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Physics Letters B 557 (2003) 303–308
PHYSICS LETTERS B
www.elsevier.com/locate/npe

Half-monopoles and half-vortices in the Yang–Mills theory

E. Harikumar, Indrajit Mitra, H.S. Sharatchandra

The Institute of Mathematical Sciences, C.I.T. Campus, Taramani P.O., Chennai 600 113, India

Received 16 January 2003; accepted 3 February 2003

Editor: M. Cvetič

Abstract

It is demonstrated that there are smooth Yang–Mills potentials which correspond to monopoles and vortices of one-half winding number. They are the generic configurations, in contrast to the integral winding number configurations like the 't Hooft–Polyakov monopole.

PACS: 14.80.Hv; 11.15.-q; 11.15.Tk

Keywords: Monopole; Poincaré–Hopf index; One-half winding number

In this Letter, we demonstrate Yang–Mills field configurations of monopoles and vortices with half the usual charges. We show that these are the generic field configurations, in contrast to the integral winding number configurations such as the 't Hooft–Polyakov monopole. In Refs. [1,2], the monopole configuration was related to the singularities of the eigenvector fields of the real symmetric matrix

$$S_{ij}(x) = B_i^a(x) B_j^a(x), \tag{1}$$

where $B_i^a = \epsilon_{ijk}(\partial_j A_k^a - \frac{1}{2}\epsilon^{abc} A_j^b A_k^c)$ is the $SO(3)$ magnetic field. Such singularities arise due to indeterminacy of the directions of the eigenvectors, and so it is crucial that the eigenvalues become degenerate at the points of singularity [3]. The topology of the configuration can be traced to these singularities. We refer to these points of singularities as the 'centres' of the topological configurations. For the 't Hooft–Polyakov monopole [4,5],

$$S_{ij} = \alpha(r^2)\delta_{ij} + \beta(r^2)x_i x_j \tag{2}$$

where α and β are functions of the distance r from the origin only. One of the eigenfunctions is the radial vector x_i, with unit winding number. This has indeterminate direction at the origin $r = 0$. But there is no contradiction because $S \propto I$ (the identity matrix) at the origin, and any vector is an eigenvector.

Note that the entries of the matrix S are smooth in the coordinates x_i at the origin. Singularities arise in spite of this, due to the eigenvalue equation.

The 't Hooft–Polyakov monopole has some exceptional features which are not generic. The first of these is that two eigenvalues are degenerate everywhere. Secondly, the entries of the matrix S are quadratic in the coordinates. Thus, in the Taylor series expansion of $S_{ij}(x)$ about the origin, linear terms are missing.

E-mail addresses: hari@imsc.res.in (E. Harikumar), indrajit@imsc.res.in (I. Mitra), sharat@imsc.res.in (H.S. Sharatchandra).

0370-2693/03/$ – see front matter
doi:10.1016/S0370-2693(03)00203-X

Both these features are a consequence of the rotational invariance of the 't Hooft–Polyakov monopole. (This rotational invariance is under simultaneous and equal rotations in physical and isospin spaces.)

In this Letter, we analyse the generic case, i.e., *we consider $S_{ij}(x)$ with linear terms in the Taylor expansion about the origin.* We find the novel feature of half-integral winding number configurations and obtain the interpretation of such configurations.

As we are interested in the eigenfunctions, we may appropriately subtract a multiple of the identity matrix from S_{ij}. Also an overall scale is irrelevant. *We will refer to the matrix after these changes as T_{ij}.*

We first illustrate the possibility and meaning of configurations with half a winding number using a 2×2 real symmetric matrix field $T_{ij}(x, y)$. The paradigm is provided by the matrix

$$T = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}. \tag{3}$$

The eigenvalues are $\lambda_\pm = \pm r$, where $r = \sqrt{x^2 + y^2}$. We denote the corresponding eigenfunctions by $\zeta_i^\pm$. The eigenfunction $\binom{\zeta_1^+}{\zeta_2^+}$ has $\zeta_1^+/\zeta_2^+ = y/(r - x)$. Thus the normalised eigenfunction has the simple form

$$\begin{pmatrix} \zeta_1^+ \\ \zeta_2^+ \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \tag{4}$$

in the polar coordinates. Here $\theta = \tan^{-1}(y/x)$.

