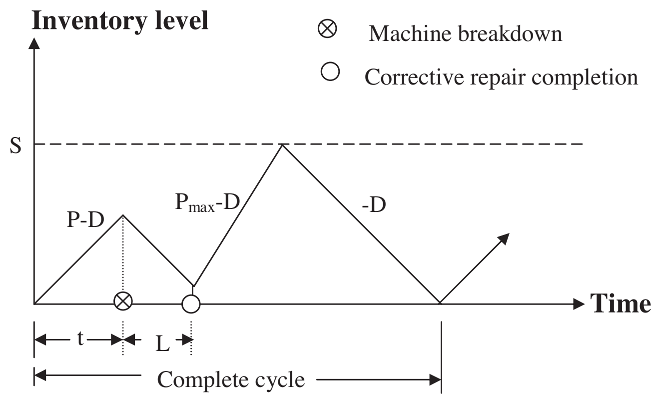

Fig. 2. Schematic diagram of the production-inventory model with machine breakdown and without shortage case.

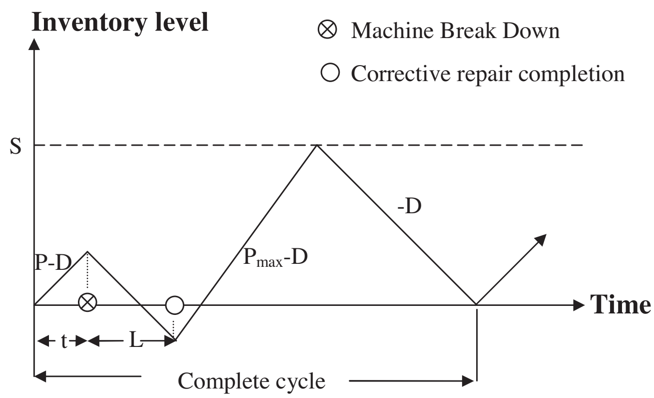

Fig. 3. Schematic diagram of the production-inventory model with machine breakdown and shortage case.

Proof. Fig. 1 indicates a complete production cycle when there is no machine breakdown during the production run. Thus, the holding cost in this case is

$$c_h \int_{S/(P-D)}^{\infty} \frac{1}{2} \frac{PS^2}{D(P-D)} dF_X(t).$$

Now, the diagrams of Fig. 4 indicates the scenario where machine failure occurs during the production run. The first diagram of Fig. 4 indicates the complete production cycle where corrective repair is completed before the depletion of inventory *i.e.* no shortage occurs during the complete production cycle. The symbol Δ's represent the area as indicated in the figure. It is to be noted that Δ_i's and t_i's used in the first and second diagrams of Fig. 4 denote different areas and time durations according to the first and second diagrams of Fig. 4.

In both diagrams of Fig. 4, $X = t$ indicates the time to machine breakdown.

According to the first diagram of Fig. 4, t_1 is the time duration which starts from the completion of corrective repair and continues to the time when inventory level reaches to the threshold level S. Again, t_2 indicates the time duration in which from stock threshold level the inventory comes to zero level satisfying customer demand. h_1 and h_2 are the heights as indicated in the first figure (Fig. 4(a)). Hence

$$\Delta_1 = \frac{1}{2}(P-D)t^2,$$
$$h_1 = (P-D)t, \quad h_2 = (P-D)t - lD.$$

Therefore, $\Delta_2 = \frac{1}{2}(h_1 + h_2)l = \frac{1}{2}\{2(P-D)t - lD\}l.$

Now, $t_1 = \frac{S-h_2}{P_{\max}-D} = \frac{S-(P-D)t+lD}{P_{\max}-D}$. Then

$$\Delta_3 = \Delta_{31} + \Delta_{32} = \frac{1}{2}(P_{\max}-D)t_1^2 + h_2 t_1 = \frac{\{S-(P-D)t+lD\}\{S+(P-D)t-lD\}}{2(P_{\max}-D)}.$$

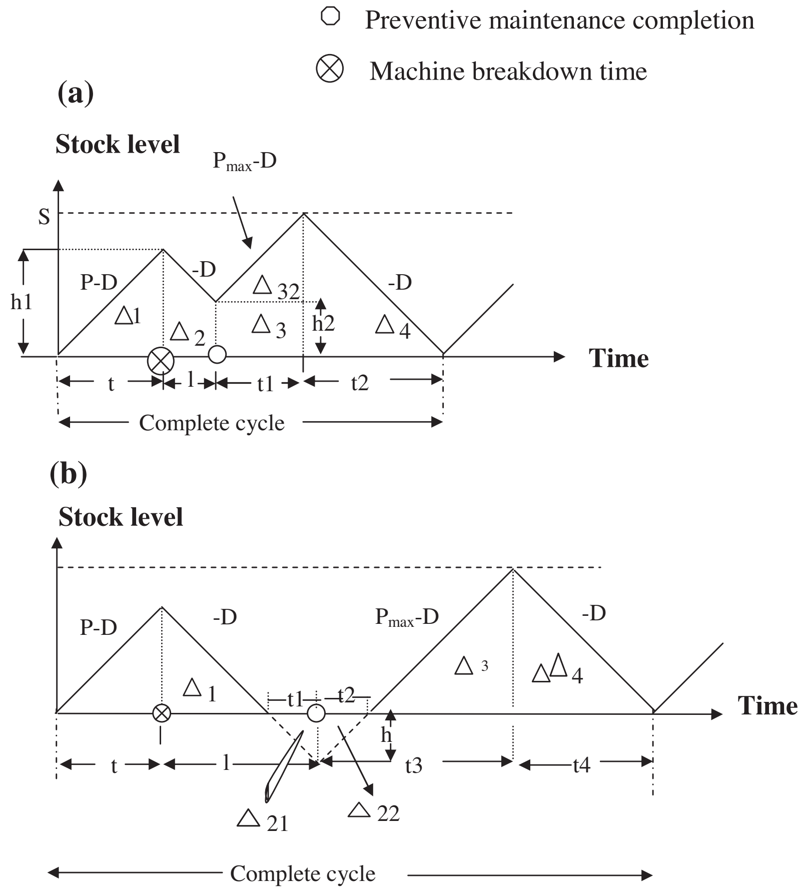

Fig. 4. Diagrammatic representation of the production-inventory model for calculation of the total holding cost incurred per production cycle.

Again, $t_2 = \frac{S}{D}$. Therefore $\Delta_4 = \frac{S^2}{2D}$.

