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# Fuzzy $\delta$-Continuous Mappings

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## 1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh in his classical paper [9]. Thereafter many investigations have been carried out, in the general theoretical field and also in different application sides, based on this concept. The idea of fuzzy topological spaces was introduced by Chang [3]. The idea is more or less a generalization of ordinary topological spaces. Different aspects of such spaces have been developed, by several investigators. This paper is also devoted to the development of the theory of fuzzy topological spaces.

In this paper we first generalize the idea of *continuity as a local property* in fuzzy setting. Then we generalize mainly the concept of $\delta$-continuity of a function due to Noiri [4] in fuzzy setting. It is seen that fuzzy $\delta$-continuity and fuzzy continuity are independent concepts; and fuzzy $\delta$-continuity implies fuzzy almost continuity due to Azad [1], but not conversely. Also it is seen that the idea of fuzzy continuity, fuzzy $\delta$-continuity and fuzzy almost continuity are equivalent in case when the domain of definition and the range space of the function are both fuzzy semi-regular spaces due to Azad [1].

It is observed that, if a fuzzy point [8] belongs to the closure of a fuzzy set of an universal set then every fuzzy neighbourhood of the fuzzy point [8] must intersect that fuzzy set, but not conversely [cf. Example 3.2]. This is a variation of fuzzy topology from ordinary topology.

## 2. FUZZY SETS

Let $X$ be a space of points. A fuzzy set in $X$ is a mapping from $X$ into the closed unit interval $I$ on the real line, i.e., a fuzzy set is member of $I^X$. Let $\lambda \in I^X$. The subset of $X$ in which $\lambda$ assumes nonzero values, is known as the



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support of $\lambda$. Ordinary subsets of $X$ are those members of $I^X$ which take values only 0 and 1. The null fuzzy set 0 is the mapping from $X$ into $I$ which assumes only the value 0 and the set $X$ is denoted by the mapping 1 from $X$ into $I$, which takes the value 1 only (cf. [9]).

DEFINITION 2.1. A fuzzy set $\lambda$ in $X$ is said to be countable iff the support of $\lambda$ is countable (provided $X$ is infinite) in $X$.

The complement $\lambda'$ of a fuzzy set $\lambda$ is also a member of $I^X$, defined by $\lambda'(x) = 1 - \lambda(x)$ $(x \in X)$ (cf. [9]). A member $\lambda$ of $I^X$ is contained in a member $\mu$ of $I^X$, denoted $\lambda \leqslant \mu$, iff $\lambda(x) \leqslant \mu(x)$ $(x \in X)$ (cf. [9]). Let $\{\lambda_i : i \in \mathscr{A}$ (an indexing set$\}$ be a family of fuzzy subsets of $X$. Then their union and intersection (denoted respectively as $\bigcup \lambda_i$ and $\bigcap \lambda_i$) are defined, respectively, as

$$\left.\begin{aligned} \bigcup \lambda_i(x) &= \sup_{i \in \mathscr{A}} \lambda_i(x) \\ \bigcap \lambda_i(x) &= \inf_{i \in \mathscr{A}} \lambda_i(x) \end{aligned}\right\} \quad (x \in X)$$

(cf. [9]). Let $\lambda, \mu \in I^X$, then $(\lambda \cap \mu)' = \lambda' \cup \mu'$ and $(\lambda \cup \mu)' = \lambda' \cap \mu'$ (cf. [9]). Two fuzzy sets $\lambda$ and $\mu$ intersect each other iff $\min(\lambda(x), \mu(x)) \neq 0$ for at least one $x \in X$. If $X$ and $Y$ are two sets of points and if $\lambda$ and $\mu$ are fuzzy subsets of $X$ and $Y$, respectively, then $\lambda \times \mu$ (the cartesian product of two fuzzy sets) is a fuzzy set of $X \times Y$ defined by,

$$\lambda \times \mu(x, y) = \min(\lambda(x), \mu(y)), \qquad [(x, y) \in X \times Y]$$

(cf. [9]).

A fuzzy point $p$ in $X$ is a fuzzy set in $X$ defined by,

$$\begin{aligned} p(x) &= y\,(y \in (0, 1)) && \text{for } x = x_p \\ &= 0 && \text{otherwise } (x \in X) \end{aligned}$$

$x_p$ and $y$ are, respectively, the support and the value of $p$ (cf. [8]). A fuzzy point $p$ is said to belong to a fuzzy set $\lambda$ of $X$ iff, $p(x_p) < \lambda(x_p)$ and $p(x) \leqslant \lambda(x)$ if $x \neq x_p$ $(x \in X)$. In this case we shall use the notation $p \in \lambda$ (cf. [6]).

RESULT 2.1 (cf. [6, Theorem 2.2]). A fuzzy set $\lambda$ in $X$ is the union of all its fuzzy points.

Let $f: X \to Y$ be a mapping. If $\lambda$ is a fuzzy set in $X$, then $f(\lambda)$ is a fuzzy set in $Y$ defined by,

$$\begin{aligned} f(\lambda)(y) &= \operatorname*{Sup}_{z \in f^{-1}(y)} \lambda(z) && \text{if } f^{-1}(y) \neq \varnothing \\ &= 0 && \text{otherwise } (y \in Y) \end{aligned}$$

(cf. [9]). If $p$ is a fuzzy point in $X$, then $f(p)$ is also a fuzzy point in $Y$ defined by, $f(p)(y_p) = \sup_{z \in f^{-1}(y_p)} p(z) = a$ (where $y_p = f(x_p)$ and $f(p)(y) = 0$, for $y \neq y_p$, and $a$ is the value of $P$). If $\mu$ is a fuzzy set in $Y$, then $f^{-1}(\mu)$ is a fuzzy set in $X$ defined by, $f^{-1}(\mu)(x) = \mu(f(x))$, $(x \in X)$ (cf. [9]). If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are two mappings, then the product $f_1 \times f_2$ of $f_1$ and $f_2$ is a mapping from $X_1 \times X_2$ into $Y_1 \times Y_2$ defined by $f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$ $[(x_1, x_2) \in X_1 \times X_2]$.

