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Inertial oscillations and multiple boundary layers in an unsteady rotating flow

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An asymptotic analysis is made of the unsteady boundary layer flow generated impulsively in an incompressible homogeneous viscous fluid bounded by an infinite porous plate with uniform suction or blowing to describe the manner of the transient approach to the ultimate steady state. It is shown that the initial motion for small times describes the general features of the unsteady boundary-layer flow and consists of three boundary layers on the plate which are unaffected by rotation. In subsequent large times, the effect of rotation manifests itself through inertial oscillations of frequency 2Ω and the generation of diffused waves propagating outward from the plate with velocity $2(\Omega\nu)^{1/2}a_1$. It is found that the final steady boundary layers are established in a dimensional time $(4/\Omega)(S^4/4E^2+4)^{-1/2}$ which decreases with an increase in the suction parameter. Several limiting cases of interest are recovered from this analysis.

I. INTRODUCTION

In a recent paper, Debnath and Mukherjee[1] made an initial value investigation of the boundary layer flow in an incompressible homogeneous viscous rotating fluid bounded by a porous plate with uniform suction or blowing. They presented some qualitative and quantitative information about (i) the structure of the steady and unsteady flow field, (ii) development of the three distinct boundary layers of thicknesses of the order $\nu/U^*\alpha_1$, $\nu/U^*\alpha_2$, and $\nu/U^*\alpha_3$ with $\alpha_1 > \alpha_3 > \alpha_2$, and (iii) the significant effects of rotation, suction, or blowing. It has been shown that this work is a generalization of the corresponding steady problem considered by Gupta[2] and incorporates some new results of interest. Further, the resonant behavior of the solution at the critical frequency $\omega = 2\Omega$ was in accord with the recent findings of Faller and Kaylor.[3] It has also been inferred that the solution describes the oscillatory boundary layer flow for all frequencies of the imposed oscillations of the plate except for the case of blowing and the resonant frequency $\omega = 2\Omega$. It is apparent from physical considerations that the suction of the plate prevents the oscillations from spreading far away from the boundary by viscous diffusion for all frequencies of the imposed oscillations. On the contrary, the blowing promotes the spreading of the shear oscillations far away from the plate and hence the oscillatory boundary layer flow confined to the boundary layers does not exist for the case of blowing and $\omega = 2\Omega$.

In spite of our earlier work, it is of special interest to discuss the manner of the transient approach to the ultimate steady state and to describe the various physical processes involved in the transient approach. In order to discover some new information about the unsteady boundary layer flows bounded by an infinite porous plate in the rotating coordinate system considered in our paper,[1] an asymptotic analysis of the problem for small and large times is carried out. It is shown that immediately after the impulsive motion is imposed on the plate, an unsteady boundary layer flow builds up in the vicinity of the plate and that it consists of three boundary layers on the plate which remain unaffected by rotation. At subsequent large times, the effect of rotation manifests itself through inertial oscillations and the propagation of diffused waves traveling outward from the plate with the velocity $2(\Omega\nu)^{1/2}a_1$ which is modified by the suction or blowing. It is also found that the ultimate boundary layers are established in a dimensional time $(4/\Omega)[(S^4/4E^2)+4]^{-1/2}$ which decreases with an increase in the suction parameter. Several limiting results of interest are also discussed.

II. ASYMPTOTIC ANALYSIS

Since this is a sequel to our earlier work,[1] it may be fair to avoid duplication of the mathematical formulation of the problem and its method of solution. However, to make this paper self-contained to some extent, we shall quote necessary results from that paper without further proof.

Making reference to our paper,[1] the Laplace transform $\bar{q}(z, p)$ of the unsteady velocity field $q(z, t)$ is

$$\bar{q}(z,p) = \left(\frac{a}{p-i\sigma} + \frac{b}{p+i\sigma} - \frac{U}{U^*}\frac{1}{p}\right) \times \exp\left\{-\frac{z}{2}[S + (S^2 + 4iE + 2pE)^{1/2}]\right\}. \quad (1)$$

The early time behavior of the velocity distribution $q(z, t)$ is determined by the corresponding behavior of $\bar{q}(z, p)$ for large $|p|$. Consequently, the function $\bar{q}(z, p)$ for large $|p|$ is given by

$$\bar{q}(z,p) \sim \left(\frac{a}{p-i\sigma} + \frac{b}{p+i\sigma} - \frac{U}{U^*}\frac{1}{p}\right) \times \exp\left[-z\left(\frac{S + (S^2+2pE)^{1/2}}{2}\right)\right]. \quad (2)$$