Thus, the holding cost in this case is

$$c_h \int_0^{S/(P-D)} \left\{ \int_0^{(P-D)t/D} \left[\frac{1}{2}(P-D)t^2 + \frac{1}{2}\{2(P-D)t - lD\}l + \frac{\{S-(P-D)t+lD\}\{S+(P-D)t-lD\}}{2(P_{\max}-D)} + \frac{S^2}{2D} \right] dG(l) \right\} dF_X(t).$$

Again, the second diagram *i.e.*, Fig. 4(b) indicates the machine breakdown case when shortages occur due to longer corrective repair time. According Fig. 4(b), t_1 is the time between the events when the inventory level comes to zero level after the failure of the machine and the time of completion of the corrective repair action. t_2 indicates the time between the time of completion of the corrective repair action and the time when the inventory from negative level comes to zero level again. t_3 is the time duration of attaining from zero level to stock threshold level S. Again, t_4 is the time duration in which the inventory from threshold level comes to zero level. It is to be noted that t_2 of Fig. 4(a) is same as t_4 of Fig. 4(b). Hence

$$\Delta_1 = \frac{1}{2D}(P-D)Pt^2,$$

$$t_1 = \frac{lD-(P-D)t}{D},$$

$$h = lD-(P-D)t, t_2 = \frac{lD-(P-D)t}{P_{\max}-D},$$

$$\Delta_{21} = \frac{1}{2}t_1 h, \Delta_{22} = \frac{1}{2}t_2 h.$$

Thus $\Delta_2 = \Delta_{21} + \Delta_{22} = \frac{P_{\max}\{lD-(P-D)t\}^2}{2D(P_{\max}-D)}$

$$t_3 = \frac{S}{P_{\max}-D} \text{ and } t_4 = \frac{S}{D}.$$

Therefore, $\Delta_3 = \frac{t_3 S}{2}$ and $\Delta_4 = \frac{S^2}{2D}$.

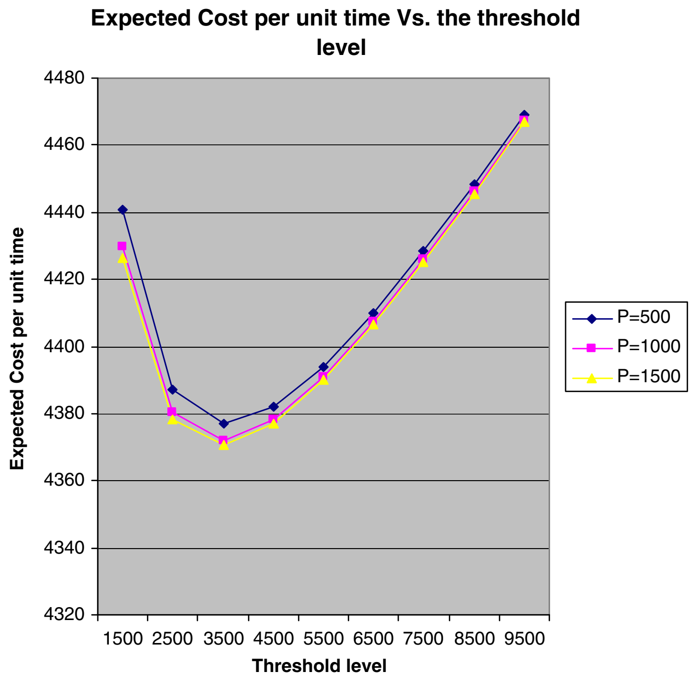

Fig. 5. Convexity of the expected cost per unit time with respect to the threshold level S for a given production rate.

Thus, the holding cost in this case is

$$\int_0^{S/(P-D)}\int_{(P-d)t/D}^{\infty}(\Delta_1+\Delta_2+\Delta_3)dG(l)dF_X(t)=\int_0^{S/(P-D)}\int_{(P-d)t/D}^{\infty}\left[\frac{(P-D)Pt^2}{2D}+\frac{S^2P_{\max}}{2D(P_{\max}-D)}\right]dG(l)dF_X(t).$$

Combining all these holding costs of these three cases we get the required expected holding cost in a complete production cycle. This completes the lemma. □

Again, the other expected costs involved in this models are given below.

The expected corrective repair cost per cycle is

$$CR=c_r\int_0^{S/(P-D)}\int_0^{\infty}l\,dG(l)dF_X(t).$$

The expected resumption cost per cycle is

$$RC=c_m\int_0^{S/(P-D)}dF_X(t)=c_mF_X\left(\frac{S}{P-D}\right).$$

The expected preventive maintenance cost per cycle is

$$PMC=M\int_{S/(P-D)}^{\infty}dF_X(t)=M\overline{F_X}\left(\frac{S}{P-D}\right).$$

The expected production cost per cycle is

$$PC=\int_0^{\frac{S}{P-D}}\left[\int_0^{\frac{(P-D)t}{D}}\left\{PtC(P)+P_{max}C(P_{max})\frac{S-(t+l)D}{P_{max}-D}\right\}dG(l)+\int_{\frac{(P-D)t}{D}}^{\infty}\left\{PtC(P)+P_{max}C(P_{max})\frac{S+lD-(P-D)t}{P_{max}-D}\right\}dG(l)\right]dF_X(t)$$
$$+\frac{PS}{P-D}C(P)\int_{\frac{S}{P-D}}^{\infty}dF_X(t).$$

The expected shortage cost per cycle is [see Fig. 5(b)]

$$SC=c_s\int_0^{S/(P-D)}\int_{(P-D)t/D}^{\infty}\frac{P_{\max}(lD-(P-D)t)^2}{2D(P_{\max}-D)}dG(l)\,dF_X(t).$$

Hence, the total expected cost per cycle is given by

$$\begin{aligned}V(P,S) &= C_0 + HC + CR + RC + PMC + PC + SC \\ &= c_0 + c_r\int_0^{S/(P-D)}\int_0^{\infty} l\,dG(l)dF_X(t) + c_m F_X\left(\frac{S}{P-D}\right) + M\overline{F_X}\left(\frac{S}{P-D}\right) + c_s\int_0^{S/(P-D)}\int_{(P-D)t/D}^{\infty}\frac{P_{\max}(lD-(P-D)t)^2}{2D(P_{\max}-D)}dG(l)dF_X(t) \\ &+ c_h\int_0^{S/(P-D)}\left\{\int_0^{(P-D)t/D}\left[\frac{1}{2}(P-D)t^2 + \frac{1}{2}(2(P-D)t-lD)l + \frac{(S-(P-D)t+lD)(S+(P-D)t-lD)}{2(P_{\max}-D)} + \frac{1}{2}\frac{S^2}{D}\right]dG(l)\right. \\ &+ \left.\int_{(P-D)t/D}^{\infty}\left[\frac{(P-D)Pt^2}{2D} + \frac{S^2P_{\max}}{2D(P_{\max}-D)}\right]dG(l)\right\}dF_X(t) + c_h\int_{S/(P-D)}^{\infty}\frac{1}{2}\frac{PS^2}{D(P-D)}dF_X(t) \\ &+ \int_0^{\frac{S}{P-D}}\left[\int_0^{\frac{(P-D)t}{D}}\left\{PtC(P) + P_{max}C(P_{max})\frac{S-(t+l)D}{P_{max}-D}\right\}dG(l) + \int_{\frac{(P-D)t}{D}}^{\infty}\left\{PtC(P) + P_{max}C(P_{max})\frac{S+lD-(P-D)t}{P_{max}-D}\right\}dG(l)\right]dF_X(t) \\ &+ \frac{PS}{P-D}C(P)\int_{\frac{S}{P-D}}^{\infty}dF_X(t). \end{aligned} \tag{1}$$