## 3. Fuzzy Topological Spaces

Let $X$ be a space of points. A family $\tau X$ of fuzzy subsets of $X$ is called a fuzzy topology on $X$ if (i) 0 and 1 belong to $\tau X$, (ii) intersection of any two members of $\tau X$ is also a member of $\tau X$, and (iii) union of any collection of members of $\tau X$ is in $\tau X$ (cf. [3]) $(X, \tau X)$ is called a fuzzy topological space (fts). Members of $\tau X$ are defined as open fuzzy subsets of $X$. Complement of any open fuzzy subset is defined as a closed fuzzy subset of $X$.

Example 3.1. Let $X$ be a noncountable set of points and let $\tau X$ consists of 0 and 1 and all those members of $X$ whose complements in $X$ are countable fuzzy subsets of $X$. Then it is easy to see that $\tau X$ forms a fuzzy topology on $X$ and we call this as fuzzy topology of countable complements.

The closure $\mathrm{Cl}\,\lambda$ and the interior $\mathrm{Int}\,\lambda$ of a fuzzy set $\lambda$ of $X$ are defined respectively as, $\mathrm{Cl}\,\lambda = \mathrm{Inf}\{\mu : \mu \geqslant \lambda,\ \mu' \in \tau X\}$,

$$\mathrm{Int}\,\lambda = \sup\{\mu : \mu \leqslant \lambda,\ \mu \in \tau X\}.$$

A fuzzy set $\lambda$ in a fts $X$ is called fuzzy regularly open if $\lambda = \mathrm{Int}(\mathrm{Cl}\,\lambda)$ and $\lambda$ is called fuzzy regularly closed if $\lambda = \mathrm{Cl}(\mathrm{Int}\,\lambda)$ (cf. [1]).

A fuzzy set $\lambda$ of a fts $X$ is fuzzy regularly open iff its complement in $X$ is fuzzy regularly closed (cf. [1]).

A subfamily $B$ of $\tau X$ in a fts $(X, \tau X)$ is called a base for $\tau X$ if each member of $\tau X$ is the union of some members belonging to $B$. A subfamily $S$ of $\tau X$ is called a subbase of $\tau X$ if the collection of all finite intersections of members of $S$ forms a base for $\tau X$.

A fuzzy topological space $(X, \tau X)$ is called a fuzzy semi-regular space iff the collection of all fuzzy regular open sets of $X$ forms a base for fuzzy topology $\tau X$ (cf. [1]).

A fuzzy set $\lambda$ in a fts $(X, \tau X)$ is a neighbourhood (or nbd in short) of a fuzzy set $\mu$ iff there is a member $\nu \in \tau X$ such that $\mu \leqslant \nu \leqslant \lambda$ (cf. [3]).

Let $p$ be a fuzzy point in a fts $(X, \tau X)$. A fuzzy set $\lambda$ is called a fuzzy nbd of $p$ iff there is a member $\nu \in \tau X$ such that $p \in \nu \leqslant \lambda$. If $\lambda \in \tau X$, it is called a fuzzy open nbd of $p$ in $(X, \tau X)$ (cf. [6]).

## Fuzzy Continuity

Definition. Let $f$ be a mapping from a fts $(X, \tau X)$ into a fts $(Y, \tau Y)$, $f$ is said to be fuzzy continuous iff the inverse image of any fuzzy set is a member of $\tau X$ (cf. [3]).

Result 3.1. A mapping $f$ from a fts $(X, \tau X)$ into a fts $(Y, \tau Y)$ is fuzzy continuous *iff* for each fuzzy point $p$ in $X$ and each fuzzy neighbourhood (nbd) $\lambda$ of $f(p)$ in $(Y, \tau Y)$, there is a fuzzy nbd $\mu$ of $p$ in $(X, \tau X)$ such that $f(\mu) \leqslant \lambda$.

*Proof.* Let $f$ be fuzzy continuous. Let $p$ be a fuzzy point in $X$ and $\lambda$ be a fuzzy nbd of $f(p)$ in $Y$. There is a member $\nu$ of $\tau Y$ such that $f(p) \in \nu \leqslant \lambda$ and so, $p \in f^{-1}(\nu) \leqslant f^{-1}(\lambda)$ (it can be seen easily) $f$ is fuzzy continuous, $\nu \in \tau Y$ implies $f^{-1}(\nu) \in \tau X$ and so $\mu = f^{-1}(\lambda)$ is a fuzzy nbd of $p$ in $X$ and $f(\mu) = f(f^{-1}(\lambda)) \leqslant \lambda$ (cf. [3]).

