Harmonic oscillator in presence of nonequilibrium environment

Jyotipratim Ray Chaudhuri,[1,a)] Pinaki Chaudhury,[2,b)] and Sudip Chattopadhyay[3,c)]
[1]*Department of Physics, Katwa College, Katwa, Burdwan 713130, India*
[2]*Department of Chemistry, University of Calcutta, Kolkata 700009, India*
[3]*Department of Chemistry, Bengal Engineering and Science University, Shibpur, Howrah 711103, India*

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Based on a microscopic Hamiltonian picture where the system is coupled with the nonequilibrium environment, comprising of a set of harmonic oscillators, the Langevin equation with proper microscopic specification of Langevin force is formulated analytically. In our case, the reservoir is perturbed by an external force, either executing rapid or showing periodic fluctuations, hence the reservoir is not in thermal equilibrium. In the presence of external fluctuating force, using Shapiro–Loginov procedure, we arrive at the linear coupled first order differential equations for the two-time correlations and examine the time evolution of the same considering the system as a simple harmonic oscillator. We study the stochastic resonance phenomena of a Kubo-type oscillator (assumed to be the system) when the bath is modulated by a periodic force. The result(s) obtained here is of general significance and can be used to analyze the signature of stochastic resonance.
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I. INTRODUCTION

The damped harmonic oscillator in the presence of sinusoidal forcing is a prototype and exactly solvable problem, particularly when the damping force is linearly proportional to the velocity. This model is widely applied for modeling various phenomena in physics, engineering, and chemistry. For example, an *L*-*C*-*R* circuit with an ac voltage source is exactly model with an equation that describes a damped force harmonic oscillator; the phenomena of resonance are neatly described. Intricacy arises when noise begins to start playing, though the problem of the damped harmonic oscillator driven by Gaussian white noise has been solved. In the majority of the treatments of harmonic oscillator in the presence of noises, whether white or colored, the stochastic differential equation is written either phenomenologically or with one concerning an equilibrium thermal bath at a finite temperature, which generates the damping and fluctuations. As the inherent noise of the medium is of internal origin, the dissipative force, which the harmonic oscillator is subjected to, in the course of its motion in the medium and the stochastic force acting on the system as a result of random impact from the constituents of the medium arise from a common mechanism. From a microscopic point of view, the system-reservoir description developed over the decades suggests that the coupling of the system and the reservoir coordinates determines both the noise and the dissipative terms in the Langevin equation describing the motion of the system. It is, therefore, not difficult to anticipate that these two entities get related through a fluctuation-dissipation relation. These systems are sometimes classified as thermodynamically closed systems in contrast with the systems driven by external noise in nonequilibrium statistical mechanics. However, when the reservoir is modulated via an external noise, it is likely that it induces fluctuations in the polarization of the reservoir. These fluctuations in turn may drive the system in addition to the usual internal noise of the reservoir. Since the polarization fluctuations of the reservoir crucially depend on its response functions, one can further envisage a connection between the dissipation of the system and response function of the reservoir due to external driving from a microscopic point of view. Consequently, one can expect a different dynamical behavior of the damped harmonic oscillator in the presence of a heat bath, which is externally modulated and not in thermal equilibrium. Two worth studying features of the linear system are the various moments and resonance phenomena.

It should be mentioned here that due to its potential applicability in biology, physics, and chemistry, stochastic resonance (SR), which is an interesting phenomenon exhibited by nonlinear dynamic systems driven by a combination of a periodic signal and a random force,[1] has been the subject of many experimental and theoretical studies.[2] The synthesis of the theoretical tools for such studies remains an active domain of research. In contrast with determining chaos, which denotes a random type of behavior in deterministic systems, SR shows a deterministiclike behavior in random system. These peculiar features show that determinism and randomness are complementary to each other, rather than being contradictory.[3] In a wide sense, the characteristic of SR lies in the nonmonotonic dependence of the output signal or some function of it, e.g., moments, autocorrelation function, power spectrum, or signal-to-noise ratio. The interesting feature of SR is that noise, which usually appears as a destructive factor, may play a constructive role. There are many interesting physical phenomena such as noise-induced transition,[4] noise-induced transport,[5] noise-induced pattern formation,[6] noise-induced resonance,[7] resonance activation,[8] coherent SR,[9] etc, where SR appears and shows the ordered

a)Electronic mail: jprc_8@yahoo.com.
b)Electronic mail: pinakc@rediffmail.com.
c)Electronic mail: sudip_chattopadhyay@rediffmail.com.