Again, by conditioning on the time to machine failure, the expected length of each production cycle is given by

$$\begin{aligned}T(P,S) &= \int_0^{\infty}E(\text{duration of the cycle}|X=t)f_X(t)dt = \int_0^{S/(P-D)}\left[\int_0^{(P-D)t/D}\left\{t + l + \frac{S-(P-D)t+lD}{P_{\max}-D} + \frac{S}{D}\right\}dG(l)\right. \\ &+ \left.\int_{(P-D)t/D}^{\infty}\left\{\left(t + \frac{(S+lD)P_{\max}-D(P-D)t}{D(P_{\max}-D)}\right\}dG(l)\right]dF_X(t) + \int_{S/(P-D)}^{\infty}\frac{PS}{D(P-D)}dF_X(t),\end{aligned} \tag{2}$$

where the first two terms of the above expression indicate the cycle length with machine breakdown scenario with and without shortage case respectively, where as the last term indicates without machine breakdown scenario. These three scenario are indicated in the schematic diagrams: Figs. 2, 3, 1, respectively. Therefore, by Renewal Reward theorem (Ross, 1981), [24] the expected cost per unit time in the steady state is

$$C(P,S) = \lim_{t\to\infty}\frac{E[\text{the total cost on }(0,t]]}{t} = \frac{V(P,S)}{T(P,S)}. \tag{3}$$

Our objective is the joint determination of the optimal production rate $P^*(D < P < P_{\max})$ and the optimal threshold level $S^*(<S_u$, where S_u is the upper bound of S) which minimize the expected cost per unit time $C(P,S)$.

5. Optimal solution of the model

In this section we will develop a constrained non-linear programming problem and will discuss different methods of its solution and the nature of the solution.

5.1. Optimal solution under general failure and general repair time distribution

The problem of our interest is

$$\begin{aligned}\text{P1}:\ &\text{Minimize}f C(P,S) = \frac{V(P,S)}{T(P,S)} \\ &\text{subject to} h_1(P,S) \equiv P - D > 0, \\ &\qquad h_2(P,S) \equiv P_{\max} - P \geqslant 0, \\ &\qquad h_3(P,S) \equiv S_{\mathrm{u}} - S \geqslant 0,\end{aligned} \tag{4}$$

where $P_{\max}$, the capacity of the machine, S_u, the upper bound of the threshold level are specified in advance by the decision maker. It is a constrained optimization problem. If (P^*,S^*) is the local optimal solution of the above mentioned problem **P1** then it must satisfy the above inequality constraints. Let us define the associate lagrangian function $L(P,S)$ as

$$L(P,S) = C(P,S) - m_1h_1(P,S) - m_2h_2(P,S) - m_3h_3(P,S),$$

where m_1, m_2, $m_3(\geqslant 0)$ are lagrangian multipliers. The Karush–Kuhn–Tucker necessary conditions for minimum are

$$\frac{\partial C(P,S)}{\partial S} + m_3 = 0, \tag{5}$$

$$\frac{\partial C(P,S)}{\partial P} - m_1 + m_2 = 0, \tag{6}$$

$$m_1(P-D) = 0,\quad m_2(P_{\max}-P) = 0,\quad m_3(S-S_{\mathrm{u}}) = 0,\quad m_1, m_2, m_3 \geqslant 0.$$

The above necessary conditions will be sufficient conditions for minimization problem **P1** if $C(P,S)$ is convex and $h_1(P,S)$, $h_2(P,S)$, $h_3(P,S)$ are concave functions with respect to P and S. Then the local optimal solutions of (P^*,S^*) will be the global optimal solutions. Since $P > D$, so clearly $m_1 = 0$. Again, if $P_{\max} > P$ and $S_u > S$ then $m_2 = 0$ and $m_3 = 0$. The optimal values of P and S can be obtained by solving the nonlinear Eqs. (5) and (6). Here, the simultaneous Eqs. (5) and (6) can be solved for (P,S) numerically following a two steps approach. In the first step, we fix P and solve Eq. (5) for S. In the next step, we substitute S in Eq. (6) and determine P. Then by putting the values of P and S we can determine the corresponding cost function $C(P,S)$ from Eq. (4). Based on this approach following solution algorithm can be proposed.

Algorithm 1.

Step 1: Input the parameter values in appropriate units and take ϵ_1 and ϵ_2 as the accuracy parameters. Let i be the counter parameter. Set $i = 0$ and $P = P^{(0)}$ as the initial value of P.
Step 2: Put $i = i + 1$, $P = P^{(i-1)}$.
Step 3: Then solve Eq. (5) for S. Let the solution be $S^{(i)}$.
Step 4: Put $S = S^{(i)}$ in Eq. (6) and solve that equation for P. Let the solution be $P^{(i)}$.
Step 5: Compute $C(P^{(i)},S^{(i)})$ from Eq. (4).
Step 6: If $|P^{(i)} - P^{(i-1)}| < \epsilon_1$ and $|S^{(i)} - S^{(i-1)}| < \epsilon_2$ then $P^* = P^{(i)}$ and $S^* = S^{(i)}$. Stop. Otherwise, go to **Step 6**.
Step 7: Repeat **Step 2** to **Step 6**.

Since, the cost function is complex and the production rate is variable it is difficult to prove analytically the convexity of the cost function $C(P,S)$ with respect to P and S simultaneously. So we can not guarantee analytically that any local solution will give the global minimum. But during numerical study by using computation software MATHEMATICA [25] we have checked that every local solution obtained by using this algorithm is actually a global solution. Further, if (P^*,S^*) is an interior point of the feasible region $R = (D < P \leqslant P_{\max}, S \leqslant S_u)$ then the generalized Newton's method can be applied to find it by solving the nonlinear equations

$$\frac{\partial C(P,S)}{\partial P} = 0, \tag{7}$$

$$\frac{\partial C(P,S)}{\partial S} = 0. \tag{8}$$

In this case, we can also apply the above mentioned two steps approach to solve Eqs. (7) and (8) simultaneously.

5.2. Barrier method and algorithm

Further, we can apply barrier method [23] to solve the constraint optimization problem **P1**. In this method, the original minimization problem **P1** will be replaced by a sequence of unconstrained optimization problems whose solutions converge to a solution (local minimization) of the original problem **P1**. Here we have to solve a sequence of unconstrained minimization problem of the form

$$\textbf{Minimize} \quad C_\eta(P,S) = C(P,S) + \eta B(P,S) \tag{9}$$

for the sequence of values $\eta = \eta_k \downarrow 0$, where the barrier function is given by

$$B(P,S) = \sum_{i=1}^{i=3} \frac{1}{h_i} + \frac{1}{P} + \frac{1}{S}. \tag{10}$$

$\eta B(P,S) \downarrow 0$ and consequently $C_\eta(P,S)$ approaches to $C(P,S)$, so the corresponding local minimum of $C_\eta(P,S)$ converges to the local minimum of the original problem. Starting with the feasible point $(\widehat{P},\widehat{S})$ in the interior of R, we will always remain within the feasible region R, for this reason this method is also known as interior point method. The conditions of optimality are

$$\nabla C_\eta(P,S) = \mathbf{0}, \tag{11}$$

where $\nabla = \left(\frac{\partial}{\partial P}, \frac{\partial}{\partial S}\right)$ and $\mathbf{0} = (0,0)$. Based on the barrier method, we propose the following algorithm to solve the problem **P1**.

Algorithm 2.