Conversely, let the given condition hold for $f$. Let $\lambda \in \tau Y$ and $p$ be any fuzzy point in $f^{-1}(\lambda)$. Then $f(p) \in f(f^{-1}(\lambda)) \leqslant \lambda$. [If $p \in \lambda$, then $f(p) \in f(\lambda)$ $(\lambda \in I^X)$. In fact let $f(x_p) = y_p$ and $p(x_p) = a(\in (0, 1))$ $p \in \lambda \Rightarrow p(x_p) < \lambda(x_p)$. Now,

$$f(p)(y_p) = \sup_{z \in f^{-1}(y_p)} p(z) = a$$
$$= p(x_p) < \lambda(x_p) \leqslant \sup_{z \in f^{-1}(y_p)} \lambda(z) = f(\lambda)(y_p)].$$

By hypothesis there is a fuzzy nbd $\mu$ of $p$ such that $f(\mu) \leqslant \lambda$, i.e., $\mu \leqslant f^{-1}(f(\mu)) \leqslant f^{-1}(\lambda)$ (cf. [3]). So, $p \in \mu \leqslant f^{-1}(\lambda)$. As $\mu$ is a fuzzy nbd of $p$ there is a member $\nu$ of $\tau X$ such that $p \in \nu \leqslant \mu$. So, $p \in \nu \leqslant f^{-1}(\lambda)$. Taking union of all such relations as $p$ runs over $f^{-1}(\lambda)$, we obtain

$$f^{-1}(\lambda) = \bigcup \{p : p \text{ is a fuzzy point of } f^{-1}(\lambda)\}$$
$$\leqslant \bigcup \{\nu\} \leqslant f^{-1}(\lambda).$$

So, $f^{-1}(\lambda) = \bigcup \{\nu\} \in \tau X$. Hence $f$ is fuzzy continuous.

Fuzzy continuity as a local property.

Definition 3.1. A function $f$ from a fts $(X, \tau X)$ into a fts $(Y, \tau Y)$ is said to be fuzzy continuous at a fuzzy point $p$ in $X$ if corresponding to any fuzzy nbd $\lambda$ of $f(p)$ in $(Y, \tau Y)$, there is a fuzzy nbd $\mu$ of $p$ in $(X, \tau X)$ such that $f(\mu) \leqslant \lambda$.

RESULT 3.2. *A mapping $f$ from a* fts $(X, \tau X)$ *into a* fts $(Y, \tau Y)$ *is fuzzy continuous iff $f$ is fuzzy continuous at every fuzzy point $p$ in $X$.*

*Proof.* Follows directly from Result 3.1 and the fact that a fuzzy set is the union of all its fuzzy points.

RESULT 3.3. *If a fuzzy point $p$ in a* fts $(X, \tau X)$ *belongs to the closure of a fuzzy set $\lambda$ in $X$, then every fuzzy* nbd *of $p$ intersects $\lambda$.*

*Proof.* Let $p \in \mathrm{Cl}\,\lambda$. If $p \in \lambda$ then every fuzzy nbd of $p$ intersects $\lambda$. Let $p \notin \lambda$. Let $\mu$ be an open fuzzy nbd of $p$. If possible, let $\lambda \cap \mu = 0$. Let support of $\mu$ be $F$ $(\subseteq x)$. Then $F \cap$ support of $\lambda = \phi$. Let us define a fuzzy set $v$ in $X$ as follows:

$$v(x) = \begin{cases} 0 & \text{for those values } x \text{ of } F \text{ for which } \mu(x) = 1 \\ \mathrm{Cl}\,\lambda(x) & \text{for } x \notin F \\ \min(\mathrm{Cl}(x), 1 - \mu(x)) & \text{for those values } x \text{ of } F \text{ for which } \mu(x) < 1. \end{cases}$$

Then obviously $\lambda \leqslant v$. But $v = \mathrm{Cl}\,\lambda \cap \mu$ is a closed set and $v < \mathrm{Cl}(\lambda)$, which implies a contradiction. So $\mu \cap \lambda \neq 0$ must hold.

Unfortunately, the converse of the above result is not true.

EXAMPLE 3.2. Let $\mu_1$, $\mu_2$, and $\mu_3$ be fuzzy sets of I defined as

$$\mu_1(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{2}, \\ 2x - 1 & \frac{1}{2} \leqslant x \leqslant 1' \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & 0 \leqslant x \leqslant \frac{1}{4} \\ -4x + 2 & \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ 0 & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{4} \\ (4x - 1) & \frac{1}{4} \leqslant x \leqslant 1. \end{cases}$$

Then $\tau = \{0, \mu_1, \mu_2, \mu_1 \cup \mu_2, 1\}$ is a fuzzy topology on $I$ ) cf. [1, Example 4.5]). It is easy to observe that $\mathrm{Cl}\,\mu_3 = \mu_2'$. Let $p$ be a fuzzy point in $I$ defined by $p(x) = \frac{1}{4}$ for $x = \frac{1}{5}$ and $p(x) = 0$ for $x \neq \frac{1}{5}$ $(x \in I)$. Then $p \notin \mathrm{Cl}\,\mu_3$.

But any fuzzy nbd $\lambda$ of $p$ must satisfy one of the relations (i) $p \in \mu_2 \leqslant \lambda$, (ii) $p \in \mu_1 \cup \mu_2 \leqslant \lambda$, and (iii) $p \in 1 \leqslant \lambda$ (in this case $\lambda = 1$). But $\mu_2 \cap \mu_3 \neq 0$,

$$\mu_3 \cap (\mu_2 \cup \mu_1) \neq 0 \qquad \text{and} \qquad \mu_3 \cap 1 \neq 0.$$

So, every fuzzy nbd of $p$ intersects $\mu_3$.

DEFINITION 3.2. A fuzzy point $p$ of a fuzzy set $\lambda$ in a fts $X$ is said to be a cluster point of $\lambda$ if every fuzzy nbd containing a fuzzy point having same support as $p$ has non-null intersection with $\lambda$.

By Result 3.3 it is clear that every fuzzy point of the closure of a fuzzy set in a fts is a cluster point of the set.

