INSTANTON SOLUTIONS IN THE ANISOTROPIC σ MODEL

Susmita GANGULY
Physics Department, Calcutta University, 92, A.P.C. Road, Calcutta-700009, India

and

Dipankar RAY
Applied Mathematics Department, Calcutta University, 92, A.P.C. Road, Calcutta-700009, India

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In a recent paper a set of equations have been obtained for a nonlinear anisotropic σ model in two-dimensional euclidean space. The present note presents the complete set of solutions of these equations.

1. Introduction

In a recent paper Kundu [1] has obtained the following equations for a nonlinear anisotropic σ model in two-dimensional euclidean space.

$$\beta_1^2 + \beta_2^2 = \sin^2\beta\,(\gamma_1^2 + \gamma_2^2)\,,$$

$$\beta_1^2 + \beta_2^2 = 2\sin\beta\,(\beta_1\gamma_1 + \beta_2\gamma_2)\,,$$

$$2(\beta_{11} + \beta_{22}) - \sin\beta\,(\gamma_{11} + \gamma_{22}) - \sin 2\beta\,(\gamma_1^2 + \gamma_2^2) = 0\,,$$

$$\sin\beta[2\sin\beta\,(\gamma_{11} + \gamma_{22}) - (\beta_{11} + \beta_{22})] + 2\sin 2\beta\,(\beta_1\gamma_1 + \beta_2\gamma_2) - \cos\beta\,(\beta_1^2 + \beta_2^2) = 0\,, \quad (1)$$

where

$$\beta_1 \equiv \partial\beta(x^1, x^2)/\partial x^1\,, \quad \beta_2 \equiv \partial\beta(x^1, x^2)/\partial x^2\,,$$

$$\beta_{11} \equiv \partial^2\beta(x^1, x^2)/\partial x^{1\,2}\,,$$

and so on.

Defining

$$X \equiv \nabla\beta\,, \quad Y = \sin\beta\,\nabla\gamma\,, \quad (2a)$$

the action integral reads

$$S \equiv \frac{1}{2}\int [X^2 + Y^2 - (X - Y)^2]^2\,\mathrm{d}^2x\,. \quad (2b)$$

(β, γ) are the polar angles of the isovector $n^a \in S^2$.

Kundu [1] has obtained a particular set of solutions of eqs. (1) as follows:

$$\beta = 2\tan^{-1}[c\rho^{\pm\sqrt{3}}\,\mathrm{e}^{\alpha})^{n/2}]\,, \quad \gamma = n\alpha \quad (3)$$

where

$$\rho^2 = x'^2 + x^{2^2}\,, \quad \tan\alpha = x^2/x^1\,,$$

n and c are constants.

In the present note we seek to obtain the complete set of solutions of eqs. (1).

2. Solutions

Case 1.

β = constant.

This leads to the trivial solution

$X = 0 = Y$.

Case 2.

$\beta \neq$ constant.

Using some elementary algebra one can eliminate $(\gamma_{11} + \gamma_{22})$, $(\gamma_1^2 + \gamma_2^2)$ and $(\beta_1\gamma_1 + \beta_2\gamma_2)$ from the

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four equations of (1) to obtain an equation for β as

$$\sin\beta\,(\beta_{11}+\beta_{22})-\cos\beta\,(\beta_1^2+\beta_2^2)=0\,,$$

which is equivalent to

$$u_{11}+u_{22}=0\,,$$

where

$$u=\int \operatorname{cosec}\beta\,\mathrm{d}\beta\,,$$

i.e.

$$\beta=2\tan^{-1}(\mathrm{e}^u)\,. \tag{4}$$

Let v be the solution of the Laplace equation conjugate to u i.e.

$$u_1=v_2\,,\quad u_2=-v_1\,. \tag{5}$$

Making a transformation of variables $(x^1,x^2)\to(u,v)$ by using (5) and noting that by virtue of (4) β is a function of u only eqs. (1) reduce to

$$\beta_u^2=\sin^2\beta\,(\gamma_u^2+\gamma_v^2)\,, \tag{6a}$$

$$\beta_u^2=2\sin\beta\,\beta_u\gamma_u\,, \tag{6b}$$

$$2\beta_{uu}-\sin\beta\,(\gamma_{uu}+\gamma_{vv})-\sin2\beta\,(\gamma_u^2+\gamma_v^2)=0\,, \tag{6c}$$

$$\sin\beta[2\sin\beta\,(\gamma_{uu}+\gamma_{vv})-\beta_{uu}]$$
$$+\,2(\sin2\beta)\beta_u\gamma_u-(\cos\beta)\beta_u^2=0\,. \tag{6d}$$

From eqs. (4), (6a) and (6b)

$$\gamma_u=\tfrac{1}{2}\,,\quad \gamma_v=\pm\tfrac{1}{2}\sqrt{3}\,. \tag{7}$$

Putting eqs. (4) and (7) into (6) it is easy to see that all the eqs. (6) and hence all eqs. (1) are satisfied provided u and v satisfy (5).

3. Conclusion

Therefore if one leaves out the trivial solutions that lead to $X=0$, $Y=0$, the complete set of solutions of the set of eqs. (1) and (2) are given by

$$\beta=2\tan^{-1}(\mathrm{e}^u)\,,\quad \gamma=\tfrac{1}{2}u\pm\tfrac{1}{2}v\sqrt{3}\,, \tag{8}$$

where u and v are mutually conjugate solutions of the Laplace equation and X, Y and S are given by eqs. (2).

It is easy to check that the solution given by Kundu [1] [eqs. (3)] is a special case of (8).

References

[1] A. Kundu, Phys. Lett. 110B (1982) 61.