The occurrence of half the polar angle in (4) is significant. If we go round the origin once, the eigenvector changes the sign. It is not possible to define the vector field $\zeta_i^+(x)$ continuously everywhere. There is necessarily a discontinuity (change of sign) across a "branch cut" starting from the origin. The choice of this branch cut is arbitrary, except that it starts at the origin. If we consider the complex vector $\zeta_1^+ + i\zeta_2^+ = \exp(i\theta/2)$, the phase changes by π when we go around the origin once. In this sense, the winding number is half. We call this configuration a half-vortex. It can be checked that such a phase change takes place for the other eigenvector $\zeta_i^-(x)$ too (Fig. 1).

We emphasise that the entries of the matrix T_{ij} are smooth even at the origin. In spite of this, the eigenvalue equation gave a discontinuous eigenvector field.

It is easy to see that only half-integral winding number is possible in this case. The eigenvector

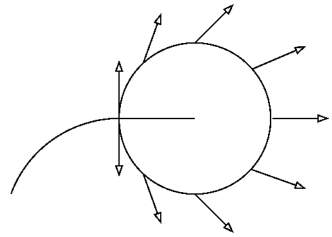


Fig. 1. A winding number half configuration: $\zeta_i^\pm$ changes sign when taken around any closed path enclosing the centre. The curved line represents the (arbitrary) line of discontinuity.

of a real symmetric matrix is real and hence a non-degenerate eigenvector, after normalisation, is ambiguous only up to a sign. Therefore, when taken continuously around a closed path, the only possible change in the eigenvector on return to the initial point is by an overall sign. This indeed happens in the present case.

We now argue that this describes the situation in the generic case too. The most general 2×2 real symmetric linear in the coordinates is

$$T = \begin{pmatrix} ax + by & cx + dy \\ cx + dy & ex + fy \end{pmatrix}. \tag{5}$$

For the eigenvalue problem, we can subtract a multiple of identity matrix from T given above. Subtracting $\frac{1}{2}((a + e)x + (b + f)y)I$, we get a symmetric matrix. We now choose the oblique system of coordinates

$$2x' = (ax + by) - (ex + fy), \qquad y' = cx + dy. \tag{6}$$

(In the generic case these are linearly independent and a valid choice of new coordinates.) With this we are back to the paradigm considered in (3).

Therefore, on considering the Taylor series expansion of the entries T_{ij} about the point of degeneracy, say $x = 0$, $y = 0$, it is clear that so long as the terms linear in x and y are present, we get the phenomenon of one-half winding number described above.

The situation will be totally different in the case where the entries are quadratic in the coordinates. The simplest example is the one analogous to the case of 't Hooft–Polyakov monopole:

$$T_{ij} = x_i x_j. \tag{7}$$

Now the eigenvectors are

$$\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

For both of these eigenvectors, the winding number around the origin is one, and the vector fields can be defined continuously everywhere (except for the singularity at the origin).

It is interesting to consider the case where

$$T=\begin{pmatrix} x^2-a^2 & xy \\ xy & y^2 \end{pmatrix}. \tag{8}$$

Now the double degeneracy is at two points, viz. $x_0=\pm a$, $y_0=0$. Around each point the Taylor series expansion has the form

$$T=\begin{pmatrix} 2X & Y \\ Y & 0 \end{pmatrix}+\text{higher order terms in } X,\ Y, \tag{9}$$

where $X=x_0(x-x_0)$ and $Y=x_0(y-y_0)$. The leading term has precisely the form of the paradigm we considered (up to a multiple of the identity matrix). So we get half a winding number around each point of degeneracy. We may conveniently choose the line joining the two centres as the branch cut. The winding number along a curve enclosing both centres is one. Indeed, as $a\to 0$, we recover from (8) the winding number one configuration considered in (7). In this limit of $a\to 0$, the pair of half winding number configurations merge together to give winding number one configuration (Fig. 2).