PRODUCT FUZZY TOLOPOGY. Let $(X_1, \tau X_1)$ and $(X_2, \tau X_2)$ be two fuzzy topological spaces. Let $X = X_1 \times X_2$ be the cartesian product of $X_1$ and $X_2$ and $p_i$ be the projection of $X$ into $X_i$. The product fuzzy space of $X_1$and $X_2$ is a set $X$ ($X = X_1 \times X_2$) equipped with the fuzzy topology generated by the family $\{p_1^{-1} \times \mu_\alpha), p_2^{-1}(\lambda_\beta)$: $\mu_\alpha \in \tau X_1$ and $\lambda_\beta \in \tau X_2\}$ (cf. [7]). The family $B = \{\mu_\alpha \times \lambda_\beta : \mu_\alpha \in \tau X_1$ and $\lambda_\beta \in \tau X_2\}$ forms a base for the product topology $\tau X$ on $X$ (cf. [1]).

The following result is due to Azad.

RESULT. *If $\lambda$ is a fuzzy set in a* fts *$X$ and $\mu$ is a fuzzy set in a* fts *$Y$, then* $\mathrm{Cl}\,\lambda \times \mathrm{Cl}\,\mu \geqslant \mathrm{Cl}(\lambda \times \mu)$ *and* $\mathrm{Int}\,\lambda \times \mathrm{Int}\,\mu \leqslant \mathrm{Int}(\lambda \times \mu)$.

DEFINITION. A fuzzy space $X$ is product related to another fuzzy space $Y$ if for any fuzzy set $\nu$ of $X$ and $\delta$ of $Y$ whenever $\lambda' \not\geqslant \nu$ and $\mu' \not\geqslant \delta$ implies $\lambda' \times 1 \cup 1 \times \mu' \geqslant \nu \times \delta$, where $\lambda \in \tau X$ and $\mu \in \tau Y$ there exists $\lambda_1 \in \tau X$ and $\mu_1 \in \tau Y$ such that $\lambda'_1 \geqslant \nu$ or $\mu'_1 \geqslant \delta$ and $\lambda'_1 \times 1 \cup 1 \times \mu'_1 = \lambda' \times 1 \cup 1 \times \mu'$ (cf. [1]).

The following result is due to Azad.

RESULT. *If $X$ and $Y$ are fuzzy topological spaces such that $X$ is product related to $Y$, then for a fuzzy set $\lambda$ of $X$ and a fuzzy set $\mu$ of $Y$,*

(a) $\mathrm{Cl}(\lambda \times \mu) = \mathrm{Cl}\,\lambda \times \mathrm{Cl}\,\mu$ *and*

(b) $\mathrm{Int}(\lambda \times \mu) = \mathrm{Int}\,\lambda \times \mathrm{Int}\,\mu$.

## 4. FUZZY δ-CONTINUOUS MAPPING

DEFINITION 4.1. A fuzzy point $p$ is a fuzzy $\delta$-cluster point of a fuzzy set $\lambda$ in a fts $X$ iff every fuzzy regularly open set containing a fuzzy point $q$ (having same support as $p$) has non-null intersection with $\lambda$.

DEFINITION 4.2. Let $\lambda$ be a fuzzy subset of a fts $X$. Let $\mu$ be a fuzzy subset of $X$ satisfying the following conditions;

(a) every fuzzy point $p$ in $\mu$ is a fuzzy $\delta$-cluster point of $\lambda$,

(b) if $\nu$ is a fuzzy set of $X$ such that $\mu \leqslant \nu$, then there is a fuzzy point $p$ in $\nu$ which is not a $\delta$-cluster point of $\lambda$ (it is clear that support of $\mu$ is a proper subset of the support of $\nu$ $(\subseteq X)$ in this case).

Any fuzzy subset of $X$ having same support as $\mu$ is defined to be a $\delta$-closure of $\lambda$.

We denote a $\delta$-closure of a fuzzy set $\lambda$ in a fts by $[\lambda]_\delta$. Thus a property related with the notation $[\lambda]_\delta$, will always imply that the property holds for all $\delta$-closures of $\lambda$.

DEFINITION 4.3. A fuzzy set $\lambda$ of a fts $X$ is said to be fuzzy $\delta$-closed if $\lambda$ is equal to one of its $\delta$-closures. Complement of a fuzzy $\delta$-closed set is said to be fuzzy $\delta$-open.

*Note.* $\delta$-closure of a fuzzy set in a fts is not unique. However, it is unique up to its support. Any fuzzy regularly open set of a fts $X$ is a fuzzy open set of $X$. Therefore, the set of all fuzzy regularly open sets is subset of the set of all fuzzy open sets of a fts. So, if a fuzzy point $p$ of a fuzzy set $\lambda$ in a fts $X$ is a fuzzy cluster point of $\lambda$ then $p$ is also a fuzzy $\delta$-cluster point of $\lambda$.

A fuzzy regularly closed set is not always a fuzzy $\delta$-closed set. For example, consider $\mu_1$ and $\mu_2$ as in Example 3.2 and the fuzzy topology $\tau = \{0, \mu_1, \mu_2, \mu_1 \cup \mu_2, 1\}$ on $I$. Then $\mu_1$ and $\mu_2$ are fuzzy regularly open sets (cf. [1, Example 4.5]). Therefore $\mu_1'$ and $\mu_2'$ are fuzzy regularly closed set. Again $\mu_1' \cap \mu_2 \neq 0$ and $\mu_1' \cap \mu_1 \neq 0$. By easy calculation one can observe that $\delta$-closures of the fuzzy regularly closed set $\mu_1'$ are the whole set $I$ and all the fuzzy sets of $I$ having same supports as $I$.