role of noise. It first seemed that all the three ingredients, such as nonlinearity, periodicity, and random forcing, are required for the onset of SR. However, later it became clear that SR may appear in the absence of force[10] and periodic forcing[11] even in a linear system.[12,13]

In our present study, we consider a system which is coupled to a nonequilibrium heat bath owing to its being driven by an external random force. Actually, we considered the externally driven Langevin dynamics of a microscopic Hamiltonian system to study the various moments and associated phenomena. The treatment has been extended to the study of SR in a linear system: Kubo oscillator. In our development, the Kubo oscillator (harmonic oscillator with random frequency) is coupled with a bath, which is externally modulated via sinusoidal force. At this point, we want to mention that the analysis of SR in linear systems was previously restricted to an overdamped oscillator with color multiplicative noise.[14–17] There are few examples of the analysis of SR in an underdamped oscillator either related to additive noise[17] or involving no external field.[18] In our study of SR, color noise appears multiplicatively apart from an additive internal noise.

II. THEORETICAL MODEL

In our present development we consider the system is to be coupled to a harmonic heat bath with characteristic frequency set $\{\omega_j\}$. At this point it is pertinent to mention the fact that the system-bath coupling is usually nonlinear in nature. Initially at $t=0$, the bath is in thermal equilibrium at temperature T. Later on, at $t=0_+$, an external noise agency is switched on, which starts modulating the bath. The Hamiltonian for the composite system can be written as

$$H = H_S + H_B + H_{SB} + H_{\text{int}} = \frac{p^2}{2} + V(x) + \sum_{j=1}^{N} \left\{ \frac{p_j}{2} + \frac{1}{2}\omega_j^2 (q_j - c_j f(x_j))^2 \right\} + H_{\text{int}}. \quad (1)$$

In Eq. (1), q_j and p_j are bath variables, c_j is the coupling constant, and $f(x)$ is some smooth well-behaved function of the system variable. H_{int} takes care of the interaction between the bath and the external fluctuation $\epsilon(t)$,

$$H_{\text{int}} = \sum_{k=1}^{N} \kappa_j q_j \epsilon(t), \quad (2)$$

with κ_j being the strength of the interaction. The external noise $\epsilon(t)$ is considered to be stationary, Gaussian with the statistical properties,

$$\langle \epsilon(t) \rangle = 0; \quad \langle \epsilon(t)\epsilon(t') \rangle = 2D\psi(t-t'), \quad (3)$$

where D is the strength of the external noise and ψ is some decaying memory kernel. $\langle \cdots \rangle_e$ implies averaging over each realization of $\epsilon(t)$. In what follows, for the sake of simplicity, we shall assume the linear system-bath coupling [i.e., $f(x) = x$]. Now eliminating the bath degrees of freedom in the usual way, we get the Langevin equation for the system particle,

$$\dot{x} = v, \qquad \dot{v} = -V'(x) - \int_0^t dt' \gamma(t'-t) v(t') + f(t) + \pi(t), \quad (4)$$

where the memory kernel $\gamma(t)$ and the Langevin force term $f(t)$ are given, respectively, by the following expressions:

$$\gamma(t) = \sum_{j=1}^{N} c_j \omega_j^2 \cos(\omega_j t) \quad (5)$$

and

$$f(t) = \sum_{j=1}^{N} c_j \omega_j^2 \left\{ [x_j(0) - c_j x(0)]\cos(\omega_j t) + \frac{v_j(0)}{\omega_j} \sin(\omega_j t) \right\}. \quad (6)$$

In Eq. (4), $\pi(t)$ is the fluctuating force generated due to the external stochastic force of the bath, $\epsilon(t)$, and is given by

$$\pi(t) = \int_0^t dt' \varphi(t'-t) \epsilon(t'), \quad (7)$$

with

$$\varphi(t) = \sum_{j=1}^{N} c_j \omega_j \kappa_j \sin(\omega_j t). \quad (8)$$