Step 1: Input the model parameters in appropriate units, the accuracy parameter (or, termination scaler) $\epsilon(>0)$, the barrier parameter $\eta_0(>0)$ and the reduction parameter θ $(0 < \theta < 1)$.
Step 2: Set $\eta \leftarrow \eta_0$ and take $(\widehat{P},\widehat{S})$ as an initial solution of (9).
Step 3: To calculate the new target point $(\widehat{P},\widehat{S})$, apply Newton's method in Eq. (11).
Step 4: If $\eta B(\widehat{P},\widehat{S}) < \epsilon$ then stop and assign $P^* = \widehat{P}, S^* = \widehat{S}$. Otherwise, go to **Step 5**.
Step 5: Set $\eta \leftarrow \theta\eta$ and go to **Step 2**.

It should be noted that using this Algorithm 2 we can get only the local optimal solution which consequently gives the local minimum cost $C(P^*,S^*)$. During numerical study we will observe that these local optimal solutions are actually the global solutions.

6. Analysis of the model

This section is intended to derive some analytical results under some specific probabilistic distributions.

6.1. The model under exponential failure and exponential repair

In this paper, production rate is considered as variable. Since, stress of the machine depends on the production rate, without loss of generality, failure rate can be assumed as a function of production rate. We assume that the time to machine failure follows the exponential distribution $F_X(t,P) = 1 - e^{-\lambda(P)t}$, $\lambda(P) > 0$ and the corrective repair time follows the exponential distribution $G(l) = 1 - e^{-\mu l}$, $\mu > 0$, where the hazard rate $\lambda(P) = \frac{f_X(t,P)}{1-F_X(t,P)}$ is an increasing function of P and μ is the repair rate parameter. Substituting the above mentioned distributions in Eq. (1) we get the expected cycle length as

$$T_1(P,S) = \frac{SP_{\max}}{D(P_{\max}-D)} + \frac{P_{\max}}{\mu(P_{\max}-D)} + \frac{P_{\max}-P}{\lambda(P)(P_{\max}-D)} - \left[\frac{1}{\lambda(P)} + \frac{P_{\max}}{\mu(P_{\max}-D)}\right] \times e^{-\frac{\lambda(P)S}{P-D}}. \tag{12}$$

Now, for any given P ($D < P \leqslant P_{\max}$) say, P_1 let $\lambda(P_1) = \lambda_1$ and $C(P_1) = C_1$. Then Eq. (12) can be re-written as

$$T_1(P_1,S) = \frac{SP_{\max}}{D(P_{\max}-D)} + \frac{P_{\max}}{\mu(P_{\max}-D)} + \frac{P_{\max}-P_1}{\lambda_1(P_{\max}-D)} - \left[\frac{1}{\lambda_1} + \frac{P_{\max}}{\mu(P_{\max}-D)}\right] e^{-\frac{\lambda_1 S}{P_1-D}}$$

and we get the following proposition.

Proposition 2. *For any given P ($D < P \leqslant P_{max}$) and given $\lambda_1 = \lambda(P_1)$, the expected cycle length $T_1(P_1,S)$ is concave with respect to the threshold level S.*

Proof. The proof is straight forward as the second order partial differentiation of $T_1(P_1,S)$ with respect to S is negative. □

6.2. The model under general failure and exponential repair

Let us suppose that the production rate P is fixed in advance. Then we are left to determine the optimal threshold level S^* which minimizes the expected cost per unit time in the steady state. Let the corrective repair time follows exponential distribution as given in the previous section. Let $t_0 = \frac{S}{P_1-D}$, is the production time if there is no machine breakdown during the production run, where $P = P_1$ ($D < P_1 \leqslant P_{\max}$) is fixed in advance. Since $P_{\max} - P \geqslant 0$ and $S_u - S \geqslant 0$, so it is easily shown that this production time t_0 is bounded by both upper and lower bounds. If $\underline{t_0}$ and $\overline{t_0}$ denote the lower bound and upper bound of t_0, respectively, then $\underline{t_0} = \frac{S}{P_{\max}-D}$ and $\overline{t_0} = \frac{S_u}{P-D}$. Our aim is, therefore, to determine the optimal production time $t_0^*(\underline{t_0} \leqslant t_0^* \leqslant \overline{t_0})$ which minimizes the expected cost per unit time. Let $C_2(t_0) = \frac{V_2(t_0)}{T_2(t_0)}$ in the steady state under exponential repair where

$$\begin{aligned} V_2(t_0) = {} & c_0 + \frac{c_r}{\mu}F_X(t_0) + c_m F_X(t_0) + M\overline{F_X}(t_0) + \frac{c_s P_{\max} D}{\mu^2(P_{\max}-D)}\int_0^{t_0} e^{-\mu(P_1-D)\frac{t}{D}} dF_X(t) \\ & + c_h\left[\frac{t_0^2(P_1-D)^2 P_{\max}}{2D(P_{\max}-D)} - \frac{P_{\max}D}{\mu^2(P_{\max}-D)}\right]F_X(t_0) + c_h\int_0^{t_0}\left\{\frac{1}{2}(P_1-D)t^2 - \frac{(P_1-D)^2 t^2}{2(P_{\max}-D)} + \frac{(P_1-D)P_{\max}t}{\mu(P_{\max}-D)}\right\}dF_X(t) + c_h \\ & \times \int_0^{t_0}\frac{DP_{max}}{\mu^2(P_{max}-D)}e^{-\mu(P_1-D)\frac{t}{D}}dF_X(t) + \frac{c_h P_1 t_0^2(P_1-D)}{2D}\overline{F_X}(t_0) + \int_0^{t_0}\left[\left\{P_1 tC(P_1) + P_{max}C(P_{max})\frac{(P_1-D)t_0 - tD}{P_{max}-D}\right.\right. \\ & \left. - \frac{P_{max}C(P_{max})D}{\mu(P_{max}-D)}\right\} + \left\{\frac{P_1 t P_{max}C(P_{max})}{P_{max}-D} + \frac{2P_{max}C(P_{max})D}{\mu(P_{max-D})}\right\}e^{-\frac{\mu(P-D)t}{D}}\Bigg] dF_X(t_0) + P_1 t_0 C(P_1)\overline{F}_X(t_0) \end{aligned}$$

and

$$T_2(t_0) = \int_0^{t_0}\left[t + \frac{t_0(P_1-D)P_{\max} - D(P_1-D)t}{D(P_{\max}-D)} + \frac{P_{\max}}{\mu(P_{\max}-D)}\right]dF_X(t) + \frac{P_1}{D}t_0\overline{F_X}(t_0).$$

We assume that the expected cost rate function $C_2(t_0)$ is continuous in $t_0 \in [\underline{t_0}, \overline{t_0}]$ and is bounded by following relation.

A11 $c_h(P_1-D)\overline{t_0} < C_2(t_0) < c_h(P_1-D)\underline{t_0} + \chi$,

where $\chi = \mathrm{Min}\{C(P_{\max})D,\ C(P_{\max})((P_1 - 2D)\underline{t_0} - D)\}$.