DEFINITION 4.4. A mapping $f$ from a fts $X$ into a fts $Y$ is said to be fuzzy $\delta$-continuous at a fuzzy point $p$ in $X$, if for each fuzzy open nbd $\lambda$ of $f(p)$, there exists an open fuzzy nbd $\mu$ of $p$ such that $f(\operatorname{Int}(\operatorname{Cl}\mu)) \leqslant \operatorname{Int}(\operatorname{Cl}\lambda)$.

THEOREM 4.1. *Let $f$ be a mapping from a* fts $(X, \tau X)$ *into a* fts $(Y, \tau Y)$. *Then the following two conditions are equivalent*

(a) *$f$ is fuzzy $\delta$-continuous,*

(b) *for each fuzzy point $p$ in $X$ and each fuzzy regular open set $\lambda$ containing $f(p)$, there exists a fuzzy regular open set $\mu$ containing $p$ such that $f(\mu) \leqslant \lambda$.*

*Proof.* (a)$\Rightarrow$(b) Let $p$ be a fuzzy point in $X$ and $\lambda$ be a fuzy regularly open set containing $f(p)$. Since every fuzzy regularly open set is a fuzzy open set, it follows from (a) that there exists a fuzzy open nbd $\mu_1$ of $p$ such that, $f(\mathrm{Int}(\mathrm{Cl}\,\mu_1)) \leqslant \mathrm{Int}(\mathrm{Cl}\,\lambda) = \lambda$. Taking $\mathrm{Int}(\mathrm{Cl}\,\mu_1) = \mu$ (being a fuzzy regularly open set containing $p$). We obtain (b).

(b)$\Rightarrow$(a) Let $p$ be a fuzzy point of $X$ and $\lambda$ be a fuzzy open nbd containing $f(p)$. Then $\mathrm{Int}(\mathrm{Cl}\,\lambda)$ is a fuzzy regular open set containing $f(p)$ ($\lambda \leqslant \mathrm{Int}(\mathrm{Cl}\,\lambda)$). By (b), there is a fuzzy regularly open set $\mu$ containing $p$ such that $f(\mu) \leqslant \mathrm{Int}(\mathrm{Cl}\,\lambda)$ (i.e., since a fuzzy regular open set is a fuzzy open set and $\mathrm{Int}(\mathrm{Cl}\,\mu) = \mu$) there is a fuzzy open set $\mu$ containing $p$ such that $f(\mathrm{Int}(\mathrm{Cl}\,\mu)) \leqslant \mathrm{Int}(\mathrm{Cl}\,\lambda)$).

THEOREM 4.2. *Let $f$ be a surjection from a* fts $(X, \tau X)$ *onto a* fts $(Y, \tau Y)$. *Then the following implications hold.*

(a) *$f$ is fuzzy δ-continuous*
$\Rightarrow$(b) *$f([\lambda]_\delta) \leqslant [f(\lambda)]_\delta$ for every fuzzy set $\lambda$ in $X$.*
$\Rightarrow$(c) *$[f^{-1}(\mu)]_\delta \leqslant f^{-1}([\mu]_\delta)$ for every fuzzy set $\mu$ in $Y$.*
$\Rightarrow$(d) *for every fuzzy δ-closed set $\mu$ in $Y$, $f^{-1}(\mu)$ is fuzzy δ-closed in $X$.*
$\Rightarrow$(e) *for every fuzzy δ-open set $\mu$ in $Y$, $f^{-1}(\mu)$ is fuzzy δ-open in $X$.*

*Proof.* (a)$\Rightarrow$(b) Let $p$ be a fuzzy point in $Y$ such that $p \in f([\lambda]_\delta)$. Then $p = f(q)$, where $q \in [\lambda]_\delta$. [Let $f: X \to Y$ be a map and $A$ be a fuzzy subset of $X$. Let $q$ be a fuzzy point of $f(A)$ defined by $q(y) = a$ for $y = y_q$ ($a \in (0, 1)$) and

$$q(y) = 0 \qquad \text{for} \quad y \neq y_q \quad (y \in Y).$$

Then

$$a = q(y_q) < f(A)(y_q) = \sup_{z \in f^{-1}(y_q)} A(z).$$

Choose $\varepsilon > 0$ so small that $a < \sup_{z \in f^{-1}(y_q)} A(z) - \varepsilon$. But there is a $z_1 \in f^{-1}(y)_q)$ such that

$$(a <) \sup_{z \in f^{-1}(y_q)} A(z) - \varepsilon < A(z_1).$$

Let us define a fuzzy point $p$ in $X$ such that $p(x) = a$ for $x = z_1$ and $p(x) = 0$ for $x \neq z_1$ ($x \in X$). Then $p \in A$ and $f(p)(y) = \sup_{z \in f^{-1}(y)} p(z)$, if $f^{-1}(y)$ is non-void

$$= 0 \qquad \text{otherwise,}$$

so that $f(p)\,(y) = a$ for $y = y_q$ and $f(p)(y) = 0$ for $y \neq y_q$ $(y \in Y)$. Therefore, $f(p) = q$].

Let $v$ be a fuzzy regularly open set containing $p$. Then there is a fuzzy regular open set $\mu$ containing $q$ such that $f(\mu) \leqslant v$ (by Theorem 4.1). $q \in [\lambda]_\delta \Rightarrow \mu \cap \lambda \neq 0$, i.e., $f(\mu \cap \lambda) \neq 0$. But $f(\mu \cap \lambda) = f(\mu) \cap f(\lambda) \neq 0$ (cf. [1]) and

$$f(\mu) \leqslant v \Rightarrow v \cap f(\lambda) \neq 0 \Rightarrow p \in [f(\lambda)]_\delta.$$

(b) $\Rightarrow$ (c) $f^{-1}(\mu)$ is a fuzzy set in $X$ for every fuzzy set $\mu$ in $Y$. By (b), $f([f^{-1}(\mu)]_\delta) \leqslant [f(f^{-1}(\mu))]_\delta$, so that $f([f^{-1}(\mu)]_\delta) \leqslant [\mu]_\delta$. [$f$ being a surjection, $f^{-1}(y)$ is not empty for all $y \in Y$. Then,

$$f(f^{-1}(\mu))(y) = \sup_{z \in f^{-1}(y)} \{f^{-1}(\mu)(z)\} = \sup_{z \in f^{-1}(y)} \{\mu(f(z))\}$$
$$= \mu(y) \qquad \text{for} \quad y \in Y]$$

Thus $[f^{-1}(\mu)]_\delta \leqslant f^{-1}([\mu]_\delta)$.