Using the table of the inverse Laplace transform due to

Campbell and Foster,[4] the solution for $q(z, t)$ is

$$q(z, t) \sim \frac{a}{2}\exp\left(i\sigma t - \frac{Sz}{2}\right)\left\{\exp[\zeta(c+id)] \times \operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} + (c+id)t^{1/2}\right] + \exp[-\zeta(c+id)]\operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} - (c+id)t^{1/2}\right]\right\} + \frac{b}{2}\exp\left(-i\sigma t - \frac{Sz}{2}\right)\left\{\exp[\zeta(c-id)] \times \operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} + (c-id)t^{1/2}\right] + \exp[-\zeta(c-id)]\operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} - (c-id)t^{1/2}\right]\right\} - \frac{U}{U^*}\left[\operatorname{erfc}\left(\frac{\zeta}{2t^{1/2}} + \frac{tS^2}{2E}\right) + \exp(-Sz) \times \operatorname{erfc}\left(\frac{\zeta}{2t^{1/2}} - \frac{tS^2}{2E}\right)\right], \tag{3}$$

where

$$\zeta = z\left(\frac{E}{2}\right)^{1/2} \quad \text{and} \quad \left(\frac{S^2}{2E} \pm i\sigma\right)^{1/2} = c \pm id,$$

with

$$c = (1/2E^{1/2})[S^4 + 4E^2\sigma^2)^{1/2} + S^2]^{1/2}, \tag{4}$$

and

$$d = (1/2E^{1/2})[(S^4 + 4E^2\sigma^2)^{1/2} - S^2]^{1/2}. \tag{5}$$

It follows from solution (3) that immediately after the impulsive motion is imposed on the plate, an unsteady boundary layer flow builds up in the vicinity of the plate. And the solution consists of the Rayleigh layer of thickness of the order $(\nu t)^{1/2}$ and two more distinct boundary layers of thicknesses of the order ν/w_0 and

$$\delta = \nu\{w_0 + (1/\sqrt{2})[w_0^2 + (w_0^4 + 16\nu^2\omega^2)^{1/2}]^{1/2}\}^{-1}. \tag{6}$$

The most significant of these boundary layers is that they remain unaffected by rotation. However, the last two layers are radically modified by suction and, in fact, the thickness of these layers decreases with an increase in suction. When $w_0 \equiv 0$, the early time boundary layer solution (3) consists of the Stokes layer of thickness of the order $(\nu/\omega)^{1/2}$ and the Rayleigh layer.

To investigate the asymptotic nature of the solution for large time t, we recall solution (18) from the paper[1] and rewrite it in the form

$$q(z, t) = \frac{a}{2}\exp\left(i\sigma t - \frac{Sz}{2}\right)\left\{\exp[\zeta(a_1+ib_1)] \times \operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} + (a_1+ib_1)t^{1/2}\right] + \exp[-\zeta(a_1+ib_1)]\operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} - (a_1+ib_1)t^{1/2}\right]\right\} + \frac{b}{2}\exp\left(-i\sigma t - \frac{Sz}{2}\right)\left\{\exp[\zeta(a_2+ib_2)] \times \operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} + (a_2+ib_2)t^{1/2}\right] + \exp[-\zeta(a_2+ib_2)]\operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} - (a_2+ib_2)t^{1/2}\right]\right\} - \frac{U}{2U^*}\exp\left(-\frac{Sz}{2}\right)\left\{\exp[\zeta(a_3+ib_3)] \times \operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} + (a_3+ib_3)t^{1/2}\right] + \exp[-\zeta(a_3+ib_3)]\operatorname{erfc}\left[\frac{\zeta}{2t^{1/2}} - (a_3+ib_3)t^{1/2}\right]\right\}, \tag{7}$$

where

$$(a_1 + ib_1) = \left[\frac{S^2}{2E} + i(2+\sigma)\right]^{1/2}, \tag{8}$$

$$(a_2 + ib_2) = \left[\frac{S^2}{2E} + i(2-\sigma)\right]^{1/2}, \tag{9}$$

and

$$a_3 + ib_3 = \left(\frac{S^2}{2E} + 2i\right)^{1/2}, \tag{10}$$

with

$$a_1 = \left\{\left[\frac{S^4}{16E^2} + \left(1+\frac{\sigma}{2}\right)^2\right]^{1/2} + \frac{S^2}{4E}\right\}^{1/2}, \tag{11}$$

$$b_1 = \left\{\left[\frac{S^4}{16E^2} + \left(1+\frac{\sigma}{2}\right)^2\right]^{1/2} - \frac{S^2}{4E}\right\}^{1/2}, \tag{12}$$

$$a_2 = \left\{\left[\frac{S^4}{16E^2} + \left(1-\frac{\sigma}{2}\right)^2\right]^{1/2} + \frac{S^2}{4E}\right\}^{1/2}, \tag{13}$$

$$b_2 = \left\{\left[\frac{S^4}{16E^2} + \left(1-\frac{\sigma}{2}\right)^2\right]^{1/2} - \frac{S^2}{4E}\right\}^{1/2}, \tag{14}$$