Let us denote the numerator of the derivative of $C_2(t_0)$ with respect to t_0 divided by the factor $\overline{F_X}(t_0)$ as $W_2(t_0)$ we get,

$$\begin{aligned}W_2(t_0) = &\left[\left(\frac{c_r}{\mu}+c_m-M\right)r_X(t_0)+\frac{c_S P_{\max} D}{\mu^2(P_{\max}-D)}e^{-\mu(P_1-D)\frac{t_0}{D}}r_X(t_0)+\frac{c_h(P_1-D)^2 t_0 P_{\max}}{D(P_{\max}-D)\overline{F_X}(t_0)}-\frac{c_h(P_1-D)^2 t_0}{D(P_{\max}-D)}P_{\max}\right.\\&+c_h\left\{\frac{(P_1-D)P_{\max}t_0}{\mu(P_{\max}-D)}-\frac{DP_{\max}}{\mu^2(P_{\max}-D)}\left(1-e^{-\mu((P_1-D)\frac{t_0}{D}}\right)\right\}r_X(t_0)+c_h\frac{p_1 t_0}{D}(P_1-D)\\&+\left\{P_{\max}C(P_{\max})\left(\frac{(P_1-2D)t_0}{P_{\max}-D}-\frac{D}{\mu(P_{\max}-D)}\right)+\left(\frac{P_1 t_0 P_{\max}C(P_{\max})}{P_{\max}-D}+\frac{2P_{\max}DC(P_{\max})}{\mu(P_{\max}-D)}\right)e^{-\frac{\mu(P_1-D)t_0}{D}}\right\}r_X(t_0)\\&\left.+\frac{P_{\max}(P_1-D)C(P_{\max})}{(P_{\max}-D)\overline{F_X}(t_0)}-\frac{P_{\max}(P_1-D)C(P_{\max})}{P_{\max}-D}+P_1C(P_1)\}\right]T_2(t_0)-\left[\frac{P_{\max}}{\mu(P_{\max}-D)}r_X(t_0)+\frac{(P_1-D)P_{\max}}{D(P_{\max}-D)\overline{F_X}(t_0)}\right.\\&\left.+\frac{P_1}{D}-\frac{(P_1-D)P_{\max}}{D(P_{\max}-D)}\right]V_2(t_0), \qquad (13)\end{aligned}$$

where $r_X(\cdot)=\frac{f_X(\cdot)}{\overline{F_X}(\cdot)}$ is the hazard rate of the breakdown time distribution. We assume that $r_X(t)$ is differentiable and increasing in t. Therefore, the underlying breakdown time distribution has IFR (Increasing Failure Rate) property which is quite appropriate to describe the machine breakdown phenomenon in the initial state. Let us assume the relation between holding cost and shortage cost as

A12 $c_h < c_S < c_h\left(e^{\mu(P_1-D)\frac{t_0}{D}}-1\right) \leqslant c_h\left(e^{\mu(P_1-D)\frac{\overline{t_0}}{D}}-1\right).$

Proposition 3. *Suppose distribution function $F_X(t)$ has strictly IFR property satisfying*

A13 $r'_X(t_0) > \frac{\mu(P_1-D)}{D}r_X(t_0).$

Under assumptions **A11** *and* **A12**

(i) *if $W_2(\underline{t_0}) < 0$ and $W_2(\overline{t_0}) > 0$, then there exists a unique optimal production time $t_0^*\left(\underline{t_0} < \frac{D}{\mu(P_1-2D)} < t_0^* < \overline{t_0}\right)$ satisfying the non-linear equation $W_2(t_0^*)=0$ and the corresponding minimum expected cost rate in the steady state is*

$$\begin{aligned}C_2(t_0^*) = &c_h(P_1-D)t_0^*+\left[\left(\frac{c_r}{\mu}+c_m-M\right)r_X(t_0^*)+\frac{c_S P_{\max}}{\mu^2(P_{\max}-D)}e^{-\mu(P_1-D)\frac{t_0^*}{D}}r_X(t_0^*)-\frac{c_h DP_{\max}}{\mu^2(P_{\max}-D)}\times\left(1-e^{-\mu(P_1-D)\frac{t_0^*}{D}}\right)r_X(t_0^*)\right.\\&+\left\{P_{\max}C(P_{\max})\left(\frac{(P_1-2D)t_0^*}{P_{\max}-D}-\frac{D}{\mu(P_{\max}-D)}\right)+\left(\frac{P_1 t_0^* P_{\max}C(P_{\max})}{P_{\max}-D}+\frac{2P_{\max}DC(P_{\max})}{\mu(P_{\max}-D)}\right)e^{-\frac{\mu(P_1-D)t_0^*}{D}}\right\}r_X(t_0^*)\\&\left.+\frac{P_{\max}(P_1-D)C(P_{\max})}{(P_{\max}-D)\overline{F_X}(t_0^*)}-\frac{P_{\max}(P_1-D)C(P_{\max})}{P_{\max}-D}+P_1C(P_1)\right]\Bigg/\left[\frac{P_{\max}}{\mu(P_{\max}-D)}r_X(t_0^*)+\frac{(P_1-D)P_{\max}}{D(P_{\max}-D)\overline{F_X}(t_0^*)}\right.\\&\left.+\frac{P_1}{D}-\frac{(P_1-D)}{D(P_{\max}-D)}\right], \qquad (14)\end{aligned}$$

(ii) *if $W_2(\underline{t_0}) \geqslant 0$, then the optimal production time is $t_0^*=\underline{t_0}$,*
(iii) *if $W_2(\overline{t_0}) \leqslant 0$, then the optimal production time is $t_0^*=\overline{t_0}$.*

Proof. Clearly $W_2(t_0)$ is continuous and derivable function of t_0. Differentiating $W_2(t_0)$ with respect to t_0, we get

$$\begin{aligned}W'_2(t_0) = &\left[\left(\frac{c_r}{\mu}+c_m-M\right)r'_X(t_0)+(c_S-C_h)\frac{P_{\max}}{\mu^2(P_{\max}-D)}e^{-\mu(P_1-D)\frac{t_0}{D}}r'_X(t_0)\right.\\&\left.+\frac{P_{\max}(P_1-D)}{\mu(P_{\max}-D)}e^{-\mu(P_1-D)\frac{t_0}{D}}\left\{-c_S+c_h\left(e^{\mu(P_1-D)\frac{t_0}{D}}-1\right)\right\}r_X(t_0)+\frac{c_h(P_1-D)^2P_{\max}}{D(P_{\max}-D)\overline{F_X}(t_0)}+\frac{c_h(P_1-D)(P_{\max}-P_1)}{(P_{\max}-D)}\right]T_2(t_0)\\&+\left\{\frac{P_{\max}C(P_{\max})(P_1-2D)}{P_{\max}-D}+\frac{P_{\max}C(P_{\max})P_1}{P_{\max}-D}e^{-\frac{\mu(P-1-D)t_0}{D}}\right\}r_X(t_0)T_2(t_0)\\&+\frac{P_{\max}(P_1-D)}{D(P_{\max}-D)\overline{F_X}(t_0)}r_X(t_0)[\{c_h(P_1-D)t_0+DC(P_{\max})\}T_2(t_0)-V_2(t_0)]+\frac{P_{\max}r'_X(t_0)}{\mu(P_{\max}-D)}[\{c_h(P_1-D)t_0\\&+(\mu(P_1-2D)t_0-D)C(P_{\max})\}T_2(t_0)-V_2(t_0)]\\&+\left\{\frac{P_1 t_0 P_{\max}C(P_{\max})}{P_{\max}-D}+\frac{2P_{\max}DC(P_{\max})}{\mu(P_{\max}-D)}\right\}\left\{r'_X(t_0)-\frac{\mu(P_1-D)}{D}r_X(t_0)\right\}e^{-\frac{\mu(P_1-D)t_0}{D}}T_2(t_0)+c_h\\&\times\frac{DP_{\max}}{\mu^2(P_{\max}-D)}r'_X(t_0)e^{-\mu(P_1-D)\frac{t_0}{D}}T_2(t_0), \qquad (15)\end{aligned}$$