(c) $\Rightarrow$ (d). Let $\mu$ be a fuzzy $\delta$-closed set in $Y$. Then $[\mu]_\delta = \mu$. (c) $\Rightarrow [f^{-1}(\mu)]_\delta \leqslant f^{-1}([\mu]_\delta) = f^{-1}(\mu)$. So, $f^{-1}(\mu)$ is fuzzy $\delta$-closed set in $X$.

(d) $\Rightarrow$ (e) Let $\mu$ be a fuzzy $\delta$-open set in $Y$. Then $\mu'$ is fuzzy $\delta$-closed set in $Y$. (d) $\Rightarrow f^{-1}(\mu')$ is fuzzy $\delta$-closed set in $X$. But $f^{-1}(\mu') = (f^{-1}(\mu))'$ (cf. [3]) so $f^{-1}(\mu)$ is fuzzy $\delta$-open in $X$.

THEOREM 4.3. *If $f$ is a mapping from a* fts $(X, \tau X)$ *into a* fts $(Y, \tau Y)$ *and $g$ is a mapping from $Y$ into a* fts $(Z, \tau Z)$, *then so is $g \circ f$.*

*Proof.* Straightforward.

The following is an example of a fuzzy $\delta$-continuous mapping that is not fuzzy continuous.

EXAMPLE 4.1. Let $\{I_\alpha : \alpha \in \Lambda\}$ be the usual interval base of the relative topology on [0, 1] induced by the set of reals $R$. We define a fuzzy topology $\tau_1$ on [0, 1] generated by the base consisting of 0, 1 and $\{I_{\alpha\beta} : \alpha \in \Lambda,\ \beta \in (0, 1)\}$, where

$$I_{\alpha\beta}(x) = \beta \qquad \text{for all } x \in I_\alpha$$
$$= 0 \qquad \text{otherwise.}$$

Then $\tau_1$ forms a fuzzy topology on $I$ (cf. [5]). Let $\tau_2$ be the fuzzy topology on $I$ such that the complement of any member of $\tau_2$ is a countable fuzzy subset in $I$. Let $f: (I, \tau_1) \rightarrow (I, \tau_2)$ defined by $f(x) = x$ $(x \in X)$. Then it is easy to show that $f$ is not continuous [e.g., let $\mu$ be a fuzzy subset of $I$ defined by

$$\mu(x)=\begin{cases}1 & \text{for} \quad x\in(0,1]\\ \beta & x=0.\end{cases}$$

Then $\mu\in\tau_2$. But $f^{-1}(\mu)=\mu$ is not open in $(I,\tau_1)$]. $f$ is fuzzy $\delta$-continuous.

Let $p$ be a fuzzy point in $I$. Then $f(p)=p$ (it can be seen easily). Let $\mu$ be an open set in $(I,\tau_2)$ containing $p$. Let $\gamma$ be the least non-null value that $\mu$ assumes (then $\gamma\in(0,1]$) and $F$ be the support of $\mu$. Choose $\alpha_1\in\Lambda$ in such a way that $F\supset I_{\alpha_1}$. Consider the member $I_{\alpha_1\gamma}$ of $\tau_1$. Obviously $p\in I_{\alpha_1\gamma}$ and $I_{\alpha_1\gamma}\leqslant\mu$. Then $\mathrm{Cl}(I_{\alpha_1\gamma})\leqslant\mathrm{Cl}(\mu)$. So, $\mathrm{Int}(\mathrm{Cl}\,I_{\alpha_1\gamma})\leqslant\mathrm{Int}(\mathrm{Cl}\,\mu)$. But $f(\lambda)=\lambda$ holds for all fuzzy subsets $\lambda$ of $I$ by definition of $f$. Hence, $f\,(\mathrm{Int}(\mathrm{Cl}\,I_{\alpha_1\gamma}))\leqslant\mathrm{Int}(\mathrm{Cl}\,\mu)$.

The following is an example of a fuzzy continuous function which is not fuzzy $\delta$-continuous.

EXAMPLE 4.2. Let $\mu_1,\mu_2$ be fuzzy subsets of $I$ defined by

$$\mu_1(x)=\begin{cases}0, & 0\leqslant x\leqslant\frac{1}{2}\\ 2x-1, & \frac{1}{2}\leqslant x\leqslant 1;\end{cases}$$

and

$$\mu_2(x)=\begin{cases}1, & 0\leqslant x\leqslant\frac{1}{4}\\ -4x+2, & \frac{1}{4}\leqslant x\leqslant\frac{1}{2}\\ 0, & \frac{1}{2}\leqslant x\leqslant 1.\end{cases}$$

Let $\tau_1=\{0,\mu_1',1\}$ and $\tau_2=\{0,\mu_1,\mu_2,\mu_1\cup\mu_2,1\}$ be two fuzzy topologies on $I$. Let $f\colon(I,\tau_1)\to(I,\tau_2)$ defined by $f(x)=x/2$ $(x\in I)$. Then it is easy to see that

$$f^{-1}(0)=0,\qquad f^{-1}(1)=1,\qquad f^{-1}(\mu_1)=0,$$
$$f^{-1}(\mu_2)=\mu_1',\qquad f^{-1}(\mu_1\cup\mu_2)=\mu_1'.$$