$$a_3 = \frac{1}{2E^{1/2}}[(S^4 + 16E^2)^{1/2} + S^2]^{1/2}, \tag{15}$$

and

$$b_3 = \frac{1}{2E^{1/2}}[(S^4 + 16E^2)^{1/2} - S^2]^{1/2}. \tag{16}$$

In view of the asymptotic expansion of the complementary error function with complex argument z in the form

$$\operatorname{erfc}(z) \sim \frac{\exp(-z^2)}{z(\pi)^{1/2}} \qquad \text{as } |z| \to \infty, \tag{17}$$

together with the result $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$, it turns out that the asymptotic solution for large t has the form

$$\begin{aligned} q(z,t) \sim\ & a \exp\left[i\sigma t - \frac{Sz}{2} - \zeta(a_1 + ib_1)\right] \\ & + b \exp\left[-i\sigma t - \frac{Sz}{2} - \zeta(a_2 + ib_2)\right] \\ & - \frac{U}{U^*} \exp\left[-\frac{Sz}{2} - \zeta(a_3 + ib_3)\right] \\ & - \frac{\zeta}{2(\pi t)^{1/2}} \exp\left[-\left(2it + \frac{Sz}{2} + \frac{S^2 t}{2E} + \frac{\zeta^2}{4t}\right)\right] \\ & \times \left\{ a\left[(a_1 + ib_1)^2 t - \frac{\zeta^2}{4t}\right]^{-1} \right. \\ & + b\left[(a_2 + ib_2)^2 t - \frac{\zeta^2}{4t}\right]^{-1} \\ & \left. - \frac{U}{U^*}\left[(a_3 + ib_3)^2 t - \frac{\zeta^2}{4t}\right]^{-1} \right\}. \end{aligned} \tag{18}$$

The first three terms of (18) represent the steady-state solution consisting of three distinct boundary layers of thicknesses of the order

$$\frac{\nu}{U^*}\left[\frac{S}{2} + \left(\frac{E}{2}\right)^{1/2} a_r\right]^{-1}, \quad (r = 1, 2, 3)$$

as also shown in the paper.[1]

The most distinctive feature of the asymptotic solution is that the last three terms in (18) confirm the existence of inertial oscillations of frequency 2Ω which decay exponentially within the ultimate steady-state boundary layers. The nature of these oscillations persists even when $\sigma = 0$.

Finally, it follows from the arguments of the complementary error function involved in solution (7) that the three ultimate boundary layers are established through the inertial oscillations in nondimensional times of the order $4(a_r^2 + b_r^2)^{-1}$, $(r = 1, 2, 3)$. When $\sigma = 0$, these times coalesce into the time of the order $(4/\Omega)(S^4/4E^2 + 4)^{-1/2}$ which decreases with an increase in the suction parameter.

To clearly demonstrate the generation of diffused waves in the fluid, it is necessary to mention some properties of the complementary error function with complex argument due to Lebedev[5] and Strand.[6] For any complex number $z = x + iy$, $\overline{\operatorname{erfc}(z)} = \operatorname{erfc}(\bar{z})$ and

$$\operatorname{erfc}(z) = \exp(-2ixy) f(x, y), \quad x > 0,\ y \geq 0, \tag{19}$$

where $f(x, y)$ is a complex function given by

$$f(x, y) = \sum_{n=0}^{\infty} (xy)^{2n} [\gamma_n(x) - i(n+1)\gamma_{n+1}(x)], \tag{20}$$

which tends to zero as $x \to \infty$ and

$$\gamma_{n+1}(x) = \frac{2}{(2n+1)}\left(\frac{\exp(-x^2)}{\pi^{1/2}(n+1)!\,x^{2n+1}} - \frac{\gamma_n(x)}{(n+1)}\right), \quad n = 0, 1, 2, \cdots, \tag{21}$$

where $\gamma_0(x) = \operatorname{erfc}(x)$.

Using these results, solution (7) can be put into the form

$$\begin{aligned} q(z,t) =\ & \frac{a}{2} \exp\left(-2it - \frac{Sz}{2}\right) [\exp(\zeta a_1) f(\eta + a_1 t^{1/2}, b_1 t^{1/2}) \\ & + \exp(-\zeta a_1) \bar{f}(\eta - a_1 t^{1/2}, b_1 t^{1/2})] \\ & + \frac{b}{2} \exp\left(-2it - \frac{Sz}{2}\right) [\exp(\zeta a_2) f(\eta + a_2 t^{1/2}, b_2 t^{1/2}) \\ & + \exp(-\zeta a_2) \bar{f}(\eta - a_2 t^{1/2}, b_2 t^{1/2})] \\ & - \frac{U}{2U^*} \exp\left(-2it - \frac{Sz}{2}\right) [\exp(\zeta a_3) \\ & \times f(\eta + a_3 t^{1/2}, b_3 t^{1/2}) + \exp(-\zeta a_3) \\ & \times \bar{f}(\eta - a_3 t^{1/2}, b_3 t^{1/2})], \end{aligned} \tag{22}$$