The form of Eq. (4) reveals that the system is driven by two fluctuating forces, $f(t)$ and $\pi(t)$. $\pi(t)$ is a dressed noise originating due to the bath modulation by external noise $\varepsilon(t)$ and $f(t)$ is the thermal noise due to system-bath coupling. To define the statistical properties of $f(t)$, we assume that the initial distribution is such that the bath is equilibrated at $t=0$ in the presence of the system but in the absence of the external noise $\epsilon(t)$ such that

$$\langle f(t) \rangle = 0, \qquad \langle f(t) f(t') \rangle = k_B T \gamma(t-t'), \quad (9)$$

where is the Boltzmann constant, T is the equilibrium temperature, and $\langle \cdots \rangle$ implies the usual average over the initial distribution, which is assumed to be a canonical distribution of Gaussian form,

$$P = N \exp\left\{ - \frac{v_j^2(0) + \omega_j^2 (x_j(0) - c_j x(0))^2}{2 k_B T} \right\},$$

where N is the normalization constant. Now at $t=0_+$, the external noise agency is switched on to modulate the bath. Here, we define an effective Gaussian noise $\xi(t) = f(t) + \pi(t)$, the statistical properties of which can be described by

$$\langle\langle \xi(t) \rangle\rangle = 0,$$
$$\langle\langle \xi(t)\xi(t') \rangle\rangle = k_B T \gamma(t-t') + 2D_\epsilon \int_0^t dt'' \int_0^{t'} dt''' \times \varphi(t-t'')\varphi(t-t''')\psi(t''-t'''). \quad (10)$$

In Eq. (10) $\langle\langle \cdots \rangle\rangle$ means that we have taken two averagings independently, an average over the initial distribution of bath variables and another one average over each realization of $\varepsilon(t)$. It is important to note that Eq. (10) is not a fluctuation-dissipation relation due to the appearance of external noise intensity, rather it serves as a thermodynamic consistency relation.

To obtain a finite result in the continuum limit, i.e., for $N\to\infty$, the coupling functions $c_i=c(\omega)$ and $\kappa_i=\kappa(\omega)$ are chosen as $c(\omega)=c_0/\omega\sqrt{\tau_c}$ and $\kappa(\omega)=\kappa_0\omega\sqrt{\tau_c}$. Consequently, $\gamma(t)$ and $\varphi(t)$ reduce to

$$\gamma(t)=\frac{c_0^2}{\tau_c}\int d\omega\rho(\omega)\cos(\omega t) \tag{11}$$

and

$$\varphi(t)=c_0\kappa_0\int d\omega\rho(\omega)\sin(\omega t), \tag{12}$$

where c_0 and κ_0 are constants and τ_c is the correlation time of the heat bath. $\rho(\omega)$ is the density of modes of the heat bath, which is assumed to be Lorentzian,

$$\rho(\omega)=\frac{2\tau_c}{\pi(1+\omega^2\tau_c^2)}. \tag{13}$$

For $\tau_c\to 0$ we obtain a δ-correlated noise process. The above assumption resembles broadly the behavior of the hydrodynamical modes in a microscopic system and is frequently used by the chemical physics community.[19–21] With these forms of $\rho(\omega)$, $c(\omega)$, and $\kappa(\omega)$, we have the expressions for $\varphi(t)$ and $\gamma(t)$, respectively, as

$$\varphi(t)=\frac{c_0\kappa_0}{\tau_c}\exp\left(-\frac{|t|}{\tau_c}\right),$$

$$\gamma(t)=\frac{c_0^2}{\tau_c}\exp\left(-\frac{|t|}{\tau_c}\right).$$

If we assume that ϵ is a δ-correlated noise, i.e., $\langle\epsilon(t)\epsilon(t')\rangle=2D_\varepsilon\delta(t-t')$, then the correlation function of $\pi(t)$ is given by

$$\langle\pi(t)\pi(t')\rangle=\frac{D_\epsilon c_0^2\kappa_0^2}{\tau_c}\exp\left(-\frac{|(t-t')|}{\tau_c}\right), \tag{14}$$

where we neglected the transient terms $(t,t'>\tau_c)$. This equation shows how the heat bath dresses the external noise. Although the external noise is a δ-correlated one, the system encounters it as an exponentially correlated noise with the same correlation time as that of the internal noise but with a strength that depends on the coupling term κ_0 and the external noise strength D_ε. On the other hand, if the external noise follows the Ornstein–Uhlenbeck process,

$$\langle\epsilon(t)\epsilon(t')\rangle=\frac{D_\epsilon}{\tau_\epsilon}\exp\left(-\frac{|(t-t')|}{\tau_\epsilon}\right),$$

the correlation function of $\pi(t)$ is found to be

$$\langle\pi(t)\pi(t')\rangle_e=\frac{D_\epsilon c_0^2\kappa_0^2}{(\tau_\epsilon/\tau_c)^2-1}(\tau_\epsilon/\tau_c)\times\left\{\frac{1}{\tau_c}\exp\left(-\frac{|(t-t')|}{\tau_\epsilon}\right)-\frac{1}{\tau_\epsilon}\exp\left(-\frac{|(t-t')|}{\tau_c}\right)\right\}, \tag{15}$$