Thus $f$ is fuzzy continuous. Now in $(I,\tau_1)$, $\mu_1'$ is not a fuzzy regular open set and in $(I,\tau_2)$, $\mu_1$ and $\mu_2$ are fuzzy regular open sets (cf. [1]). Therefore, for a fuzzy point $p$ in $I$ such that $f(p)\in\mu_1$ or $f(p)\in\mu_2$, there does not exist any fuzzy regular open set $\lambda$ (in $(I,\tau_1)$) containing $p$ such that $f(\lambda)\leqslant\mu_1$ or $f(\lambda)\leqslant\mu_2$. Hence $f$ is not $\delta$-continuous.

*Remark.* It follows from Examples 5.1 and 5.2 that fuzzy $\delta$-continuity and fuzzy continuity are independent concepts.

THEOREM 4.4. *Let $X_1, X_2$ and $Y_1, Y_2$ be fuzzy topological spaces such that $Y_1$ is product related to $Y_2$ and $X_1$ is product related to $X_2$. Then the product $f_1 \times f_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$ of fuzzy δ-continuous mappings $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ is fuzzy δ-continuous.*

*Proof.* Let $(p_1, p_2)$ be a fuzzy point in $X_1 \times X_2$. Let $\lambda$ be a fuzzy open set in $Y_1 \times Y_2$ containing $(f_1(p_1), f_2(p_2))$. Then $\lambda$ is the union

$$\bigcup_{\alpha \in \mathscr{A}, \beta \in \mathscr{B}} (\lambda_\alpha \times \lambda_\beta),$$

where $\mathscr{A}$ and $\mathscr{B}$ are indexing sets and $\lambda_\alpha$'s and $\lambda_\beta$'s are fuzzy open sets of $Y_1$ and $Y_2$, respectively. To obtain the result we have to show that there exists a fuzzy open set $\mu$ in $X_1 \times X_2$ containing $(p_1, p_2)$ such that, $f_1 \times f_2(\mathrm{Int}(\mathrm{Cl}\, \mu)) \leqslant \mathrm{Int}(\mathrm{Cl}\, \lambda)$ i.e.,

$$\mathrm{Int}(\mathrm{Cl}\, \mu) \leqslant (f_1 \times f_2)^{-1}(\mathrm{Int}(\mathrm{Cl}\, \lambda)) \qquad \text{[cf. [3].} \qquad \text{(A)}$$

Now

$$\begin{aligned} \mathrm{Cl}\, \lambda &= \mathrm{Cl}\left(\bigcup (\lambda_\alpha \times \lambda_\beta)\right) \\ &\geqslant \bigcup (\mathrm{Cl}(\lambda_\alpha \times \lambda_\beta)). \end{aligned}$$

So,

$$\begin{aligned} \mathrm{Int}(\mathrm{Cl}\, \lambda) &\geqslant \mathrm{Int}\left(\bigcup \mathrm{Cl}(\lambda_\alpha \times \lambda_\beta)\right) \\ &\geqslant \bigcup (\mathrm{Int}(\mathrm{Cl}(\lambda_\alpha \times \lambda_\beta))) \\ &= \bigcup (\mathrm{Int\,Cl}\, \lambda_\alpha \times \mathrm{Int\,Cl}\, \lambda_\beta). \end{aligned}$$

Now

$$\begin{aligned} &(f_1 \times f_2)^{-1}(\mathrm{Int\,Cl}\, \lambda) \\ &\quad \geqslant (f_1 \times f_2)^{-1}\left[\bigcup (\mathrm{Int\,Cl}\, \lambda_\alpha \times \mathrm{Int\,Cl}\, \lambda_\beta)\right] \\ &\quad = \bigcup [(f_1 \times f_2)^{-1}(\mathrm{Int\,Cl}\, \lambda_\alpha \times \mathrm{Int\,Cl}\, \lambda_\beta)] \\ &\quad = \bigcup [f_1^{-1}(\mathrm{Int\,Cl}\, \lambda_\alpha) \times f_2^{-1}(\mathrm{Int\,Cl}\, \lambda_\beta)] \\ &\quad \geqslant f_1^{-1}(\mathrm{Int\,Cl}\, \lambda_{\alpha_1}) \times f_2^{-1}(\mathrm{Int\,Cl}\, \lambda_{\beta_1}) \end{aligned}$$

[where $\alpha_1 \in \mathscr{A}$ and $\beta_1 \in \mathscr{B}$ are such that $(f(p_1), f(p_2)) \in \lambda_{\alpha_1} \times \lambda_{\beta_1}$ (cf. [Theorem 3.1, 8])],

$$\geqslant \mathrm{Int\,Cl}\, \mu_{\alpha_1} \times \mathrm{Int\,Cl}\, \mu_{\beta_1} = \mathrm{Int\,Cl}(\mu_{\alpha_1} \times \mu_{\beta_1}).$$

[Since $f_1$ and $f_2$ are $\delta$-continuous mappings and $\lambda_{\alpha_1}$ is fuzzy open set in $Y_1$ containing $f(p_1)$ and $\lambda_{\beta_1}$ is fuzzy open set in $Y_2$ containing $f(p_2)$, there exist fuzzy open sets $\mu_{\alpha_1}$ (containing $p_1$) in $X_1$ and $\mu_{\beta_1}$ (containing $p_2$) in $X_2$ satisfying

$$\mathrm{Int}(\mathrm{Cl}\,\mu_{\alpha_1}) \leqslant f_1^{-1}(\mathrm{Int}\,\lambda_{\alpha_1}))$$

$$\mathrm{Int}(\mathrm{Cl}\,\mu_{\beta_1}) \leqslant f_2^{-1}(\mathrm{Int}(\mathrm{Cl}\,\lambda_{\beta_1})).]$$

Taking $\mu_{\alpha_1} \times \mu_{\beta_1} = \mu$ we obtain (A).