when $\zeta > \max(2a_1 t, 2a_2 t, 2a_3 t)$

$$\begin{aligned} q(z,t) =\ & a \exp\left[i\sigma t - \frac{Sz}{2} - \zeta(a_1 + ib_1)\right] \\ & + b \exp\left[-i\sigma t - \frac{Sz}{2} - \zeta(a_2 + ib_2)\right] \\ & - \frac{U}{U^*} \exp\left[-\frac{Sz}{2} - \zeta(a_3 + ib_3)\right] \\ & + \frac{a}{2} \exp\left(-2it - \frac{Sz}{2}\right) [\exp(\zeta a_1) f(\eta + a_1 t^{1/2}, b_1 t^{1/2}) \\ & - \exp(-\zeta a_1) f(a_1 t^{1/2} - \eta, b_1 t^{1/2})] \\ & + \frac{b}{2} \exp\left(-2it - \frac{Sz}{2}\right) [\exp(\zeta a_2) f(\eta + a_2 t^{1/2}, b_2 t^{1/2}) \\ & - \exp(-\zeta a_2) f(a_2 t^{1/2} - \eta, b_2 t^{1/2})] \\ & - \frac{U}{2U^*} \exp\left(-2it - \frac{Sz}{2}\right) [\exp(\zeta a_3) \\ & \times f(\eta + a_3 t^{1/2}, b_3 t^{1/2}) - \exp(-\zeta a_3) \\ & \times f(a_3 t^{1/2} - \eta, b_3 t^{1/2})] \end{aligned} \tag{23}$$

when $\zeta < \min(2a_1 t, 2a_2 t, 2a_3 t)$, where $\bar{f}(x, y)$ is the complex conjugate of $f(x, y)$ and $\eta = \zeta / 2t^{1/2}$.

In particular, when $\sigma = 0$, the above results reduce to

$$\begin{aligned} q(z,t) =\ & \frac{1}{2}\left(a + b - \frac{U}{U^*}\right) \exp\left(-2it - \frac{Sz}{2}\right) \\ & \times [\exp(\zeta a_1) f(\eta + a_1 t^{1/2}, b_1 t^{1/2}) \\ & + \exp(-\zeta a_1) f(\eta - a_1 t^{1/2}, b_1 t^{1/2})], \end{aligned} \tag{24}$$

when $\zeta > 2a_1t$, and

$$q(z, t) = \left(a + b - \frac{U}{U^*}\right) \exp\left[-\frac{Sz}{2} - \zeta(a_1 + ib_1)\right]$$
$$+ \frac{1}{2}\left(a + b - \frac{U}{U^*}\right) \exp\left(-2it - \frac{Sz}{2}\right)$$
$$\times [\exp(\zeta a_1) f(\eta + a_1 t^{1/2}, b_1 t^{1/2})$$
$$- \exp(-\zeta a_1) f(a_1 t^{1/2} - \eta, b_1 t^{1/2})], \qquad (25)$$

when $\zeta < 2a_1t$.

These results clearly indicate the generation and propagation of diffused waves traveling outward from the plate with velocity $2a_1(\Omega\nu)^{1/2}$ which is modified by the suction. In fact, Eq. (24) represents the unsteady flow behind the wave front $z = (2/E)^{1/2} 2a_1 t$, whereas (25) describes the flow ahead of it. Eventually, these waves decay within the final steady boundary layers.

Finally, the solution for the case of blowing and $\sigma \neq 2$ can readily be obtained from the above analysis by replacing S by $-S_1$ where $S_1 > 0$. An inspection of the results for the case of blowing indicates that the diffusive waves travel outward from the plate with velocity $2(\Omega\nu)^{1/2} \times (1 + \sigma/2)^{1/2}$ when the speed of the blowing does not exceed the value $2(\Omega\nu)^{1/2}(2 + \sigma)^{1/2}$. Obviously, there exists a relationship between the wave speed and the maximum blowing speed. Further, if the blowing parameter S_1 is such that $S_1^2 < 4E$, the thicknesses of the boundary layers associated with the ultimate flow field do not exceed the value $\nu/U\alpha$, which decreases with an increase in rotation, where

$$\alpha = \frac{1}{2}\left[\frac{2E(2 + \sigma) + S_1^2}{2}\right]^{1/2}.$$

The physical significance of this conclusion is that if the blowing is not very large, the thickening effect of blowing on the boundary layers is counterbalanced by the thinning effect of rotation so that the depth of penetration of vorticity remains confined to the plate.

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