where we again neglected the transient terms. The dressed noise $\pi(t)$ now has a more complicated form of the correlation function with two correlation terms τ_c and τ_ϵ. If the external noise correlation time is much larger than that of the internal noise $(\tau_\epsilon\gg\tau_c)$, which is a much more realistic physical situation, the dressed noise is dominated by the external noise and one may obtain from Eq. (15)

$$\langle\pi(t)\pi(t')\rangle_e=\frac{D_\epsilon c_0^2\kappa_0^2}{\tau_\epsilon}\exp\left(-\frac{|(t-t')|}{\tau_\epsilon}\right). \tag{16}$$

On the other hand, when the external noise correlation time is smaller that the internal one, we recover Eq. (14). In what follows, we shall focus on the situation when $(\tau_\epsilon\gg\tau_c)$. Thus, in terms of the effective noise $\xi(t)$, the Langevin Eq. (4) can be written as

$$\begin{aligned}\dot{x}&=v,\\ \dot{v}&=-V'(x)-\int_0^t dt'\,\gamma(t'-t)v(t')+\xi(t),\end{aligned} \tag{17}$$

which reduces to

$$\begin{aligned}\dot{x}&=v,\\ \dot{v}&=-V'(x)-\gamma v+\xi(t),\end{aligned} \tag{18}$$

where we assumed that the internal noise $f(t)$ is δ-correlated and the internal dissipation is Markovian so that

$$\gamma(t)=2c_0^2\delta(t-t')=2\gamma\delta(t-t'),$$

$$\langle f(t)\rangle=0,$$

$$\langle f(t)f(t')\rangle=2\gamma k_BT\delta(t-t'),$$

with $\gamma=c_0^2$. The effective noise $\xi(t)$ thus follows the statistical properties

$$\begin{aligned}\langle\langle\xi(t)\rangle\rangle&=0,\\ \langle\langle\xi(t)\xi(t')\rangle\rangle&=\frac{D_R}{\tau_R}\exp\left(-\frac{|(t-t')|}{\tau_R}\right),\end{aligned} \tag{19}$$

where

$$\begin{aligned}D_R&=\gamma(k_BT+D_\epsilon\kappa_0^2),\\ \tau_R&=\frac{D_\epsilon}{D_R}\gamma\kappa_0^2\tau_c,\end{aligned} \tag{20}$$

with D_R and τ_R being the strength and correlation time of the effective noise $\xi(t)$, respectively.

III. HARMONIC OSCILLATOR AND ITS MOMENTS

We now consider that our system is an underdamped linear oscillator of unit mass with constant frequency ω. Then the equation of motion, Eq. (18), reduces to

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x = \xi(t). \tag{21}$$

Since $dx^2/dt=2xv$ and $(d/dt^2)v^2(t)=2v(dv/dt)$, one can write after averaging over each realization of $\xi(t)$ that

$$\frac{d}{dt}\langle\langle x^2(t)\rangle\rangle = 2\langle\langle x(t)v(t)\rangle\rangle \tag{22}$$

and

$$\frac{d}{dt}\langle\langle v^2(t)\rangle\rangle = -2\omega^2\langle\langle x(t)v(t)\rangle\rangle - 2\gamma\langle\langle v^2(t)\rangle\rangle + 2\langle\langle \xi(t)v(t)\rangle\rangle. \tag{23}$$

We have to calculate the two correlations $\langle\langle x(t)v(t)\rangle\rangle$ and $\langle\langle \xi(t)v(t)\rangle\rangle$ separately. By the Shapiro–Loginov procedure[22] [for exponentially correlated noise $\xi(t)$] we have

$$\frac{d}{dt}\langle\langle \xi(t)g[\xi(t)]\rangle\rangle = \left\langle\left\langle \xi(t)\frac{dg}{dt}\right\rangle\right\rangle - \lambda\langle\langle \xi(t)g[\xi(t)]\rangle\rangle, \tag{24}$$

where $g[\xi(t)]$ is some functional of $\xi(t)$. In the present case, the exponential correlator is given by Eq. (19) with $\sigma = D_R/\tau_R$; $\lambda=1/\tau_R$. In the limiting case, $\sigma\to\infty$ and $\lambda\to\infty$, with σ/λ remaining equal to the correlation D_R; Eq. (19) reduces to

