Generation of Magneto-Hydrodynamic Waves by Moving Pressure Distribution*

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Two-dimensional magneto-hydrodynamic surface waves generated by a moving oscillatory pressure distribution in an inviscid, incompressible and electrically conducting fluid of finite uniform depth are investigated. In the ultimate steady state, there exist six progressive waves, four original gravity waves and two other magnetic waves, all of which are mainly due to magnetic force. The effect of this force on the waves produced is studied in detail. It is found that, due to this force, disturbance in the wake is increased.

Key Words: Linear Waves, Asymptotic Solution, Gravity Waves, Magnetic Waves, Singularity of Linear Solution, Effect of Magnetic Force on Gravity Waves

1. Introduction

Debnath[1] has investigated the generation and propagation of two-dimensional magneto-hydrodynamic surface waves due to oscillatory pressure distribution acting on the surface of an inviscid electrically conducting fluid of finite depth at rest. The corresponding problem in a running stream appears to remain uninvestigated though there are a number of such papers for nonconducting fluids. The gravity wave problem generated by an oscillatory pressure distribution in a running stream has been investigated by Debnath and Rosenblat[2] and the problem of capillary-gravity waves generated by an oscillatory moving pressure distribution has been investigated by Pramanik[3]. The aim of the present study is to investigate the unsteady waves generated by a moving oscillatory pressure distribution in a conducting fluid of finite uniform depth. In this problem, the asymptotic waves are determined by asymptotic estimation of the integral in the exact Fourier transform solution and these asymptotic waves arise from the residue contribution at the poles of the integrands.

The main difficulty in these asymptotic estimations lies in finding the poles of the integrands which are, in turn, the roots of the frequency equation when only the gravitational force is present. The difficulty is not as acute as when surface tension or magnetic force is taken into account. However, the roots are determined by a special geometrical method as used in Ref.(3).

There are two dimensionless parameters among the parameters of the problem in Ref.(3) which characterise the wave pattern. The roots of the frequency equation are determined by considering the case of repeated roots, which is called the critical condition of the problem. This critical condition can be expressed by some curve in the plane of the parameters and the roots for all cases are determined by a special geometrical consideration. In the present case, there appear three such parameters, a, b, c, given by

$$a=\omega\left(\frac{h}{g}\right)^{1/2},\quad b=\frac{V}{(gh)^{1/2}},\quad c=\frac{a_1^2}{gh},$$

where V and ω are the uniform velocity and the frequency of the pressure distribution, respectively, h is the finite depth of the fluid and $a_1=B_0/(4\pi\rho)^{1/2}$ where B_0 is the magnetic field component along the free surface of the fluid. It is found that the case of repeated roots is a surface in the space of parameters a, b, c. However, it is noted that this surface has more or less a cylindrical shape in the c-direction; thus, a section made by a plane c=constant has the same form for all possible values of c. Consideration

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is therefore focused on the plane of parameters a and b for a certain fixed value of c. It is found for $c=2$ that the critical curve divides the entire positive quadrant of the plane into seven distinct regions. The roots in each region are found and the ultimate waves in each region are ascertained. It is also found that for any point on the critical curve, the behaviours of the waves are singular. It is like $t^{1/2}$ for a large value of t for points where two roots coincide, while it is like $t^{2/3}$ for a large value of t where three roots coincide. The waves have a constant amplitude away from the critical curves.

In the ultimate steady-state condition, there are six progressive waves: four original gravity waves and the rest effect of conduction. It is found that the gravity waves are not influenced by the magnetic force except for a change in the phase and amplitude. The magnetic force in conjunction with gravitational force creates two new waves, one propagating in the upstream side and the other, in the downstream side. For various values of the parameters, the group velocities of the waves are calculated and some physically interesting conclusions are made.

2. Formulation and Formal Solution

We take the x-axis along the free surface and the y-axis vertically upwards. We consider an unbounded $(-\infty<x<\infty)$ region of an inviscid, incompressible, electrically conducting, field-free fluid of uniform density ρ occupying the space $-h\leq y\leq 0$ to which a constant magnetic field $\boldsymbol{B}_0=(B_0, 0)$ is applied for $y>0$. Both the magnetic permeability and the dielectric constant are the same as those in the free space and the fixed boundary is free from any field. For the system being initially at rest, waves are produced by the continuous application of the pressure distribution $p(x, t)=f(x)e^{i\omega t}$ which at the same time moves along the positive direction of the x-axis with uniform velocity V.