Let us also consider the matrix

$$T=\begin{pmatrix} x^2-a^2 & ay \\ ay & 0 \end{pmatrix}. \tag{10}$$

This again has the same two points of degeneracy as the matrix in (8). However, in the present case the winding numbers are $\pm 1/2$, respectively. (The configuration around $(-a,0)$ is related to our paradigm in (9) by reflection about the X-axis: $Y\to -Y$. So it has the winding number $-1/2$.) In the limit $a\to 0$, the eigenvectors are now $\binom{1}{0}$ and $\binom{0}{1}$, and each of them has a vanishing winding number.

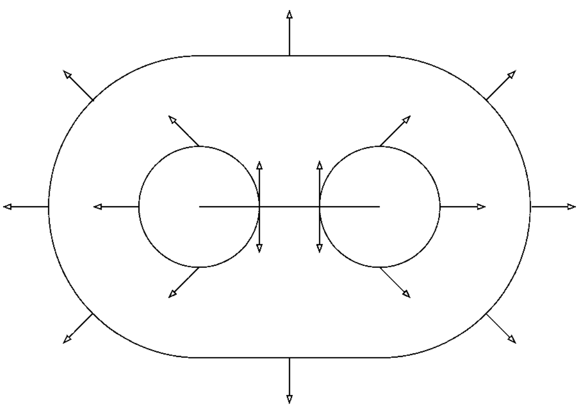


Fig. 2. Two winding number half configurations give a winding number one configuration at large distances.

We have considered 2×2 matrices though the S_{ij} relevant for the Yang–Mills theory are 3×3 matrices. We regard the matrix in (3) as a block of the 3×3 matrix

$$T=\begin{pmatrix} x & y & 0 \\ y & -x & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Then the interpretation is that we have a vortex with one-half winding number centred on the z-axis and extending indefinitely along it. To justify this interpretation, we have to exhibit an Yang–Mills potential which will give rise to T_{ij} as considered in (11). It has been shown in Ref. [6] that in the generic situation where the 3×3 matrix B_i^a is invertible and smooth, there exists a smooth A_i^a which will reproduce such a B_i^a. So for the case here, A_i^a can be constructed as a Taylor series expansion about the origin. We will present such a series for a different example below. We will also discuss the finiteness of energy (per unit length) there.

We now show that monopoles of one-half winding number also occur. The paradigm in this case is provided by the 3×3 real symmetric matrix

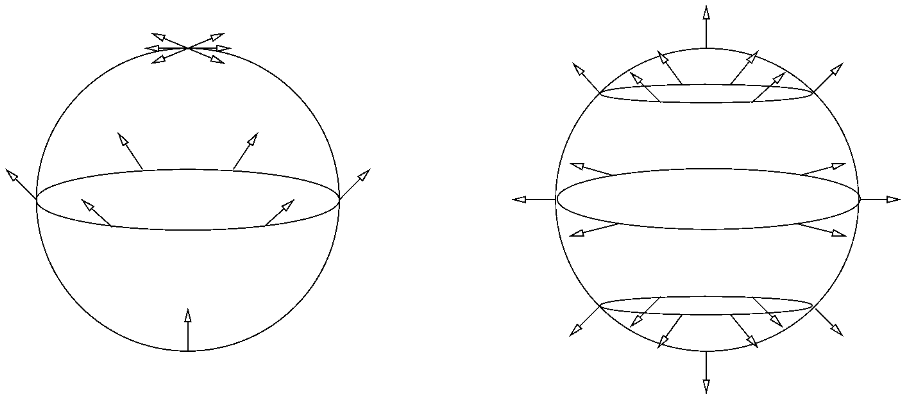
$$T=\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & -2z \end{pmatrix}. \tag{12}$$