$$\langle\langle \xi(t)\xi(t')\rangle\rangle = 2D_R\delta(t-t'). \tag{25}$$

Now using Eq. (24) we get from Eqs. (22) and (23)

$$\frac{d}{dt}\langle\langle \xi(t)x(t)\rangle\rangle = \langle\langle \xi(t)v(t)\rangle\rangle - \lambda\langle\langle \xi(t)x(t)\rangle\rangle,$$

$$\frac{d}{dt}\langle\langle \xi(t)v(t)\rangle\rangle = -(\gamma+\lambda)\langle\langle \xi(t)v(t)\rangle\rangle - \omega^2\langle\langle \xi(t)x(t)\rangle\rangle + \sigma, \tag{26}$$

$$\frac{d}{dt}\langle\langle v(t)x(t)\rangle\rangle = \langle\langle v^2(t)\rangle\rangle - \omega^2\langle\langle x^2(t)\rangle\rangle - \gamma\langle\langle v(t)x(t)\rangle\rangle + \langle\langle \xi(t)x(t)\rangle\rangle.$$

Thus, the second moments (the position-position correlation function) can be determined from the following sets of closed linear first order differential equations:

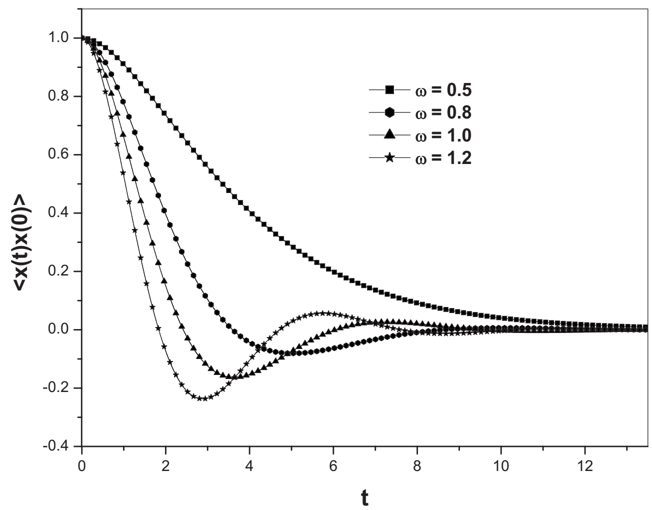

FIG. 1. Variation of position-position correlation function, $\langle\langle x(0)x(t)\rangle\rangle$, as a function of t for various $\omega[\gamma=1, \quad \lambda=10, \quad \sigma=0.1]$.

$$\frac{d}{dt}\langle\langle x(t)x(0)\rangle\rangle = \langle\langle x(0)v(t)\rangle\rangle,$$

$$\frac{d}{dt}\langle\langle v(t)x(0)\rangle\rangle = -\omega^2\langle\langle x(t)x(0)\rangle\rangle - \gamma\langle\langle v(t)x(0)\rangle\rangle + \langle\langle \xi(t)x(0)\rangle\rangle, \tag{27}$$

$$\frac{d}{dt}\langle\langle \xi(t)x(0)\rangle\rangle = -\lambda\langle\langle \xi(t)x(0)\rangle\rangle,$$

and along the same line of calculation, the velocity-velocity correlation functions can be obtained from the following sets of coupled equations:

$$\frac{d}{dt}\langle\langle v(t)v(0)\rangle\rangle = -\omega^2\langle\langle x(t)v(0)\rangle\rangle - \lambda\langle\langle v(t)v(0)\rangle\rangle + \langle\langle \xi(t)v(0)\rangle\rangle,$$

$$\frac{d}{dt}\langle\langle x(t)v(0)\rangle\rangle = \langle\langle v(t)v(0)\rangle\rangle, \tag{28}$$

$$\frac{d}{dt}\langle\langle \xi(t)v(0)\rangle\rangle = -\lambda\langle\langle \xi(t)v(0)\rangle\rangle.$$

In Figs. 1 and 2 we study time development of position-position and velocity-velocity two-time correlation functions for various ω with the initial conditions $\langle\langle x^2(0)\rangle\rangle=k_BT/2\omega^2$, $\langle\langle p^2(0)\rangle\rangle=k_BT/2$, and $\langle\langle x(0)v(0)\rangle\rangle=0$, where classical equipartition theorem has been invoked and $\langle\langle \xi(t)x(0)\rangle\rangle=0 =\langle\langle \xi(t)v(0)\rangle\rangle$. In both cases the two-time correlation functions go to zero after passing through a minimum.