Let $\varphi(x, y;t)$ and $\psi(x, y;t)$ be the velocity potential and the magnetic potential, respectively, and $\eta(x;t)$ be the surface elevation. Then, in the coordinate system that moves with the velocity V, we have the following linearised initial-value problem in Ref.(4):

$$\nabla^2\varphi=0, \quad \text{in } (-\infty<x<\infty,\ -h\leq y\leq 0,\ t\geq 0)$$

$$\nabla^2\psi=0, \quad \text{in } (-\infty<x<\infty,\ y\geq 0,\ t\geq 0)$$

$$\left.\begin{aligned} &\frac{p}{\rho}+\frac{\partial\varphi}{\partial t}-V\frac{\partial\varphi}{\partial x}+g\eta+\frac{B_0}{4\pi\rho}\frac{\partial\psi}{\partial x}=0\\ &\frac{\partial\eta}{\partial t}-V\frac{\partial\eta}{\partial x}=\frac{\partial\varphi}{\partial y}\\ &\frac{\partial\psi}{\partial y}=B_0\frac{\partial\eta}{\partial x}\end{aligned}\right\} \quad \text{at } y=0$$

$$\frac{\partial\varphi}{\partial y}=0 \quad \text{at } y=-h$$

$$|\nabla\psi|=0 \quad \text{as } y\to\infty$$

and $\varphi(x, y;0)=0, \quad \eta(x;0)=0.$

The formal solution of the problem is obtained by Fourier-transforming the above system of equations with respect to x and then using the Fourier inversion formula. The following integral representation for η with dimensionless variables can be easily obtained:

$$\eta(x;t)=\frac{1}{2\rho gh\cdot(2\pi)^{1/2}}\sum_{m=1}^{8}I_m, \tag{1}$$

where

$$I_1=\int_0^\infty F_1(\lambda)e^{i[\lambda x+(\sigma+b\lambda)t]}d\lambda,$$

$$I_2=\int_0^\infty F_2(\lambda)e^{-i[\lambda x-(\sigma-b\lambda)t]}d\lambda,$$

$$I_3=\int_0^\infty F_3(\lambda)e^{i[\lambda x-(\sigma-b\lambda)t]}d\lambda,$$

$$I_4=\int_0^\infty F_4(\lambda)e^{-i[\lambda x+(\sigma+b\lambda)t]}d\lambda,$$

$$I_5=-\int_0^\infty F_3(\lambda)e^{i[\lambda x+at]}d\lambda,$$

$$I_6=-\int_0^\infty F_4(\lambda)e^{-i[\lambda x-at]}d\lambda,$$

$$I_7=-\int_0^\infty F_1(\lambda)e^{i[\lambda x+at]}d\lambda,$$

$$I_8=-\int_0^\infty F_2(\lambda)e^{-i[\lambda x-at]}d\lambda,$$

and

$$F_1(\lambda)=\frac{q(\lambda)\cdot\lambda\tanh\lambda}{\sigma(\sigma+b\lambda-a)},$$

$$F_2(\lambda)=\frac{q(-\lambda)\cdot\lambda\tanh\lambda}{\sigma(\sigma-b\lambda-a)},$$

$$F_3(\lambda)=\frac{q(\lambda)\cdot\lambda\tanh\lambda}{\sigma(\sigma-b\lambda+a)},$$

$$F_4(\lambda)=\frac{q(-\lambda)\cdot\lambda\tanh\lambda}{\sigma(\sigma+b\lambda+a)}$$

$$\sigma^2=(\lambda+c\lambda^2)\tanh\lambda,$$

$$q(\lambda)=\frac{h}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}f(hx)e^{-i\lambda x}dx$$

$$b=\frac{V}{(gh)^{1/2}}, \quad a=\omega\left(\frac{h}{g}\right)^{1/2},$$

$$c=\frac{a_1^2}{gh}, \quad a_1=\frac{B_0}{(4\pi\rho)^{1/2}}.$$

3. Steady-State Waves

We shall determine the asymptotic values of the integrals in Eq.(1) for long times at a large distance from the pressure segment. These asymptotic values appear as contributions in the form of residue at the real poles of the integrands. These poles are solutions of the following three equations called the frequency equations:

$$\sigma-b\lambda+a=0 \tag{3.a}$$

$$\sigma-b\lambda-a=0 \tag{3.b}$$

$$\sigma+b\lambda-a=0. \tag{3.c}$$

We shall determine the roots for given values of parameters a, b and c. The roots will be determined as points of the intersection of curve $m=\sigma$ and

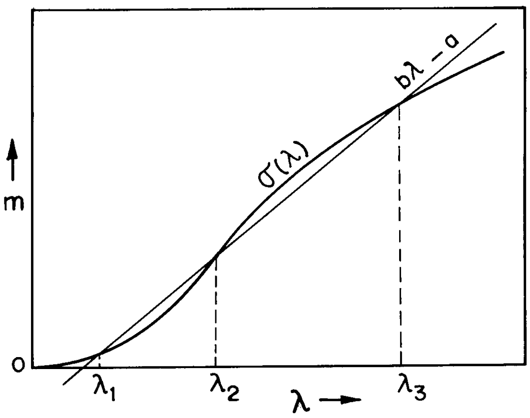

Fig. 1 Roots of (3.a)