where prime denotes the differentiation with respect to t_0. Clearly, $\frac{c_r}{\mu} + c_m > M$, that means fixed preventive maintenance cost must be less than the sum of resumption cost and mean corrective repair cost which is quite reasonable for practical problems. Hence, if $F_X(t)$ is IFR satisfying the assumption **A13**, it can be shown from Eq. (15) that $W_2(t_0)$ is strictly increasing in $t_0 \in (\underline{t_0}, \overline{t_0})$ under assumptions **A11** and **A12** *ie.* $W_2'(t_0) > 0$. Therefore, if $W_2(\underline{t_0}) < 0$ and $W_2(\overline{t_0}) > 0$, then there exists a unique optimal production time $t_0^*\left(\underline{t_0} < \frac{D}{\mu(P_1 - 2D)} < t_0^* < \overline{t_0}\right)$ which satisfies the non-linear equation $W_2(t_0^*) = 0$. Further, the equation $W_2(t_0^*) = 0$ yields the corresponding minimum expected cost rate as given in Eq. (14). Again if $W_2(\underline{t_0}) \geqslant 0$ or $W_2(\overline{t_0}) \leqslant 0$, then the function $C_2(t_0)$ is increasing or decreasing in the interval $[\underline{t_0}, \overline{t_0}]$ under assumptions **A11** and **A12**. Hence, the optimal production time is $t_0^* = \underline{t_0}$ or $t_0^* = \overline{t_0}$. This completes the proof of the proposition. □

Note 1. From the above proposition we get the optimal threshold level $S^* = t_0^*(P_1 - D)$ for any given production rate P_1 of P.

Note 2. The assumption **A12** imposes some parametric restrictions on the selection of holding cost and shortage cost.

As mentioned before, due to complicated form of the objective function, derivation of analytical solution of this problem is quite difficult. So, we can borrow numerical methods for gaining the key insights of our model.

7. Managerial explanation

In this section, to illustrate the above model from managerial point of view we analyze our proposed model through some numerical examples and investigate the integrated effects of production, maintenance and stock threshold level on the optimal lot sizing decisions. We also carry out some sensitivity analysis with respect to some important parameters. Since in this model we have considered variable production rate and since stress of the machine changes with the change of production rate, so, it is quite natural if we assume that the machine failure rate is a function of the production rate P. Hence, suppose $\lambda(P) = \alpha P^\beta$, where α and β are real positive constants. For all values of β, $\lambda(P)$ is an increasing function of P. It can be easily verified that $\lambda(P)$ is convex for $\beta > 1$ and concave for $0 < \beta < 1$. Since the maximum capacity of the machine is predetermined, we will consider $0 < \beta < 1$. Again since, $\beta < 1$, $\lambda(P)$ is concave with respect to P. It is therefore preferable to produce items in higher production speed. Further, since instead of constant production rate variable production rate is considered, so there is no loss of generality if we assume the unit production cost $C(P)$ as a convex function of the variable production rate P. Suppose it is defined as $C(P) = \delta_0 + \delta_1 P + \frac{\delta_2}{P}$, where δ_0, δ_1 and δ_2 are all positive and are to be estimated accordingly. δ_0 can be regarded as primary cost involved in the unit production cost (eg: raw material cost, etc.), $\delta_1 P$ increases linearly with the increase of the production rate (eg: tool failure or maintenance cost etc.) and $\frac{\delta_2}{P}$ decreases with the increase of P (eg: electricity cost, fuel cost, labor cost, etc.). It is to be noted that $C(P)$ is convex function with respect to P, so attains its minimum and the minimum value is $\sqrt{\frac{\delta_2}{\delta_1}}$.

Due to lack of closed form of analytical optimal solution, we find it numerically. To do so we have followed the approaches as outlined in Section 5.1. For this we formulate our problem as non-linear constrained optimization problem and solve it using **MATHEMATICA 5.2** [25].

Here we have fixed the parameters c_0, c_r, c_s, M, c_m, D, $P_{\max}$ and S_u, since we are interested on the effect of other parameters on the optimal decisions. The selection of the parametric values is based on the assumptions **A11** and **A12** and on the existing literature. By varying the key parameters α, β (failure parameters), μ (repair parameter) and c_h (holding cost) we get a set of problems. The other parametric values in appropriate units are set to : $c_0 = 750$, $c_r = 500$, $c_s = 5$, $M = 20$, $c_m = 50$, $D = 300$, $P_{max} = 2500$, $S_u = 10{,}000$, $\delta_0 = 10$, $\delta_1 = 0.0012$, $\delta_2 = 2500$. We obtain the optimal solutions by varying the parameter α with $\beta = 0.5$, $\mu = 2.5$, $c_h = 0.05$; by varying β with $\alpha = 1.0$, $\mu = 2.5$, $c_h = 0.05$; by varying μ with $\alpha = 1.0$, $\beta = 0.5$, $c_h = 0.05$ and finally by varying c_h with $\alpha = 1.0$, $\beta = 0.5$, $\mu = 2.5$. In the numerical study we have obtained the optimal solutions for the following two cases:

(i) **Example 1** (*Exponential failure and exponential repair*);
(ii) **Example 2** (*Weibull failure and exponential repair*).

7.1. Example 1 (Exponential failure and exponential repair)

Let the time to machine breakdown as well as the machine repair time follow exponential distributions with probability distribution functions

$$F_X(t) = 1 - e^{\lambda(P)t}, \quad \lambda(P) > 0,\ t > 0,$$
$$\text{and } G(l) = 1 - e^{-\mu l}, \quad \mu > 0,\ t > 0.$$

The dependence of the optimal production policy on the failure rate parameters α and β is shown in Tables 3 and 4. As α or β increases the optimal threshold level together with optimal production rate and optimal expected cost rate increases. This phenomenon can be interpreted as follows. As machine failure rate $\lambda(P) = \alpha P^{\beta}$ increases with the increase of α, β, stock threshold level should be higher to buffer against the possible stock out. Again, as the failure rate and stock threshold level increase, due to increase in the holding cost and maintenance cost, optimal expected cost rate also increases. Consequently, to decrease the holding cost production rate increases.

Table 5 shows the impact of the repair rate μ on the optimal decisions of our proposed model. As the repair rate μ increases, due to less possibility of stockout during repair, the optimal threshold level decreases which in turns decreases the expected holding cost. Thus, with the increase of μ, the optimal expected cost rate decreases due to decrease in the holding cost. Hence, by increasing repair rate manufacturer can decrease the stock threshold level and thus can minimize the expected cost of the system.