IV. KUBO OSCILLATOR AND STOCHASTIC RESONANCE

We now consider the bath to be driven by an oscillatory periodic force instead of external random noise,

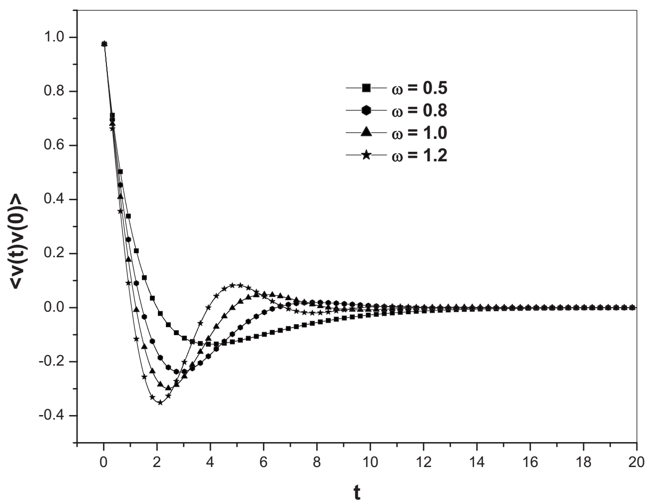

FIG. 2. Variation of velocity-velocity correlation function, $\langle\langle v(0)v(t)\rangle\rangle$, as a function of t for various $\omega[\gamma=1, \quad \lambda=10, \quad \sigma=0.1]$.

$$H_{\text{int}} = \sum_j \kappa_j x_j \sin \Omega t,$$

and obtain the equation of motion as

$$\dot{x} = v, \qquad \dot{v} = -v\gamma - V'(x) + f(t) + c_0\kappa_0 \sin(\omega t). \tag{29}$$

At this point to study the interplay between multiplicative and additive noise in the phenomena of SR, let us consider our system to be a Kubo oscillator. Since its introduction by Anderson[23] and Kubo[24] in the context of the line-shape theory, stochastic approaches of which have found wide application in condensed-phase spectroscopy,[23,24] ranging from magnetic resonance spectroscopy, nonlinear spectroscopy,[25] to single molecule spectroscopy.[26] The stochastic line-shape theory of Kubo and Anderson[26] is based on the equation of motion for the transition dipole,[23,24]

$$\frac{d\mu(t)}{dt} = i\omega(t)\mu(t),$$

where $\omega(t)$ is the stochastic frequency of the oscillator. The Kubo and Anderson approach is based on the Markovian assumption. However, single molecule studies[27,28] revealed that in some cases the underlying dynamics in the condensed-phase is highly non-Markovian in nature.[29,30]

In the present development, our system is a forced, underdamped linear oscillator with random frequency,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + [\omega^2 + \eta(t)]x = f(t) + c_0\kappa_0 \sin(\Omega t), \tag{30}$$

where the random force $\eta(t)$ is uncorrelated with the internal noise $f(t)$ and has the following statistical properties:

$$\langle \eta(t)\rangle = 0; \qquad \langle \eta(t)\eta(t')\rangle = \sigma \exp[-\lambda|t-t'|]. \tag{31}$$

One can easily obtain, by using the Shapiro–Loginov theorem, as is done in the earlier case, that the average of position $\langle\langle x(t)\rangle\rangle$ satisfies the following fourth order differential

$$\left\{\frac{d^4}{dt^4} + 2(\lambda+\gamma)\frac{d^3}{dt^3} + (2\omega^2+\lambda^2+3\lambda\gamma+\gamma^2)\frac{d^2}{dt^2} + [2\omega^2(\lambda+\gamma)+\gamma(\lambda+\gamma)\lambda]\frac{d}{dt} + [\omega^2(\omega^2+\lambda^2+\lambda\gamma)-\sigma]\right\}\langle\langle x(t)\rangle\rangle = (\omega^2+\lambda^2+\lambda\gamma-\Omega^2)c_0\kappa_0\cos(\Omega t). \tag{32}$$