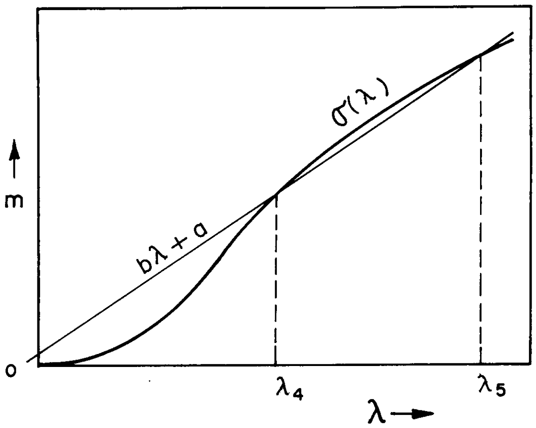

Fig. 2 Roots of (3.b)

straight lines $m=b\lambda-a$, $m=b\lambda+a$ and $m=-b\lambda+a$. It is easy to see that for all values of c, the curve $m=\sigma$ has a point of inflexion at some point $\lambda=\lambda_0>0$, and that $\frac{d^2\sigma}{d\lambda^2}$ changes its sign from positive to negative as λ increases through λ_0. Therefore, in this case, the curve has the shape shown in Fig. 1 or 2.

It can be seen from Fig. 1 that Eq. (3.a) can have a maximum of three distinct real positive roots $\lambda=\lambda_1$, λ_2, λ_3, say ($\lambda_1<\lambda_2<\lambda_3$), and Eq. (3.b) can have two such roots $\lambda=\lambda_4$, λ_5, ($\lambda_4<\lambda_5$). It is also easy to understand that Eq. (3.c) always has one real positive root $\lambda=\lambda_6$. To determine the exact set of roots for the given values of the parameters, we consider the case of repeated roots. It can be seen from Figs. 1 and 2 that there can be three such cases: $\lambda_1=\lambda_2$, $\lambda_2=\lambda_3$ and $\lambda_4=\lambda_5$. The precise conditions for the occurrence of these cases can be written respectively as:

$$\sigma'=b,\quad \sigma=b\lambda-a,\quad 0\leq\lambda<\lambda_0, \tag{4.a'}$$

$$\sigma'=b,\quad \sigma=b\lambda-a,\quad \lambda_0\leq\lambda<\lambda', \tag{4.b'}$$

$$\sigma'=b,\quad \sigma=b\lambda+a,\quad \lambda'\leq\lambda<\infty, \tag{4.c'}$$

where σ' denotes the derivatives of σ with respect to λ and λ' is the value of λ for which the straight line $m=b\lambda$ is tangent to $m=\sigma$ and λ_0 is the point of

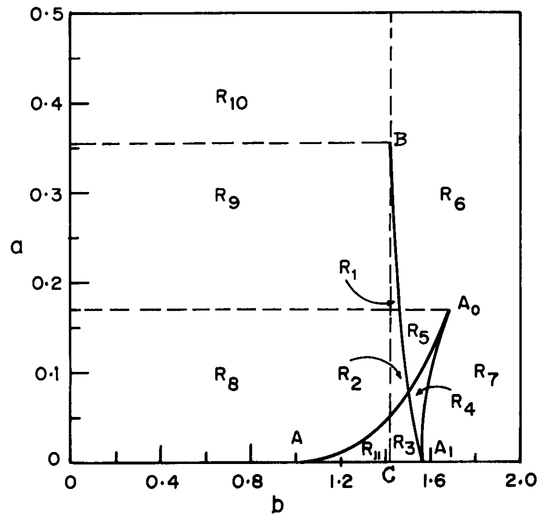

Fig. 3 Section of Critical Surface for $c=2$

inflexion of the curve $m=\sigma$. It is understood that λ_0 and λ' respectively satisfy the following equations:

$$\begin{aligned}&(4\lambda^3\operatorname{sech}^2\lambda\tanh\lambda-4\lambda^4\operatorname{sech}^2\lambda+3\lambda^4\operatorname{sech}^4\lambda)c^2\\&+(6\lambda^2\operatorname{sech}^2\lambda\tanh\lambda-8\lambda^3\operatorname{sech}^2\lambda+6\lambda^3\operatorname{sech}^4\lambda)c\\&+(2\lambda\operatorname{sech}^2\lambda\tanh\lambda-\tanh^2\lambda-4\lambda^2\operatorname{sech}^2\lambda\\&+3\lambda^2\operatorname{sech}^4\lambda)=0\end{aligned} \tag{5}$$

$$\lambda(\lambda+c\lambda^2)\operatorname{sech}^2\lambda-\lambda\tanh\lambda=0 \tag{6}$$