Here the eigenvalues are $\lambda_\pm=-r(\cos\theta\mp 1)$ and $\lambda_0=0$. In the spherical coordinates, the corresponding eigenfunctions are

$$\zeta^+=\begin{pmatrix} \cos\frac{\theta}{2}\cos\phi \\ \cos\frac{\theta}{2}\sin\phi \\ \sin\frac{\theta}{2} \end{pmatrix},\qquad \zeta^-=\begin{pmatrix} \sin\frac{\theta}{2}\cos\phi \\ \sin\frac{\theta}{2}\sin\phi \\ -\cos\frac{\theta}{2} \end{pmatrix},$$

$$\zeta^0=\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{13}$$

Comparing $\zeta_i^\pm$ with the radial vector and with the normalised Higgs in the 't Hooft–Polyakov monopole,



(a) (b)

Fig. 3. (a) A winding number half configuration in three dimensions. There is a vortex of winding number one along the positive z-axis terminating at the centre. (b) A winding number one configuration in three dimensions. The upper half of this configuration is mapped onto the entire sphere in (a) to give one-half winding number.

viz. $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we notice that essentially the angle θ is replaced by $\theta/2$. This leads to one-half winding number in the present case. This phenomenon is illustrated for ζ_i^+ in Fig. 3. In effect, the configuration in the upper half of the sphere for winding number one is mapped onto the entire sphere to give one-half winding number. We refer to such a configuration as a half-monopole. Note that the vector field ζ_i^+ is singular (indeterminate in direction) all along the positive z-axis. This is possible because T_{ij} has a double degeneracy there. This has the interpretation of a vortex (of winding number one) along the positive z-axis terminating at the origin and giving rise to a monopole. Because of this vortex, the vector field is not continuous on the sphere, and therefore one-half winding number is possible. If the vector field were smooth on the sphere, the winding number would have been only integral.

In an analogous way, ζ_i^- corresponds to a vortex of unit winding number along the negative z-axis, terminating at the origin. Finally ζ_i^0 is again a vortex of winding number one extending indefinitely along the z-direction.

That the monopole centre (point of triple degeneracy) is a terminating point of vortex centre (line of double degeneracy) is a generic situation. In fact, the generic situation is as follows. The configuration ζ_i^A, for each A, will have double degeneracy along two lines terminating at the centre. Each such line will be the centre of a vortex of winding number half. This will be elaborated elsewhere. It may also be noted that for the 't Hooft–Polyakov monopole, which is not generic due to rotational invariance, we have double degeneracy everywhere.

If we formally compute the Poincaré–Hopf index of the vector field ζ_i^+, we get it to be $-1/2$. The index for ζ_i^A is given by $M = \oint_S dS^i\, k_i^A$, where the integration is over a surface S enclosing the centre and k_i^A is the Poincaré–Hopf current [2,5,7]

$$k_i^A = \frac{1}{2}\epsilon_{ijk}\epsilon_{lmn}\zeta_l^A\partial_j\zeta_m^A\partial_k\zeta_n^A \quad \text{(no sum over } A). \tag{14}$$

We have in the present case (see Eq. (13))

$$k_i^+ = -\hat{x}_i\frac{1}{4r^2}\operatorname{cosec}\frac{\theta}{2}. \tag{15}$$

The vector field ζ_i^+ is not smooth at the north pole of the sphere. Therefore, the definition of the index M is only formal. Nevertheless, this singularity is of zero measure in the integration over S and we get the winding number to be $-1/2$. Note that the "magnetic field" k_i^+ of this half-monopole is not spherically symmetric, in contrast to the case of the Dirac monopole. It has only an axial symmetry.

In Ref. [2], it was shown that the Poincaré–Hopf current for the eigenvector ζ_i^A can be expressed as the curl of an Abelian vector potential ω_i^A:

$$k_i^A = \epsilon_{ijk}\partial_j\omega_k^A - \text{Dirac string contributions,} \tag{16}$$

where

$$\omega_i^A = \frac{1}{2}\epsilon^{ABC}\zeta_j^B\partial_i\zeta_j^C. \tag{17}$$

Here the indices A, B and C, having the values 1, 2 and 3, label the three eigenvectors. The Abelian vector potential corresponding to ζ_i^+ is

$$w_i^+ = -\hat{\phi}_i\frac{1}{2r}\sec\frac{\theta}{2}. \tag{18}$$

This potential has the Dirac string along the negative z-axis. This Dirac string is unphysical, in the sense that it does not contribute to the "magnetic field" k_i^A (see Eq. (16)). In contrast, the vortex line along the positive z-axis is physical, and, because of it, the monopole does not have spherical symmetry.