We seek a solution of the form

$$\langle\langle x(t)\rangle\rangle = \langle\langle x\rangle\rangle_0 + \langle\langle x\rangle\rangle_\Omega, \tag{33}$$

where the output signal $\langle\langle x\rangle\rangle_\Omega$ is induced by the external field $\sin(\Omega t)$ and $\langle\langle x\rangle\rangle_0$ is defined by the internal dynamics. For the purpose of discussion, we ignore the transient terms for fast fluctuations. Let us write the solution $\langle\langle x\rangle\rangle_\Omega$ in the form

$$\langle\langle x\rangle\rangle_\Omega = A\sin(\Omega t + \varphi). \tag{34}$$

Then one can easily find that

$$A = \left[\frac{\alpha_1^2+\alpha_2^2}{\alpha_3^2+\alpha_4^2}\right]^{1/2} \tag{35}$$

and

$$\varphi = \tan^{-1}\left[\frac{\alpha_1\alpha_3+\alpha_2\alpha_4}{\alpha_1\alpha_4-\alpha_2\alpha_3}\right], \tag{36}$$

where $c_0^2=\gamma$ and

$$\begin{aligned}\alpha_1 &= (2\lambda+\gamma)c_0\kappa_0\Omega,\\ \alpha_2 &= (\Omega^2-\omega^2-\lambda^2-\lambda\gamma)c_0\kappa_0,\\ \alpha_3 &= (\Omega^2-\omega^2)(\Omega^2-\omega^2-\lambda^2)-\sigma-(3\lambda\gamma+\gamma^2)+\lambda\gamma\omega^2,\\ \alpha_4 &= \Omega(\lambda+\gamma)[2(\Omega^2-\omega^2)+\gamma\lambda].\end{aligned} \tag{37}$$

Prior to further analysis, let us consider the limiting case of white Gaussian noise, which, according to Eq. (31), corresponds to $\sigma\to\infty$ and $\lambda\to\infty$, with a constant ratio σ/λ. Then, in the absence of friction, Eq. (30) reduces to

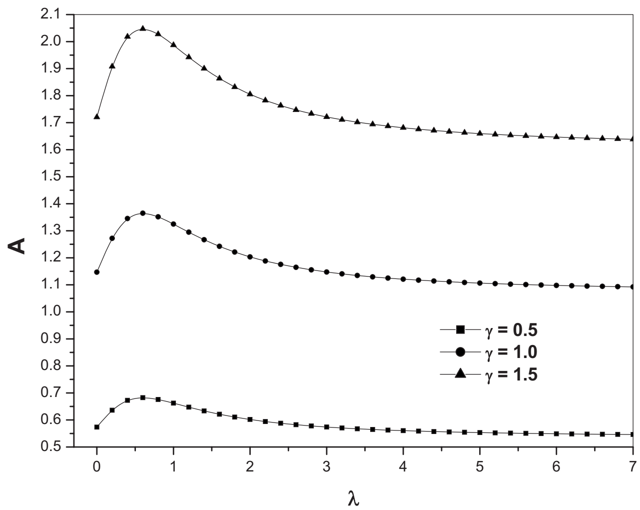

FIG. 3. Plot of A as a function of λ for various γ (all other parameters are set to be unity).

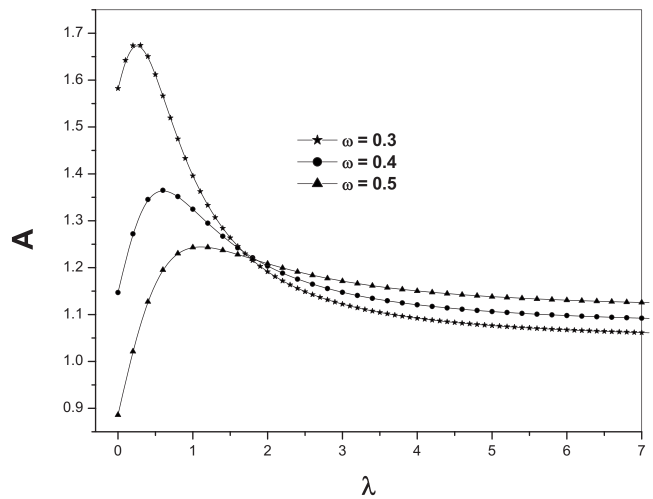

FIG. 4. Plot of A as a function of λ for various ω (all other parameters are set to be unity).