Equations (4.a'), (4.b') and (4.c') can be transformed to the following forms:

$$b=\frac{(1+2c\lambda)\tanh\lambda+(\lambda+c\lambda^2)\operatorname{sech}^2\lambda}{2\{(\lambda+c\lambda^2)\tanh\lambda\}^{1/2}},\quad 0\leq\lambda<\infty \tag{7}$$

$$\left.\begin{aligned}a&=\frac{\lambda(\lambda+c\lambda^2)\operatorname{sech}^2\lambda-\lambda\tanh\lambda}{2\{(\lambda+c\lambda^2)\tanh\lambda\}^{1/2}},\quad 0\leq\lambda<\lambda'\\&=\frac{\lambda\tanh\lambda-\lambda(\lambda+c\lambda^2)\operatorname{sech}^2\lambda}{2\{(\lambda+c\lambda^2)\tanh\lambda\}^{1/2}},\quad \lambda'\leq\lambda<\infty\end{aligned}\right\} \tag{8}$$

Equations (7) and (8) represent in the parametric form a surface called the critical surface $f(a, b, c)=0$ in the space of parameters a, b and c. The intersecting curve of this surface by a plane $c=$constant can be drawn for certain values of c. For $c=2$, this curve is shown in Fig. 3. Point A_0 corresponds to $\lambda=\lambda_0$ and point A_1, to $\lambda=\lambda'$.

It is noted that for all values of c, the third branch ends at point B with coordinates $(c^{1/2}, 1/2c^{1/2})$. It follows therefore that for other values of c, the intersecting curve remains similar in form, and only the positions of point A_0 and the straight line CB through B and parallel to the a-axis change.

The curvilinear portions C_1, C_2 and C_3, extending respectively from point A to A_0, from A_0 to A_1 and from A_1 to B, represent cases $\lambda_1=\lambda_2$, $\lambda_2=\lambda_3$ and $\lambda_4=\lambda_5$, respectively. These critical curves, the straight lines through A_0 and B parallel to the b-axis and the straight line CB, divide the entire positive quadrant of the plane of the curve into 11 regions R_n ($n=1$ to 11), as shown in Fig. 3. In each of these regions, there corresponds a definite subset of roots λ_n ($n=1$ to 6) in the sense that the values of parameters a, b and c that determine these regions also fix the roots. The roots of Eqs. (3.a), (3.b) and (3.c) can now be determined for all values of parameters a, b and c. This can be achieved geometrically by considering Figs. 1 and 3 side by side. This procedure is explained in Ref. (3). In the following, we write the distributions of these roots in various regions:

Regions:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
Roots	λ_3, λ_4	λ_3, λ_4	$\lambda_1, \lambda_2, \lambda_3$	$\lambda_1, \lambda_2, \lambda_3$	λ_3	λ_3	λ_1	λ_4	λ_4	λ_4	λ_1, λ_2
	λ_5, λ_6	λ_5, λ_6	$\lambda_4, \lambda_5, \lambda_6$	λ_6	λ_6	λ_6	λ_6	λ_6	λ_6	λ_6	λ_4, λ_6

It is noted that the roots in regions R_5 and R_6 are identical; the same holds for regions R_1 and R_2 and for regions R_8, R_9 and R_{10}. There are thus seven distinct regions. For points on the critical curve, the roots can also be similarly determined. It is noted that for each point on the critical curve, two of the roots coincide except for point A_0 where three roots coincide, thereby forming the case of a triple root.

We are now in a position to determine the asymptotic waves at far field after a long time. These asymptotic waves come as contributions to the asymptotic values of the integrals in terms of the poles of the integrands. The method for this asymptotic estimation has been explained in previous papers in Refs. (3) and (5). As an illustration, we consider the integral I_3. In general, the integrand of this integral has three different cases for different values of parameters a and b. These include the cases of three distinct poles, one single pole and one double pole and the case of one triple pole. For three distinct poles, the suitable substitution is

$$m=\sigma-b\lambda \tag{9}$$

A sketch of the curve of transformation is shown in Fig. 4. Poles λ_1, λ_2 and λ_3 are the points of intersection of curve $m=\sigma-b\lambda$ and straight line $m=-a$. Points $\lambda=\alpha_1$ and $\lambda=\alpha_2$ are the solutions to the equation $\sigma'=b$. The curve $m=\sigma-b\lambda$ may have other forms. However, it can be proved that in the case under consideration, α_1 and α_2 always exist and $m(\alpha_1)<m(\alpha_2)$. To calculate the asymptotic value of I_3 as $t\to\infty$, we break it up as follows:

$$I_3=\left(\int_0^{\alpha_1}+\int_{\alpha_1}^{\alpha_2}+\int_{\alpha_2}^{\infty}\right)\frac{q(\lambda)\lambda\tanh\lambda}{\sigma(\sigma-b\lambda+a)}e^{i\{\lambda x-(\sigma-b\lambda)t\}}d\lambda$$
$$=-\left(\int_{-\infty}^{m(\alpha_2)}+\int_{m(\alpha_2)}^{m(\alpha_1)}+\int_{m(\alpha_1)}^{0}\right)\left[\frac{\lambda q(\lambda)\tanh\lambda e^{i\lambda x}}{\sigma(\sigma'-b)}\right]\times\frac{e^{-imt}}{m+a}dm, \tag{10}$$

where the quantity within the square bracket is expressed in terms of m by Eq. (9). Now each of the integrals in Eq. (10) contains one pole. Hence, the formula referred to above gives the following asymptotic value for $t\to\infty$:

$$I_3\simeq\pi i\left[\frac{\lambda_3 q(\lambda_3)\tanh\lambda_3}{\sigma(\lambda_3)\{\sigma'(\lambda_3)-b\}}e^{i[-\{\sigma(\lambda_3)-b\lambda_3\}t+\lambda_3 x]}-\frac{\lambda_2 q(\lambda_2)\tanh\lambda_2}{\sigma(\lambda_2)\{\sigma'(\lambda_2)-b\}}e^{i[-\{\sigma(\lambda_2)-b\lambda_2\}t+\lambda_2 x]}+\frac{\lambda_1 q(\lambda_1)\tanh\lambda_1}{\sigma(\lambda_1)\{\sigma'(\lambda_1)-b\}}e^{i[-\{\sigma(\lambda_1)-b\lambda_1\}t+\lambda_1 x]}\right]$$
as $t\to\infty$

For points either on curve C_1 or on curve C_2 except point A_0, two of the roots coincide. Let us calculate the integral for the points on C_1. Here, $\lambda_1=\lambda_2=\alpha_1$ and λ_3 is distinct. We break up the integral as follows:

$$I_3=\left(\int_0^{\alpha_1-\varepsilon}+\int_{\alpha_1-\varepsilon}^{\alpha_1}+\int_{\alpha_1}^{\alpha_1+\varepsilon}+\int_{\alpha_1+\varepsilon}^{\alpha_2}+\int_{\alpha_2}^{\infty}\right)\times\frac{\lambda q(\lambda)\tanh\lambda}{\sigma(\sigma-b\lambda+a)}e^{i\{\lambda x-(\sigma-b\lambda)t\}}d\lambda \tag{11}$$

where $0<\varepsilon\ll 1$.

The first and fourth integrals, having no pole, will not contribute to the asymptotic value, and the contribution of the last integral from the distinct pole $\lambda=\lambda_3$ can be calculated as above. To calculate the second and third integrals, we make the substitution

$$m=\sigma-b\lambda+a \tag{12}$$

Now, near $\lambda=\alpha_1$, we can write

$$m\simeq\frac{1}{2}(\lambda-\alpha_1)^2\sigma''(\alpha_1)$$
$$\frac{dm}{d\lambda}=|2m\sigma''(\alpha_1)|^{1/2}\,\mathrm{sgn}\,(\lambda-\alpha_1)$$
$$m(\alpha_1-\varepsilon)=m(\alpha_1+\varepsilon)$$
$$=\frac{1}{2}\varepsilon^2\sigma''(\alpha_1)=\varepsilon_1(>0),\quad \text{say}$$

and $m(\alpha_1)=0$.

Substituting all these, the combination of the second and third integrals in Eq. (11) reduces to

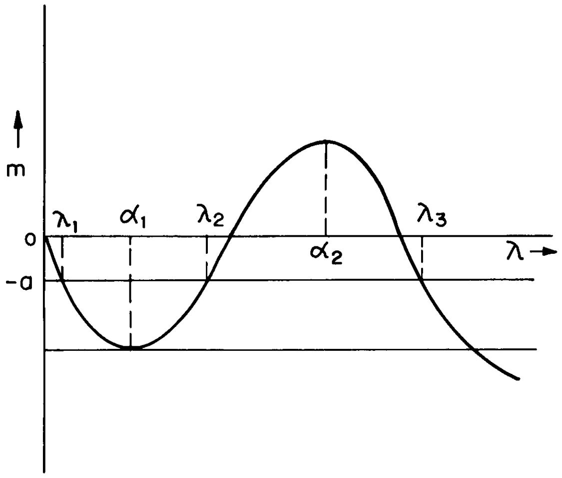

Fig. 4 Form of Curve $m=\sigma-b\lambda$

$$2e^{iat}\int_0^{\varepsilon_1}\left[\frac{\lambda q(\lambda)\tanh\lambda}{\sigma}e^{i\lambda x}\right]|2\sigma''(\alpha_1)|^{-1/2}\frac{e^{-imt}}{|m|^{3/2}}dm. \tag{13}$$

The asymptotic value of the integral in Eq.(13) remains unchanged if the upper limit is replaced with ∞; hence, using Lighthill's formula in Ref.(6), the asymptotic value is given by

$$2e^{iat}|2\sigma''(\alpha_1)|^{-1/2}\left(-\frac{3}{2}\right)!\times\frac{\alpha_1 q(\alpha_1)\tanh\alpha_1}{\sigma(\alpha_1)}e^{i\alpha_1 x+i\frac{\pi}{4}}t^{1/2}\quad\text{as } t\to\infty. \tag{14}$$

To find the asymptotic value of integral I_3 for point A_0 for which $\lambda_1=\lambda_2=\lambda_3=\lambda_0$, we use the same substitution equation (12). Then, the significant contribution to integral I_3 as $t\to\infty$ comes from the neighbourhood of point $\lambda=\lambda_0$ and the integral can be expressed as

$$I_3=\int_{\lambda_0-\mu}^{\lambda_0+\mu}\frac{\lambda q(\lambda)\tanh\lambda}{\sigma(\sigma-b\lambda+a)}e^{i\{\lambda x-(\sigma-b\lambda)t\}}d\lambda, \tag{15}$$

where $0<\mu\ll 1$.