THEOREM 4.5. *Let $X_1 \times X_2$ be the product space of two fuzzy topological spaces $(X_1, \tau X_1)$ and $(X_2, \tau X_2)$ and let $(X, \tau X)$ be an* fts. *If $f: X \to X_1 \times X_2$ is fuzzy $\delta$-continuous, then $p_i \circ f$ $(i = 1.2)$ is also fuzzy $\delta$-continuous, where $p_i: X_1 \times X_2 \to X_i$ $(i = 1.2)$ is the projection of $X_1 \times X_2$ onto $X_i$.*

*Proof.* Let $p$ be a fuzzy point in $X$ and $\lambda$ be an open set in $X_i$ containing $(p_i \circ f)(p)$ (i.e., $p_i(f(p))$ (cf. [2]). Since $p_i$ is fuzzy continuous, $p_i^{-1}(\lambda)$ is fuzzy open in $X_1 \times X_2$ and certainly $f(p) \in p_i^{-1}(\lambda)$. Since $f$ is fuzzy $\delta$-continuous, there is a fuzzy open set $\mu$ in $X$ containing $p$ such that

$$f(\mathrm{Int}(\mathrm{Cl}\,\mu)) \leqslant \mathrm{Int}(\mathrm{Cl}\,p_i^{-1}(\lambda)),$$

$p_i$ being fuzzy continuous for any fuzzy set $\nu$ of $X_i$ we have

$$\mathrm{cl}(p_i^{-1}(\nu)) \leqslant p_i^{-1}(\mathrm{Cl}\,\nu) \qquad \text{and} \qquad \mathrm{Int}\,p_i^{-1}(\nu) \geqslant p_i^{-1}(\mathrm{Int}\,\nu).$$

So, $f(\mathrm{Int}(\mathrm{Cl}\,\mu) \leqslant \mathrm{Int}(p_i^{-1}(\mathrm{Cl}\,\lambda))$. So,

$$\begin{aligned} p_i[f(\mathrm{Int}(\mathrm{Cl}\,\mu))] &\leqslant p_i(\mathrm{Int}(p_i^{-1}(\mathrm{Cl}\,\lambda))) \\ &= p_i(p_i^{-1}(\mathrm{Int}\,\mathrm{Cl}\,\lambda)) \\ &\leqslant \mathrm{Int}(\mathrm{Cl}\,\lambda) \end{aligned}$$

($p_i$ being an open map $\mathrm{Int}\,p_i^{-1}(\lambda) = p_i^{-1}(\mathrm{Int}\,\lambda)$ holds for any fuzzy set $\lambda$ of $X_i$, for detail, see Theorem 7.11 [1]).

So, $p_i \circ f(\mathrm{Int}(\mathrm{Cl}\,\mu)) \leqslant \mathrm{Int}(\mathrm{Cl}\,\lambda)$.

DEFINITION 7.1([1]). A mapping $f$ from a fuzzy topological space $(X, \tau X)$ into a fuzzy topological space $(Y, \tau Y)$ is called a fuzzy almost continuous mapping if $f^{-1}(\mu) \in \tau X$ for each fuzzy regular open set $\mu$ of $Y$.

It follows from definitions that, fuzzy $\delta$-continuity $\Rightarrow$ fuzzy almost continuity. Example 4.2 shows that the converse is not true in general.

THEOREM 4.6. *For a function from a* fts *$X$ into a* fts *$Y$, the following are true*:

(1) *If Y is fuzzy semi-regular space and f is fuzzy δ-continuous, then f is continuous.*

(2) *If X is fuzzy semi-regular space and f is fuzzy almost continuous, then f is fuzzy δ-continuous.*

*Proof.* (1) Let $p$ be a fuzzy point in $X$ and let $\lambda$ be an open fuzzy set in $Y$ containing $f(p)$. $Y$ is fuzzy semi-regular implies $\lambda = \bigcup \lambda_\alpha$, where $\lambda_\alpha$'s are fuzzy regular open sets and $f(p) \in \lambda_\alpha$ for some $\alpha$ and $f$ is fuzzy $\delta$-continuous implies there is a fuzzy regular open set $\mu$ containing $p$ such that $f(\mu) \leqslant \lambda_\alpha \leqslant \lambda$. Thus $f$ is fuzzy continuous.

(2) Let $p \in X$ and $\lambda$ be a fuzzy regular open set in $Y$ containing $f(p)$. Then $f^{-1}(\lambda)$ is fuzzy open in $X$, and $f^{-1}(\lambda) = \bigcup \lambda_\alpha$, where $\lambda_\alpha$'s are fuzzy regular open sets in $X$,

$$p \in f^{-1}(\lambda) \Rightarrow p \in \lambda_\alpha \qquad (\text{for some } \alpha) \leqslant f^{-1}(\lambda)$$

therefore,

$$f(\lambda_\alpha) \leqslant f(f^{-1}(\lambda) \leqslant \lambda.$$

So $f$ is $\delta$-continuous.

COROLLARY 4.1. *If X and Y are fuzzy semi regular spaces, then fuzzy δ-continuity, fuzzy continuity and fuzzy almost continuity of a function f: X → Y are equivalent concepts.*

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