$$\frac{d^2x}{dt^2} + \omega^2 x + \eta(t)x = f(t),$$

as $\gamma = c_0^2 = 0$. Consequently, $\langle\langle x(t)\rangle\rangle$ satisfies the equation of motion as that of simple harmonic oscillator. Interestingly, friction arises due to system-bath interaction and in the absence of internal friction, the system does not feel the external periodic forcing. On the other hand, in presence of internal damping, the average of $x(t)$ satisfies the equation

$$\frac{d^2\langle\langle x\rangle\rangle}{dt^2} + \omega^2\langle\langle x\rangle\rangle + \gamma\frac{d\langle\langle x\rangle\rangle}{dt} = c_0\kappa_0 \sin(\Omega t),$$

which is valid for δ-correlated white noise. The above equation is the equation of motion for a damped forced harmonic oscillator from which one can easily evaluate the time development of mean position. Complications arise when the multiplicative noise ceases to be white δ-correlated. For colored noise, exponentially correlated noise in our case the output signal shows nonmonotonic dependence on the noise strength σ and the correlation rate λ, which is the signature of SR. In Fig. 3, we show the dependence of the amplitude ($A = c_0\kappa_0$) of the output signal on λ for a fixed frequency ($\Omega = 0.4$) of the output signal with various γ. Indeed the amplitude of the output signal A reaches a maximum, which clearly demonstrates the nonmonotonic dependence of A on λ, which is the signature of SR. In Fig. 4, we plot the amplitude of output signal A as a function of λ for different values of frequency, Ω of the output signal, and also observe SR phenomena. It is pertinent to mention here that the system (the Kubo oscillator) is under the action of two forces, one is random force $f(t)$ and the other is periodic driving of the heat bath. This periodic driving is very interestingly manifested through the resonance behavior of the amplitude of the output signal. This resonance behavior of output amplitude would be present if, instead of the bath, the system would have been driven directly by external periodic forcing.

The observation of SR closest to ours is that by Gitterman.[31] Incidentally, Gitterman[31] studied the SR phenomena where he observed that the output signal A of an underdamped linear oscillator with a random frequency subjected to a periodic force, which shows nonmonotonic dependence on the strength of the rate of underlying colored noise. Thus, it is instructive to present a comparison of the output signal A with the variation of λ with that of observations made by Gitterman using same values of various parameters as those used in our numerical implementation. We numerically observed that the SR phenomena of Gitterman is identical to ours using same parameter sets as that used in this paper. Actually, the results of the A versus λ for Gitterman and ours are superimposed on one another over the entire range of λ (see Fig. 5). This is due to the fact that the phenomenological equation of Gitterman, Eq. (1) in Ref. 31, and our microscopically originated equation, Eq. (30), possess the same structure after averaging over internal noise, $f(t)$. Here, it is important to recall the fact that the internal noise $f(t)$ emerges due to the system-reservoir coupling. This internal noise is absent in the work of Gitterman.

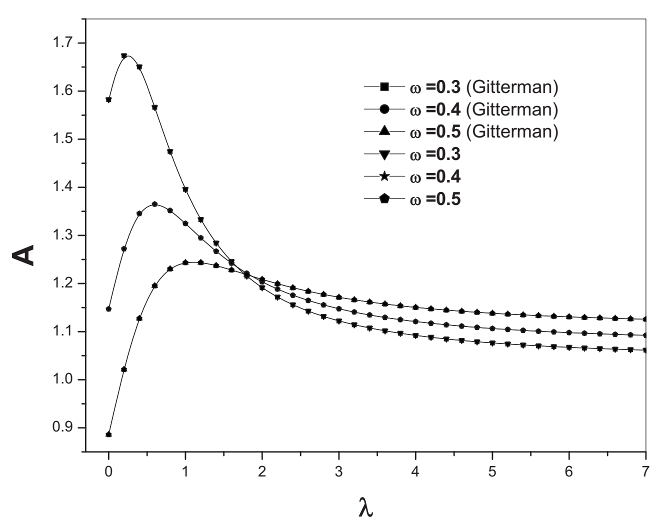

FIG. 5. Plot of A as a function of λ for various ω along with Gitterman's model.

V. SUMMARIZING REMARKS

In this paper we considered a microscopic Hamiltonian picture where the system is coupled with the environment, which is modeled by a set of harmonic oscillators. The reservoir is not kept in thermal equilibrium, instead it is perturbed by an external force, either rapidly fluctuating or periodically oscillating. We then obtain the Langevin equation with proper microscopic specification of Langevin force. Then, for the case of external fluctuating force, we consider that the system is a simple harmonic oscillator and using the Shapiro–Loginov procedure we arrive at the linear coupled first order differential equations for two-time correlations and examine the time evolution of the same. Next, we consider a Kubo-type oscillator as our system and the bath modulating force to be periodic and study the SR phenomena. This reveals the important observation that SR can be exhibited by a linear system under appropriate condition and the Kubo oscillator coupled with a bath, which is periodically modulated by external driving, meets the requirements. We plan to explore investigations of the corresponding quantum approach in the near future.

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