Near $\lambda=\lambda_0$, $m\simeq\frac{1}{3!}(\lambda-\lambda_0)^3\sigma'''(\lambda_0)$

$$\frac{dm}{d\lambda}=\left(\frac{9}{2}\right)^{1/3}|m|^{2/3}\{\sigma'''(\lambda_0)\}^{1/3}.$$

Substituting all these, integral (15) reduces to

$$-\left(\frac{2}{9}\right)^{1/3}\frac{e^{iat}}{\{\sigma'''(\lambda_0)\}^{1/3}}\int_{-\mu_1}^{\mu_1}\left[\frac{\lambda q(\lambda)\tanh\lambda}{\sigma}e^{i\lambda x}\right]\times\frac{e^{-imt}}{|m|^{5/3}\operatorname{sgn} m}dm, \tag{16}$$

where μ_1 is a positive quantity.

The asymptotic value of integral (16) remains unchanged if μ_1 is replaced with ∞; hence, using Lighthill's formula, the asymptotic value is given by

$$-2i\frac{\left(-\frac{5}{3}\right)!}{(48)^{1/6}}\frac{\lambda_0 q(\lambda_0)\tanh\lambda_0}{\sigma(\lambda_0)\{\sigma'''(\lambda_0)\}^{1/3}}e^{i(at+\lambda_0 x)}t^{2/3}\quad\text{as } t\to\infty.$$

It is thus seen that the asymptotic values of the integrals become large for points on the critical curves. It is large like $t^{1/2}$ where two roots coincide and it is like $t^{2/3}$ where three roots coincide. For points not on the critical curve, the asymptotic values of the integrals give waves with a constant coefficient. In the following, we write the system of waves for points in R_3:

$$\eta(x,t)=\frac{i\sqrt{\pi}}{\rho gh\sqrt{2}}\left[\frac{\lambda_4 q(-\lambda_4)\tanh\lambda_4}{\sigma(\lambda_4)\{\sigma'(\lambda_4)-b\}}e^{i[\{\sigma(\lambda_4)-b\lambda_4\}t-\lambda_4 x]}-\frac{\lambda_2 q(\lambda_2)\tanh\lambda_2}{\sigma(\lambda_2)\{\sigma'(\lambda_2)-b\}}e^{i[-\{\sigma(\lambda_2)-b\lambda_2\}t+\lambda_2 x]}\right]$$

$$\text{when } t\to\infty \text{ and } x\to\infty \tag{17.a}$$

$$=\frac{i\sqrt{\pi}}{\rho gh\sqrt{2}}\left[\frac{\lambda_1 q(\lambda_1)\tanh\lambda_1}{\sigma(\lambda_1)\{\sigma'(\lambda_1)-b\}}e^{i[-\{\sigma(\lambda_1)-b\lambda_1\}t+\lambda_1 x]}+\frac{\lambda_3 q(\lambda_3)\tanh\lambda_3}{\sigma(\lambda_3)\{\sigma'(\lambda_3)-b\}}e^{i[-\{\sigma(\lambda_3)-b\lambda_3\}t+\lambda_3 x]}-\frac{\lambda_5 q(-\lambda_5)\tanh\lambda_5}{\sigma(\lambda_5)\{\sigma'(\lambda_5)-b\}}e^{i[\{\sigma(\lambda_5)-b\lambda_5\}t-\lambda_5 x]}+\frac{\lambda_6 q(\lambda_6)\tanh\lambda_6}{\sigma(\lambda_6)\{\sigma'(\lambda_6)+b\}}e^{i[\{\sigma(\lambda_6)+b\lambda_6\}t+\lambda_6 x]}\right]$$

$$\text{whent } t\to\infty \text{ and } x\to-\infty. \tag{17.b}$$

The wave system for the case in which the values of the parameters of the problem are such that the point lies in other regions is easy to determine. This is the same wave system as that expressed in Eqs.(17.a) and (17.b) with only the wave corresponding to a pole not occurring in a region being deleted for that region. For convenience, we shall call a wave λ_n, which indicates the contribution from pole λ_n. Among these six waves, four waves are original gravity waves, with one (λ_4-wave) existing in the upstream side and other three (λ_3, λ_5, λ_6 waves) existing in the downstream side, and the rest are the effect of the magnetic field, one (λ_2-wave) propagating in the upstream side and the other (λ_1-wave), in the downstream side. Now, we wish to consider the particular case of $\omega=0$. The relevant frequency equation is $\sigma=b\lambda$. It is easy to see that it always has the root zero and a maximum of two nonzero roots α and β ($\alpha<\beta$). The corresponding critical condition is represented by the intersecting curve Γ of surface $f(a,b,c)=0$ by plane $a=0$ and hence it has the following representation,

$$\left.\begin{aligned}b&=\frac{\sinh\lambda}{\lambda}\\ c&=\frac{\tanh\lambda-\lambda\operatorname{sech}^2\lambda}{\lambda^2\operatorname{sech}^2\lambda}\end{aligned}\right\}\quad 0<\lambda<\infty, \tag{18}$$