In Table 6 we carry out a sensitivity analysis with respect to unit holding cost rate c_h. The optimal expected cost rate increases with the increase of c_h. Consequently, in order to decrease the optimal cost rate, the optimal threshold level decreases.

Though exponential distribution is very popular to demonstrate the failure pattern of a machine, its disadvantage is that failure rate remains constant with respect to time *i.e.* its failure rate behavior is regarded as memoryless. Memoryless property indicates that remaining life of the system is independent of its current age. This means used components are assumed to be 'as-good-as-new' components. But in the real world situations production system may not follow this memoryless property and failure rate of the machine may change throughout its life. So to fit the realistic situations more closely a more flexible lifetime distribution is considered in the next example.

7.2. Example 2 (Weibull failure and exponential repair)

Let us suppose that the time to machine failure follows Weibull distribution and its probability distribution function is given by

$$F_X[\lambda(p)t] = 1 - e^{-(\lambda(P)t)^{\gamma}} \text{ for } \gamma > 0, \quad \lambda(P) > 0,$$

which can represent the failure pattern in all the three phases (infant mortality, stable and wear-out) of the buth–tub curve. We take the same parametric values as given in the above example and consider the case of increasing failure rate that means $\gamma > 1$.

Using Algorithm 2 the convergence of the solution sequence of barrier method is shown in Table 2. From this table, we obtain a local optimal solution as $(P^*, S^*) = (1790.17, 7759.92)$ and the associated minimum cost $C(P^*, S^*) = 4277.60$. Using the computational software MATHEMATICA [25] we have also checked that this local optimal solution $(P^*, S^*) = (1790.17, 7759.92)$ is infact a global optimal solution. Table 7 demonstrates a successive iterative scheme for finding the minimum cost rate function $C(P^*, S^*)$ using Algorithm 1 which is based on two-steps approach. As before we have checked that this local solution is actually the global solution [see Tables 2, 7 and 11].

Table 3
Impact of the failure parameter β on the optimal decisions when $\gamma = 1$.

β	P^*	S^*	$C(P^*,S^*)$
0.5	1465.91	3396.43	4369.82
0.6	1717.79	3473.47	4373.67
0.7	1775.62	3513.42	4375.67
0.8	1730.26	3532.09	4376.60
0.9	1650.13	3540.60	4377.03

Table 4
Impact of the failure parameter α on the optimal decisions when $\gamma = 1$.

α	P^*	S^*	$C(P^*,S^*)$
1.0	1465.91	3396.43	4369.82
1.1	1518.85	3409.12	4370.46
1.2	1565.50	3419.86	4370.99
1.3	1606.74	3429.09	4371.45
1.4	1643.36	3437.10	4371.86
1.5	1674.99	3444.13	4372.21
1.6	1705.18	3450.34	4372.52
1.7	1731.38	3455.88	4372.79
1.8	1754.99	3460.84	4373.04
1.9	1776.33	3465.31	4373.27
2.0	1795.68	3469.36	4373.47

Table 5
Dependence of the optimal decisions on the repair-rate parameter μ when $\gamma = 1$.

μ	P^*	S^*	$C(P^*,S^*)$
2.0	1656.00	3636.27	4381.81
2.5	1465.91	3396.43	4369.82
3.0	1327.03	3252.68	4362.63
3.5	1219.83	3158.93	4357.95
4.0	2461.50	3109.12	4355.46

Table 6
Sensitivity analysis with respect to the holding cost parameter c_h when $\gamma = 1$.

c_h	P^*	S^*	$C(P^*,S^*)$
0.01	1482.74	7742.65	4277.43
0.02	1477.12	5439.56	4308.79
0.03	1472.80	4419.39	4332.58
0.04	1469.14	3811.33	4352.45
0.05	1465.91	3396.43	4369.82
0.06	1462.98	3090.20	4385.41
0.07	1460.29	2852.23	4399.66
0.08	1457.77	2660.44	4412.83
0.09	1455.41	2501.59	4425.14
0.10	1453.16	2367.23	4436.72

Table 7
Successive iterative values of P and S when $\epsilon_1 = \epsilon_2 = 0.00001$ and $\gamma = 2$ at $c_h = 0.01$.

i	$P^{(i-1)}$	$S^{(i)}$	$P^{(i)}$	$C(P^{(i)}, S^{(i)})$
1	800.00	7902.72	1790.16	4277.61
2	1790.16	7759.92	1790.17	4277.60
3	1790.17	7759.92	1790.17	4277.60
4	1790.17	7759.92	1790.17	4277.60

Table 8
Dependence of the optimal policy on the failure parameter β when $\gamma = 2$.

β	P^*	S^*	$C(P^*,S^*)$
0.5	1770.94	3404.82	4370.24
0.6	1869.53	3481.94	4374.10
0.7	1835.91	3517.89	4375.89
0.8	1756.35	3534.10	4376.71
0.9	1662.96	3541.50	4377.07

Table 9
Dependence of the optimal policy on the failure parameter α when $\gamma = 2$.

α	P^*	S^*	$C(P^*,S^*)$
1.0	1770.94	3404.82	4370.24
1.1	1807.32	3417.91	4370.90
1.2	1837.74	3428.88	4371.44
1.3	1863.44	3438.20	4371.91
1.4	1885.37	3446.20	4372.31
1.5	1904.22	3453.15	4372.66
1.6	1920.55	3459.24	4372.96
1.7	1934.81	3464.61	4373.23
1.8	1947.34	3469.38	4373.47
1.9	1958.42	3473.65	4373.68
2.0	1968.26	3477.49	4373.87

From Tables 8–11, we carry out similar experiments with respect to the parameters as that of the exponential case and get the similar characteristic of the optimal lot sizing decisions. Fig. 5 shows the convexity property of the expected cost per unit time with respect to the stock threshold level for different values of production rate. This implies the existence of the

Table 10
Influence of the repair-rate parameter μ on the optimal decisions when $\gamma = 2$.

μ	P^*	S^*	$C(P^*,S^*)$
2.0	1945.94	3645.72	4382.29
2.5	1770.94	3404.82	4370.24
3.0	1641.44	3260.42	4363.02
3.5	1538.41	3166.32	4358.32
4.0	2461.75	3150.66	4357.53

Table 11
Sensitivity analysis with respect to the holding cost parameter c_h when $\gamma = 2$.

c_h	P^*	S^*	$C(P^*,S^*)$
0.01	1790.17	7759.92	4277.60
0.02	1783.74	5452.14	4309.04
0.03	1778.80	4429.89	4332.90
0.04	1774.63	3820.58	4352.82
0.05	1770.94	3404.82	4370.24
0.06	1767.61	3097.96	4385.88
0.07	1764.55	2859.50	4400.16
0.08	1761.69	2667.30	4413.38
0.09	1759.00	2508.12	4425.73
0.10	1756.46	2373.48	4437.35