Similarly, we get the Poincaré–Hopf index for ζ_i^- as $-1/2$. In the case of ζ_i^0, notice that it spans a two-dimensional vector space as we vary ϕ. Therefore the index computed over S will be zero (three-dimensional winding number is zero). On the other hand, it makes sense to calculate the index over a two-dimensional surface. For any such surface not containing the z-axis, we get winding number one.

We now present the Taylor series expansion of A_i^a about the origin which leads to T_{ij} considered in (12). Consider first the matrix $(B)_{ia} = B_i^a$. In the symmetric gauge $(B)_{ia} = (B)_{ai}$ [2], we have $(B^2)_{ij} = S_{ij}$, so that, for the case given in (12),

$$B = I + \frac{1}{2}\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & -2z \end{pmatrix} + \cdots. \tag{19}$$

Here the ellipsis indicates terms of higher order in the coordinates. The most general Taylor expansion of A_i^a about the origin is:

$$A_i^a = a_{ai} + b_{aij}x_j + c_{aijk}x_jx_k + \cdots. \tag{20}$$

To obtain B_i^a as given in (19), it suffices to take

$$a_{ai} = 0, \qquad b_{aij} = -\frac{1}{2}\epsilon_{aij}, \tag{21}$$

$$c_{aijk} = \frac{1}{2}\left(\epsilon_{ijp}M_{pk}^a + \epsilon_{ikp}M_{pj}^a\right), \tag{22}$$

where

$$M_{31}^1 = -\frac{1}{6}, \qquad M_{32}^2 = -\frac{1}{6}, \qquad M_{33}^3 = \frac{1}{2}, \tag{23}$$

and all other M_{ij}^a are zero. Thus our solution for the gauge field is

$$A = \frac{1}{2}\begin{pmatrix} -xy/3 & z - y^2/3 & -y + yz \\ -z + x^2/3 & xy/3 & x - xz \\ y & -x & 0 \end{pmatrix} + \cdots, \tag{24}$$

where $(A)_{ia} = A_i^a$.

We now address the question of finiteness of the energy of the half-monopole, given by $E = \int d^3x\, S_{ii}/2$. As S_{ij} can be expanded in Taylor series about the origin, the energy is finite in the ultraviolet. Also the infrared finiteness of the energy resides in the scale factors of S_{ij}, such as $\alpha(r^2)$ and $\beta(r^2)$ in Eq. (2), and these can be chosen appropriately to get a finite energy. Note that the one-half winding number is due to the tensorial structure of S_{ij}, the eigenvectors ζ_i^A being unaffected by the scale factors.

In both two and three dimensions, we have seen that the phenomenon of one-half winding number is due to the generic linear terms in the Taylor expansion of S_{ij}. Nevertheless, there are crucial differences in the origin of this phenomenon in the two cases. In two dimensions, the ambiguity in the sign of the eigenvector was the underlying reason. The line of discontinuity (the "branch cut") was arbitrary, except for the starting point. In three dimensions, lines of double degeneracy terminating at the centre of the monopole were necessary to give the necessary discontinuity in the form of a vortex. But these lines of double degeneracy are rigid, in contrast to the branch cuts in two dimensions.

To conclude, we have pointed out in this Letter that vortex and monopole configurations of one-half winding number are present in the Yang–Mills theory. They arise from smooth Yang–Mills potentials, and are indeed the generic configurations in contrast to the 't Hooft–Polyakov monopole.

Note added in proof

Some of the works which discuss vortices of half-integer winding number in other contexts are given in Ref. [8].

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