This is represented in Fig. 5. Then, the following statements regarding the steady-state waves can be easily verified. The α-wave, which is the wave due to the magnetic field, exists in the upstream side, while the β-wave, which is the original gravity wave, exists in the downstream side. For points (b, c) to the left of the curve Γ, both waves exist, while for points (b, c) to the right, none of the waves propagate.

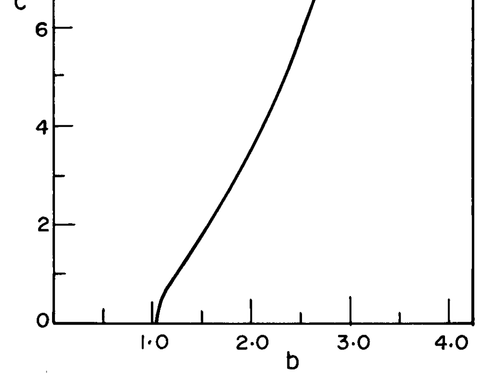

Fig. 5 Section of critical surface for $a=0$

4. Some Physical Discussions

We compute the group velocities of the waves for various values of parameters 'a' and 'b'. The group velocities relative to the moving source can be determined from the dispersion relations (3.a) to (3.c). Denoting the relative group velocity of the wave η_n, where $n=1$ to 6, by C_{gn}, we get

$$C_{g_i}=\sigma'(\lambda_i)-b, \quad i=1, 2, 3 \tag{4.a}$$

$$C_{g_i}=\sigma'(\lambda_i)-b, \quad i=4, 5 \tag{4.b}$$

$$C_{g_6}=-(\sigma'+b). \tag{4.c}$$

Relations (4.a) together with (3.a) determine the group velocities of waves η_1, η_2 and η_3. Similarly, relations (4.b) and (4.c) together with (3.b) and (3.c) determine the group velocities of waves η_4, η_5 and wave η_6 respectively. Essentially, these group velocities depend on parameters 'a' and 'b' as well as on 'c'. We consider the variation of the group velocities at fixed values of 'b' and 'c' and different values of 'a'. The result is shown in Fig. 6. The dotted lines show the corresponding curves in the gravity wave case, i.e., the case where $c=0$. The group velocities of the waves that exist for given values of parameters are present in the figure. The group velocity of η_6 though present could not be shown as it has largest magnitude.

It follows that the relative group velocities of waves η_2 and η_4 are positive while those of waves η_1, η_3, η_5 and η_6 are negative. This explains the reason for the waves propagating upstream or downstream. The phase of each of waves η_4 and η_5 propagates along the positive x-axis direction. However, as the group velocity of wave η_4 relative to the source is positive, its front remains in the upstream side and as the relative group velocity of η_5 is negative, its front remains behind the source. Similarly, the fronts of η_1 and η_3 remain in the downstream side while that of η_2 remains in the upstream side. However, their phases propagate in the negative x-axis direction and these waves appear to originate at infinity. This anomaly is only observed in a moving coordinate system. In a fixed coordinate system, the phases are actually propagating along the positive x-axis direction. The phase and also the front of wave η_6 propagate in the negative x-axis direction.

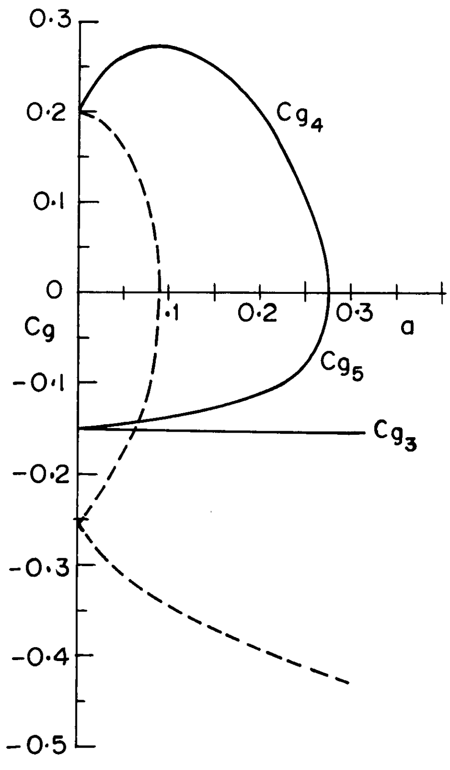

Fig. 6 Group velocities for $b=0.8$, (i) the bold curves for $c=0.4$ and (ii) the dotted curves for $c=0$

Figure 6 also explains the reason for the occurrence of singularity for certain values of the parameters. It follows from the figure that singularity occurs when the group velocities of two waves, one propagating upstream and other downstream, coincide and become equal to zero. This means that for those values of the parameters, the waves are generated but they cannot come out of the source. As a result, the energy of these waves is accumulated at the source which, in turn, gives rise to the singularity.