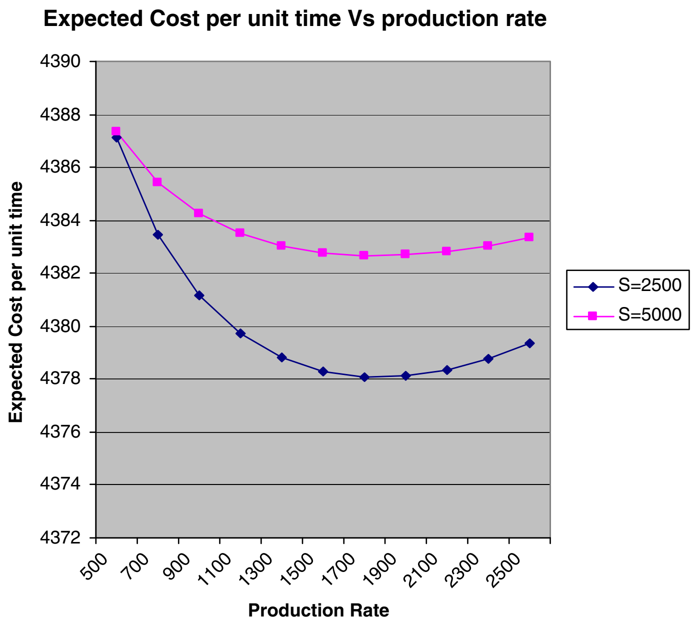

Fig. 6. Convexity of the expected cost per unit time with respect to the production rate P for a given threshold level.

optimal stock threshold level which minimizes the expected cost of the system for a given production rate. Similarly, Fig. 6 shows the convexity property with respect to P for different stock threshold level.

8. Conclusions

In this article, we have considered an imperfect EMQ model with stochastic machine breakdown and repair. A single item on a single machine with a certain capacity has been taken into consideration. Instead of constant production rate, variable production rate is considered in our model. In this paper we have considered an extra capacity of the machine. In many production system where machine is subject to stochastic breakdown or failure, extra capacity is maintained to buffer against the possible stock out during the repair. Further, as in most of the volume flexible manufacturing system, machine reliability depends on the stress of the machine, it is reasonable to assume that failure rate is an increasing function of the production rate. Here we have assumed that the failure rate is proportional to $P^{\beta}(\beta > 0)$. The model is formulated under general failure and general repair time distributions. The problem under consideration is a constrained optimization problem. Kuhn–Tucker constrained optimization method for non-linear equations and Barrier method were adopted to solve our constrained minimization problem. We have developed two Algorithms describing iterative numerical procedure using those methods. Analytically we have established the existence and uniqueness of the optimal production time t_0^* which minimizes the expected cost rate. Hence, optimal stock threshold level is obtained using the relation $S^* = t_0^*(P_1 - D)$ for any given production rate P_1 of P. Due to complexity of the objective function, some specific failure and repair time distributions are considered in order to avoid the intractability of its mathematical analysis. We were unable to find the closed analytical solutions for complex form of the objective function but were able to determine the optimal control parameters numerically. Finally, the influence of the failure parameters and repair parameter on the optimal policies are investigated through numerical examples. Through this study we have shown that by increasing repair rate manufacturer can decrease the expected cost of the system. Also, the convexity property of the expected cost function with respect to the stock threshold level and as well as variable production rate are depicted in Figs. 5 and 6, respectively. Further research considerations include the more realistic demand assumption such as variable demand rate. Moreover we have restricted ourselves into the case of machine failure only. We did not take into account the impact of usage of machine for a long time on the quality of the produced items. Hence, consideration of the quality of produced items may be an important research extension of this work.

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References

[1] F.W. Harris, Operations and Cost-factory Management Series, A.W. Shaw Co., Chicago, 1915.
[2] E.L. Porteus, Optimal lot sizing, process quality improvement and setup cost reduction, Oper. Res. 34 (1986) 137–144.
[3] M.J. Rosenblatt, H.L. Lee, Economic production cycles with imperfect production process, IIE Trans. 18 (1986) 48–55.
[4] M.J. Rosenblatt, H.L. Lee, A comparative study of continuous and periodic inspection policies in deteriorating production system, IIE Trans. 18 (1986) 2–9.
[5] H.L. Lee, M.J. Rosenblatt, Simultaneous determination of production cycle and inspection schedules in a production system, Manage. Sci. 33 (1987) 1125–1136.
[6] H.L. Lee, M.J. Rosenblatt, A production and maintenance planning model with restoration cot depending on detection delay, IIE Trans. 21 (1989) 368–375.
[7] S.T. Tseng, Optimal preventive maintenance policy for detriorating production system, IIE Trans. 28 (1996) 687–694.
[8] M.J. Liou, S.T. Tseng, T.M. Lin, The effects of inspection errors to the imperfect EMQ model, IIE Trans. 26 (1994) 42–51.
[9] V. Makis, J. Fung, An EOQ model with inspection and random machine failures, J. Oper. Res. Soc. 49 (1998) 66–76.
[10] V. Makis, Optimal lot sizing and inspection policy for an EMQ model with imperfect inspections, Nav. Res. Logis. 45 (1998) 165–186.
[11] C.H. Kim, Y. Hong, C.H. Chang, Optimal production run length and inspection schedules in a deteriorating production process, IIE Trans. 33 (2001) 421–426.
[12] H. Groenevelt, L. Pintelon, A. Seidmann, Production lot sizing with machine breakdown, Manage. Sci. 38 (1992) 104–123.
[13] H. Groenevelt, L. Pintelon, A. Seidmann, Production batching with machine breakdown and safety stocks, Oper. Res. 40 (1992) 959–971.
[14] C.H. Kim, Y. Hong, S.Y. Kim, An extended optimal lot sizing model with an unreliable machine, Prod. Plan. Control 8 (6) (1997) 577–585.
[15] T. Chakraborty, B.C. Giri, K.S. Chaudhuri, Production lot sizing with process deterioration and machine breakdown, Eur. J. Oper. Res. 185 (2008) 606–618.
[16] T. Chakraborty, B.C. Giri, K.S. Chaudhuri, Production lot sizing with process deterioration and machine breakdown under inspection schedule, Omega 37 (2009) 257–271.
[17] M. Khouja, A. Meherez, Economic production lot size model with variable production rate and imperfect quality, J. Oper. Res. Soc. 45 (1994) 1405–1417.
[18] B.C. Giri, W.Y. Yun, T. Dohi, Optimal design of unreliable production-inventory system with variable production rate, Eur. J. Oper. Res. 162 (2005) 372–386.
[19] B.C. Giri, T. Dohi, Computational aspects of an extended EMQ model with variable production rate, Comput. Oper. Res. 32 (2005) 3143–3161.
[20] K.L. Cheung, W.H. Hausman, Joint determination of preventive maintenance and safety stocks in an unreliable production environment, Nav. Res. Logis. 44 (1997) 257–271.
[21] T. Dohi, H. Okamura, S. Osaki, Optimal control of preventive maintenance schedule and safety stocks in and unreliable manufacturing environment, Int. J. Prod. Econ. 74 (2001) 147–155.
[22] T. Chakraborty, B.C. Giri, Joint determination of optimal safety stocks and production policy for an imperfect production system, Appl. Math. Model. 36 (2012) 712–722.
[23] M.S. Bazaraa, C.M. Shetty, Nonlinear Programming: Theory and Algorithms, Wiley, New York, 1979.
[24] S.M. Ross, Introduction to Probability Theory, Academic Press, New York, 1981.
[25] S. Wolfram, The MATEHMATICA Book, third ed., Wolfram Media, Cambridge University Press, Cambridge, 1996.