This analysis gives one important feature of the waves generated by a moving source. The total waves are divided into two distinct classes. The first class contains waves η_2, η_4 and η_6 which, being generated by the source, move away from the source either downstream or upstream. The second class contains waves η_1, η_3, and η_5 which exist in the downstream side and move with the source. Due to the waves in the first class, energy is transmitted into the far field from the source, while due to the waves in the second class, energy remains confined in the region behind the source. The wake may be conveniently defined as the region behind the source up to the farthest wavefront of the second class of waves. One may reasonably assume that the main disturbance due to the generated waves lies in this wake.

Comparison of the group velocities of the waves in the magnetic-gravity wave case, i.e., the case of $c\neq 0$, with those in the gravity wave case, i.e., the case of $c=0$, shows that due to the presence of the magnetic force, the group velocities relative to the source for gravity waves η_3 and η_5 decrease while that of wave η_4 increases. By representing the group velocity curves of wave η_6 in figures for various values of the parameters, one can show that the relative group velocity of wave η_6 also increases. Thus, due to the presence of a magnetic force, the relative group velocities of the waves in the first class increase and those in the second class decrease. By decreasing the relative group velocities of the waves in the second class, the length of the wake decreases and one may reasonably assume that as a result, the intensity of the disturbance in the wake increases. On the other hand, by increasing the relative group velocities of the waves in the first class, the total disturbed zone beyond the wake increases, energy is transmitted into a more distant region and as a result, the intensity of the disturbance in the region decreases. These arguments are not valid for some values of the parameters where the wavefront of the gravity wave η_5 is nearer the source than the corresponding magnetic-gravity wave. However, this is the case near the critical condition of the corresponding gravity wave where the amplitude of that wave is large and cannot be included in this analysis. This is clearly shown in Fig. 6.

To illustrate the points stated above, we draw the wave profile for some specific values of the parameters. For convenience, we present the result only in the downstream side.

We compute numerically the surface elevation η. For this, we consider a particular pressure distribution where $f(x)=p_0\delta(x)$, where p_0 is a constant. The quantity η is made dimensionless by the scaling factor $p_0/\rho g$. The dimensionless surface elevation η is computed for various values of x with $a=0.55$, $b=0.4$, $t=10$, $c=0.1$ and the results are presented in Fig. 7. The dotted curve represents the surface elevation for the case of $c=0$. The characteristic features stated above are clearly manifested in the figure. The wakes in both magnetic-gravity waves and gravity waves are clearly exhibited. Points $x=x_{3m}$ and $x=x_{5m}$ are, respectively, the wavefronts of magnetic-gravity waves η_3 and η_5. Similarly, $x=x_{3g}$ and $x=x_{5g}$ are the corresponding wavefronts for the case of $c=0$. The wave in the case of $c\neq0$ extends up to $x=x_{3m}$ and that in the case of $c=0$ extends up to $x=x_{3g}$. The intensity of the disturbance in the wake in the case of $c\neq 0$ is much larger than the corresponding intensity in the case of $c=0$. The disturbed zone in the case of $c\neq0$ outside the wake is much larger than the corresponding zone in the case of $c=0$ but the intensity is reduced. This intensity is so small that it cannot be drawn in the scale considered.

5. Conclusion

In the investigation we get the following conclusions:

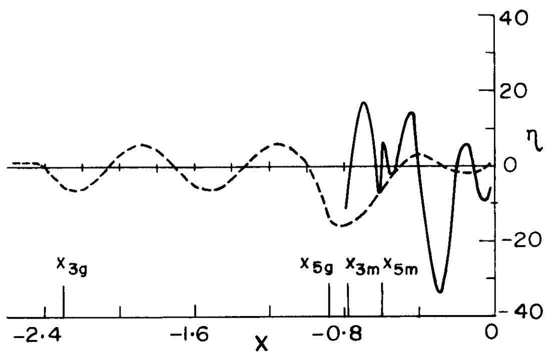

Fig. 7 Surface elevation for $b=0.4$, $a=0.55$, (i) the bold curve for $c=0.1$ and (ii) the dotted curve for $c=0$

(i) There are six progressive waves, four original gravity waves, rest are the magnetic waves. The magnetic waves are generated due to magnetic field imposed outside the air-water interface.

(ii) Among the three downstream gravity waves η_3, η_5 and η_6, two waves η_3 and η_5 propagating with the source create the main disturbance in the wake. Due to the existence of magnetic force, the disturbance in the wake is generally increased.

(iii) For certain values of the parameters, the behaviours of the waves are singular. Such singularity occurs when the group velocities of two waves, one propagating upstream and other downstream, coincide and become equal to zero.

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